A method for solving quadratic equations in real quaternion algebra by using Scilab software

Geanina Zaharia*
PhD student at Doctoral School of Mathematics
Ovidius University of Constanţa
Romania
geanina.zaharia@365.univ-ovidius.ro

Diana-Rodica Munteanu
Ovidius University of Constanţa
Faculty of Psychology and Educational Sciences
diana_munteanu@365.univ-ovidius.ro

Abstract. In this paper, we present some numerical applications for the equation $x^2 + ax + b = 0$, where a, b are two quaternionic elements in $\mathbb{H}(\alpha, \beta)$. $\mathbb{H}(\alpha, \beta)$ represents the algebra of real quaternions with parameterized coefficients by α and β. The algebra of real quaternions is an extension of complex numbers and is represented by algebraic objects called quaternions. These quaternions are composed of four components: a real part and three imaginary components. In general, $\mathbb{H}(\alpha, \beta)$ indicates a family of parameterized quaternion algebras, in which the specific values of α and β determine the specific properties and structure of the quaternion algebra. Based on well-known solving methods, we have developed a new numerical algorithm that solves the equation for any quaternions a and b in any algebra $\mathbb{H}(\alpha, \beta)$.

Keywords: quaternion, quadratic formula, solving polynomial equation.
MSC 2020: 17A35, 17A45, 15A18

Introduction

Quaternions are a number system first introduced in 1843 by Irish mathematician Sir William Rowan Hamilton. Hamilton was seeking a way to extend the complex numbers to three dimensions and realized that he could do so by adding an additional imaginary unit.

Quaternions are different from complex numbers in that they are non-commutative. Quaternions have found many practical applications in fields such as computer graphics, physics, and engineering. For instance, they are used in computer graphics to represent 3D rotations and orientations, and in aerospace engineering to model spacecraft altitude and control systems.

Quaternions are essential in control systems for guiding aircraft and rockets: each quaternion has an axis indicating the direction and a magnitude determining the size of the rotation. Instead of representing an orientation change

*. Corresponding author

through three separate rotations, quaternions use a single rotation to achieve the same transformation.

Despite their usefulness, quaternions are not as widely used as complex numbers, largely due to their non-commutative nature. However, they remain an important topic in mathematics and physics, and continue to be studied and applied in various fields to this day. ([1], [4], [6], [10], [13])

We will numerically solve the monic quadratic equation with quaternion coefficients in the algebra $\mathbb{H}(\alpha, \beta)$ using Scilab, a free and open-source software for numerical computation.

We chose to use the Scilab software to numerically solve the monic quadratic equations with quaternionic coefficients in the algebra $\mathbb{H}(\alpha, \beta)$ because Scilab is a free and open-source software, making it accessible and usable by a large number of users. Additionally, this software allows us to customize and adapt it to the specific needs and requirements of our problem. Scilab is renowned for its powerful functionality in numerical computation. It offers a wide range of mathematical and algebraic functions, including an integrated solver for polynomial equations. The built-in polynomial equation solver in Scilab provides us with the necessary tools to efficiently solve the monic quadratic equation with quaternionic coefficients. Scilab, such as Matlab, which is more widely known, has a user-friendly and intuitive interface, facilitating ease of use and navigation within the software. The programming is very intuitive and doesn't require definition of any parameters, so the main focus remains the mathematical modeling of the equations and the algorithm. This decision allows us to obtain precise and efficient results in studying and applying our new findings in quaternion algebra.

The aim of the paper is to present an innovative, efficient, and accurate method for the numerical solution of monic quadratic equations in the algebra of real quaternions using the Scilab software. We develop a new algorithm that solves these equations for any quaternionic coefficients in any algebra $\mathbb{H}(\alpha, \beta)$. Our ultimate goal is to contribute to the development and application of this knowledge in various fields such as computer graphics, physics, and engineering, opening up new research and application perspectives for quaternions and monic quadratic equations with quaternionic coefficients.

1. Preliminaries

The quadratic equation has been explored in the context of Hamilton quaternions in the works [11], [13]. In [11], the equation $x^2 + bx + c = 0$ is analyzed and explicit formulas for its roots are obtained. These formulas were subsequently used in the classification of quaternionic Möbius transformations [14], [2]. In Hamilton quaternions, every nonzero element can be inverted, while in $\mathbb{H}(\alpha, \beta)$ there exist split quaternions that cannot be inverted. In an algebraic system, finding the roots of a quadratic equation is always connected to the factorization of a quadratic polynomial [12]. In the case of real numbers ($\mathbb{R}$) and complex numbers ($\mathbb{C}$), the two problems are identical. However, in noncommutative algebra, these two problems are interconnected. Scharler et al. [15] analyzed the factorizability of a quadratic split quaternion polynomial, revealing certain information about the roots of a split quaternionic quadratic equation.

In a publication from 2022, [7] exploring algebras derived from the Cayley-Dickson process presents challenges in achieving desirable properties due to computational complexities. Hence, the discovery of identities within these algebras it gains meaning, helping to acquire new properties and making calculations easier. To this end, the study introduces several fresh identities and properties within the algebras derived from the Cayley-Dickson process. Furthermore, when certain elements serve as coefficients, quadratic equations in real division quaternion algebra can be solved, showcasing the authors ability to provide direct solutions without relying on specialized software.

In the paper [3], the author specifically focuses on deriving explicit formulas for the roots of the quadratic equation $x^2 + bx + c = 0$ where b and c are split quaternions ($\mathbb{H}_S$).

The same subject can be found in [1], where quadratic formulas for generalized quaternions are studied. It focuses on obtaining explicit formulas for the roots of quadratic equations in this specific context of generalized quaternions.

Let $\mathbb{H}(\alpha, \beta)$ be the generalized quaternion algebra over an arbitrary field $\mathbb{K}$, that is the algebra of the elements of the form $q = q_1 + q_2e_1 + q_3e_2 + q_4e_3$ where $q_i \in \mathbb{K}$, $i \in \{1, 2, 3, 4\}$, and the basis elements $\{1, e_1, e_2, e_3\}$ satisfy the following multiplication table:

(1)

$\cdot$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	α	e_3	αe_2
e_2	e_2	$-e_3$	β	$-\beta e_1$
e_3	e_3	$-\alpha e_2$	βe_1	$-\alpha\beta$

The conjugate of a quaternion is obtained by changing the sign of the imaginary part: $\overline{q} = q_1 - q_2e_1 - q_3e_2 - q_4e_3$, where $q = q_1 + q_2e_1 + q_3e_2 + q_4e_3$.

The norm of a quaternion is defined as the sum of the squares of its components, for this case, the norm is:

$$\boldsymbol{n}(q) = q \cdot \overline{q} = ||q||^2 = q_1^2 - \alpha q_2^2 - \beta q_3^2 + \alpha\beta q_4^2.$$

If for $x \in \mathbb{H}(\alpha, \beta)$, the relation $n(x) = 0$ implies $x = 0$, then the algebra $\mathbb{H}(\alpha, \beta)$ is called a division algebra, otherwise the quaternion algebra is called a split algebra. (see [4])

If α and β are negative real numbers, it becomes a division algebra, therefore the norm will be different from zero. The role of α and β is to parameterize the coefficients of the quaternion algebra $\mathbb{H}(\alpha, \beta)$. These values determine the specific properties and structure of the quaternion algebra. In the multiplication table given in equation (1), α and β appear as parameters that determine the specific structure and properties of the quaternion algebra $\mathbb{H}(\alpha, \beta)$.

The role of the norm is to provide a measure of the size of a quaternion in the algebra $\mathbb{H}(\alpha, \beta)$. The norm expression involves the coefficients q_1, q_2, q_3, q_4, and the parameters α and β. The norm plays a crucial role in determining whether the algebra $\mathbb{H}(\alpha, \beta)$ is a division algebra or a split algebra, based on whether the norm is nonzero or zero, respectively.

Split quaternions form an algebraic structure and are linear combinations with real coefficients. Every quaternion can be written as a linear combination of the elements 1, e_1, e_2, and e_3, where e_1, e_2, and e_3 are the imaginary units that satisfy the relations $e_1^2 = \alpha$, $e_2^2 = \beta$, and $e_3^2 = -\alpha\beta$.

We will now present some of the most important properties and relations of quaternions, which play a fundamental role in various fields such as physics, engineering, computer science, and applied mathematics:

- The addition is done component-wise:
  $a = a_1 \cdot 1 + a_2 e_1 + a_3 e_2 + a_4 e_3$,
  $b = b_1 \cdot 1 + b_2 e_1 + b_3 e_2 + b_4 e_3$,
  $\Rightarrow a + b = (a_1 + b_1) \cdot 1 + (a_2 + b_2) e_1 + (a_3 + b_3) e_2 + (a_4 + b_4) e_3$.
- Quaternion multiplication is not commutative:
  $a \cdot b = (a_1 b_1 + \alpha a_2 b_2 + \beta a_3 b_3 - \alpha\beta a_4 b_4) + e_1(a_1 b_2 + a_2 b_1 - \beta a_3 b_4 + \beta a_4 b_3) +$
  $e_2(a_1 b_3 + \alpha a_2 b_4 + a_3 b_1 - \alpha a_4 b_2) + e_3(a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1)$
  $b \cdot a = (a_1 b_1 + \alpha a_2 b_2 + \beta a_3 b_3 - \alpha\beta a_4 b_4) + e_1(a_2 b_1 + a_1 b_2 - \beta a_4 b_3 + \beta a_3 b_4) +$
  $e_2(a_3 b_1 + \alpha a_4 b_2 + a_1 b_3 - \alpha a_2 b_4) + e_3(a_4 b_1 + a_3 b_2 - a_2 b_3 + a_1 b_4)$
  $\Rightarrow a \cdot b \neq b \cdot a$.
- Quaternions are associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot b \cdot c$.
- The trace of the element q:
  $$t(q) = q + \overline{q}.$$
- The multiplication of a quaternion by a scalar:
  $$\alpha \cdot q = \alpha \cdot (q_1 + q_2 e_1 + q_3 e_2 + q_4 e_3) = (\alpha \cdot q_1) + (\alpha \cdot q_2) \cdot e_1 + (\alpha \cdot q_3) \cdot e_2 + (\alpha \cdot q_4) \cdot e_3.$$
- The inverse of a non-zero quaternion q is given by
  $$q^{-1} = \frac{\overline{q}}{||q||^2} = \frac{q_1 - q_2 e_1 - q_3 e_2 - q_4 e_3}{q_1^2 - \alpha q_2^2 - \beta q_3^2 + \alpha\beta q_4^2}.$$
- The dot product of two quaternions can be defined as $q \cdot r = (qr + rq)/2$.

These are just some of the many important relations and properties of quaternions. All these properties make quaternions a powerful tool in mathematics and practical applications.

2. Known results

In [16] and [17], to find the root of the equation $f(x_t) = 0$, the Newton-Raphson method relies on the Taylor series expansion of the function around the estimate x_i to find a better estimate x_{i+1}:

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \mathcal{O}(h^2),$$

where x_{i+1} is the estimate of the root after iteration $i+1$ and x_i is the estimate at iteration i. $\mathcal{O}(h^2)$ means the order of error of the Taylor series around the point x_i. Assuming $f(x_{i+1}) = 0$ and rearranging:

$$x_{i+1} \approx x_i - \frac{f(x_i)}{f'(x_i)}.$$

The procedure is as follows. Setting an initial guess x_0, a tolerance ε_s, and a maximum number of iterations N:

At iteration i, calculate $x_i \approx x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}$ and ε_r. If $\varepsilon_r \leq \varepsilon_s$ or if $i \geq N$, stop the procedure. Otherwise, repeat.

In [10], the authors present specific formulas to solve the monic quadratic equation $x^2 + bx + c = 0$ with $b, c \in \mathbb{H}(\alpha, \beta)$, where $\alpha = -1$, $\beta = -1$, the real division algebra, according to the multiplication table presented in (1). In the following we present the results we will use in developing our solutions, and a proof of lemma 2:

Lemma 2.1 ([10], Lemma 2.1)**.** *Let $A, B, C \in \mathbf{R}$ with the following properties: $C \neq 0$, $A < 0$ implies $A^2 < 4B$.*

Then the equation of order 3:

$$y^3 + 2Ay^2 + (A^2 - 4B)y - C^2 = 0 \tag{2}$$

has exactly one positive solution y.

Lemma 2.2 ([10], Lemma 2.2)**.** *Let $A, B, C \in R$ such that: $B \geq 0$ and $A < 0$ implies $A^2 < 4B$ then the real system:*

$$\begin{cases} Y^2 - (A + W^2)Y + B = 0, \\ W^3 + (A - 2Y)W + C = 0 \end{cases} \tag{3}$$

has at most two solutions (W, Y) with $W \in \mathbf{R}$ and $Y \geq 0$ as follows:

- *(i)* $W = 0$, $Y = \frac{A \pm \sqrt{A^2 - 4B}}{2}$ *provided that $C = 0$, $A^2 \geq 4B$;*
- *(ii)* $W = \pm\sqrt{2\sqrt{B} - A}$, $Y = \sqrt{B}$ *provided that $C = 0$, $A^2 < 4B$.*
- *(iii)* $W = \pm\sqrt{z}$, $Y = \frac{W^3 + AW + C}{2W}$ *provided that $C \neq 0$ and z is the unique positive solution of the real polynomial:*

$$z^3 + 2Az^2 + (A^2 - 4B)z - C^2 = 0.$$

Proof. Let $A, B, C \in \mathbf{R}$ such that $B \geq 0$ and $A < 0 \implies A^2 < 4B$.
We want to show that the real system has at most two solutions (W, Y) with $W \in \mathbf{R}$ and $Y \geq 0$ as follows:

(i) $W = 0$, $Y = \frac{A \pm \sqrt{A^2 - 4B}}{2}$ provided that $C = 0$, $A^2 \geq 4B$;

(ii) $W = \pm\sqrt{2\sqrt{B} - A}, Y = \sqrt{B}$ provided that $C = 0$, $A^2 < 4B$;

(iii) $W = \pm\sqrt{z}, Y = \frac{W^3 + AW + C}{2W}$ provided that $C \neq 0$ and z is the unique positive solution of the real polynomial:

$$z^3 + 2Az^2 + (A^2 - 4B)z - C^2 = 0.$$

From Lemma 2.1, we know that the polynomial $z^3 + 2Az^2 + (A^2 - 4B)z - C^2 = 0$ has exactly one positive solution z when $C \neq 0$.

For the cases (i) and (ii), when $C = 0$, the first equation becomes a quadratic equation in Y. If $A^2 \geq 4B$, there are two real solutions for Y, and if $A^2 < 4B$, there is one real solution for Y. Since $W = 0$, these solutions correspond to the cases 1. and 2. in the lemma.

The case (iii), when $C \neq 0$, we can express Y as a function of W using the second equation: $Y = \frac{W^3 + AW + C}{2W}$. Substituting this expression for Y in the first equation, we obtain a polynomial equation in W^2 of degree 3. Since z is the unique positive solution of this polynomial, there are two solutions for W: $W = \pm\sqrt{z}$. These solutions correspond to the case 3. in the lemma.

In conclusion, the real system (3) has at most two solutions (W, Y) with $W \in \mathbf{R}$ and $Y \geq 0$ as described in the lemma. □

Theorem 2.3 ([10], Theorem 2.3)**.** *The solution of the quadratic equation* $x^2 + bx + c = 0$ *can be obtained in the following way:*

Case 1. If $b, c \in \mathbf{R}$ *and* $b^2 < 4c$ *then:*

$$x = \frac{1}{2}(-b + e \cdot e_1 + f \cdot e_2 + g \cdot e_3), \tag{4}$$

where $e^2 + f^2 + g^2 = 4c - b^2$ *where* $e, f, g \in \mathbf{R}$.

Case 2. If $b, c \in \mathbf{R}$ *and* $b^2 \geq 4c$ *then:*

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \tag{5}$$

Case 3. If $b \in \mathbf{R}$, $c \notin \mathbf{R}$ *then:*

$$x = \frac{-b}{2} \pm \frac{m}{2} \mp \frac{c_1}{m} \cdot e_1 \mp \frac{c_2}{m} \cdot e_2 \mp \frac{c_3}{m} \cdot e_3, \tag{6}$$

where $c = c_0 + c_1 \cdot e_1 + c_2 \cdot e_2 + c_3 \cdot e_3$, *and*

$$m = \sqrt{\frac{b^2 - 4c_0 + \sqrt{(b^2 - 4c_0)^2 + 16(c_1^2 + c_2^2 + c_3^2)}}{2}}. \tag{7}$$

Case 4. If $b \notin \mathbf{R}$ then:

$$x = \frac{(-Re(b))}{2} - (b' + W)^{-1}(c' - Y), \tag{8}$$

where $b' = b - Re(b) = Im(b)$, $c' = c - (Re(b)/2)(b - (Re(b))/2)$, where (W, Y) are chosen in the following way:

(i) $W = 0$, $Y = (A \pm \sqrt{A^2 - 4B})/2$ *provided that* $C = 0$, $A^2 \geq 4B$;

(ii) $W = \pm\sqrt{2\sqrt{B} - A}$, $Y = \sqrt{B}$ *provided that* $C = 0$, $A^2 < 4B$;

(iii) $W = \pm\sqrt{z}$, $Y = (W^3 + AW + C)/2W$ *provided that* $C \neq 0$ *and* z *is the unique positive solution of the equation:*

$$z^3 + 2Az^2 + (A^2 - 4B)z - C^2 = 0,$$

where $A = |b'|^2 + 2Re(c')$, $B = |c'|^2$ *and* $C = 2Re(\overline{b'}c')$.

Corollary 2.4 ([10], Corollary 2.4)**.** *The equation has an infinity of solutions if $b, c \in \mathbf{R}$ and $b^2 < 4c$.*

Corollary 2.5 ([10], Corollary 2.6)**.** *The equation has an unique solution if and only if:*

1. $b, c \in \mathbf{R}$ *and* $b^2 - 4c = 0$;
2. $b \notin \mathbf{R}$ *and* $C = 0 = A^2 - 4B$.

Corollary 2.6. *If the quadratic equation $x^2 + bx + c = 0$ has real coefficients b and c, and $b^2 < 4c$, then the solution of the equation can be expressed as $x = \frac{1}{2}(-b + e \cdot e_1 + f \cdot e_2 + g \cdot e_3)$, where $e^2 + f^2 + g^2 = 4c - b^2$ and $e, f, g \in \mathbf{R}$.*

Corollary 2.7. *If the quadratic equation $x^2 + bx + c = 0$ has real coefficients b and c, and $b^2 \geq 4c$, then the solutions of the equation are $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$.*

Corollary 2.8. *If b and c are the coefficients of the quadratic equation $x^2 + bx + c = 0$, such that $b \notin \mathbf{R}$, then the solution of the equation can be expressed as:*

$$x = \frac{(-Re(b))}{2} - (b' + W)^{-1}(c' - Y),$$

where $b' = b - Re(b) = Im(b)$, $c' = c - (Re(b)/2)(b - (Re(b))/2)$, and (W, Y) are chosen such that:

- $W = 0$, $Y = (A \pm \sqrt{A^2 - 4B})/2$ *if* $C = 0$ *and* $A^2 \geq 4B$;
- $W = \pm\sqrt{2\sqrt{B} - A}$, $Y = \sqrt{B}$ *if* $C = 0$ *and* $A^2 < 4B$;
- $W = \pm\sqrt{z}$, $Y = (W^3 + AW + C)/2W$ *if* $C \neq 0$ *and* z *is the unique positive solution of the equation* $z^3 + 2Az^2 + (A^2 - 4B)z - C^2 = 0$, *where* $A = |b'|^2 + 2Re(c')$, $B = |c'|^2$ *and* $C = 2Re(\overline{b'}c')$.

3. The solutions of the second-degree equation in real quaternions

It is important to mention that the algebra $\mathbb{H}(\alpha,\beta)$ is a mathematical construction, and its properties can vary depending on the values chosen for α and β. When we take negative values for α and β in the algebra $\mathbb{H}(\alpha,\beta)$), it becomes a division algebra. This means that every nonzero element in the algebra can be inverted. Multiplication and inversion of elements can be performed using the specific rules of this algebra.

Therefore, for the algebra $\mathbb{H}(\alpha,\beta)$, we will take negative values for α and β, thus making it a division algebra, and the norm will be nonzero. If the values of α and β are positive, we no longer have a division algebra because the norm is zero.

Next, we will describe the solution of a monic quadratic equation in the algebra of real quaternions. This statement provides an explicit formula for finding the solutions of the equation and explains how to perform the necessary calculations. It presents the general formula for the solution of the monic quadratic equation, where the equation's coefficients are represented as real quaternions, and the solution is a linear combination of the imaginary units of the quaternions. This formula is presented in a detailed manner, specifying the values of each component of the solution in terms of the coefficients and other terms involved in the equation.

Proposition 3.1. *Let $b = b_0+b_1\cdot e_1+b_2\cdot e_2+b_3\cdot e_3$ and $c = c_0+c_1\cdot e_1+c_2\cdot e_2+c_3\cdot e_3$ where b, c are two quaternionic elements in $\mathbb{H}(\alpha,\beta)$ and knowing W and Y of the Theorem 2.3 the solution of the second degree equation $x^2 + bx + c = 0$ is of the form*

$$x = x_1 + x_2e_1 + x_3e_2 + x_4e_3, \tag{9}$$

where:

$$x_1 = -t-[Wc_1-YW-b_2c_2\alpha-b_3c_3\beta+b_4c_4\alpha\beta-t(Wt-b_2^2\alpha-b_3^2\beta+b_4^2\alpha\beta)]/m,$$
$$x_2 = (Wc_2 - b_2c_1 + b_2Y + b_3c_4\beta - b_4c_3\beta - tb_2(W - t))/m,$$
$$x_3 = (Wc_3 - b_2c_4\alpha - b_3c_1 + b_3Y + b_4c_2\alpha - tb_3(W - t))/m,$$
$$x_4 = (Wc_4 - b_2c_3 + b_3c_2 + b_4c_1 + b_4Y - tb_4(W - t))/m$$

with $t = \frac{b_1}{2}$ and

$$m = W^2 - \alpha b_2^2 - \beta b_3^2 + \alpha\beta b_4^2.$$

Proof. Let $b = b_1 + b_2 \cdot e_1 + b_3 \cdot e_2 + b_4 \cdot e_3$ and $c = c_1 + c_2 \cdot e_1 + c_3 \cdot e_2 + c_4 \cdot e_3$. for this case, the norm is:

$$\boldsymbol{n}(a) = a\bar{a} = a_1^2 - \alpha a_2^2 - \beta a_3^2 + \alpha\beta a_4^2.$$

We compute the necessary elements for applying the theorem: $Re(b) = b_1$. Therefore,

$$b' = b - Re(b) = Im(b) = b_2 \cdot e_1 + b_3 \cdot e_2 + b_4 \cdot e_3$$

and

$$\begin{aligned} c' &= c - (Re(b)/2)(b - (Re(b))/2) \\ &= c_1 + c_2 \cdot e_1 + c_3 \cdot e_2 + c_4 \cdot e_3 - \frac{b_1}{2}\left(b_1 + b_2 \cdot e_1 + b_3 \cdot e_2 + b_4 \cdot e_3 - \frac{b_1}{2}\right) \\ &= \left(c_1 - \frac{b_1^2}{2} + \frac{b_1^2}{4}\right) + \left(c_2 - \frac{b_1 b_2}{2}\right) e_1 + \left(c_3 - \frac{b_1 b_3}{2}\right) e_2 + \left(c_4 - \frac{b_1 b_4}{2}\right) e_3 \\ &= \left(c_1 - \frac{b_1^2}{4}\right) + \left(c_2 - \frac{b_1 b_2}{2}\right) e_1 + \left(c_3 - \frac{b_1 b_3}{2}\right) e_2 + \left(c_4 - \frac{b_1 b_4}{2}\right) e_3. \end{aligned}$$

Using all the above and $C = 2Re(\overline{b'}c')$, we find

$$\begin{aligned} C = 2Re((-b_2 \cdot e_1 - b_3 \cdot e_2 - b_4 \cdot e_3) \cdot ((c_1 - \frac{b_1^2}{4}) + (c_2 - \frac{b_1 b_2}{2})e_1 \\ + (c_3 - \frac{b_1 b_3}{2})e_2 + (c_4 - \frac{b_1 b_4}{2})e_3)). \end{aligned}$$

The real part is obtained only by multiplying terms of the same kind, therefore we obtain:

$$C = -2b_2c_2\alpha + b_1b_2^2\alpha - 2b_3c_3\beta + b_1b_3^2\beta + 2b_4c_4\alpha\beta - b_1b_4^2\alpha\beta$$

and $A = |b'|^2 + 2Re(c') = (-\alpha b_2^2 - \beta b_2^3 + \alpha\beta b_4^2) + 2(c_1 - \frac{b_1^2}{4})$. Then $A = -\alpha b_2^2 - \beta b_2^3 + \alpha\beta b_4^2 + 2c_1 - \frac{b_1^2}{2}$

Computing $B = |c'|^2$ we get

$$B = \left(c_1 - \frac{b_1^2}{4}\right)^2 - \alpha\left(c_2 - \frac{b_1 b_2}{2}\right)^2 - \beta\left(c_3 - \frac{b_1 b_3}{2}\right)^2 + \alpha\beta\left(c_4 - \frac{b_1 b_4}{2}\right)^2.$$

We denote $\frac{b_1}{2} = t$ and obtain:

$$B = (c_1 - t^2)^2 - \alpha(c_2 - tb_2)^2 - \beta(c_3 - tb_3)^2 + \alpha\beta(c_4 - tb_4)^2.$$

We compute W and Y according to the cases of the theorem. By denoting $m = |b' + W| = W^2 - \alpha b_2^2 - \beta b_3^2 + \alpha\beta b_4^2$ and cu $t = b_1/2$, we apply equation (8) and we find

$$\begin{aligned} x_1 &= -t - (Wc_1 - YW - b_2c_2\alpha - b_3c_3\beta + b_4c_4\alpha\beta \\ &\quad - t(Wt - b_2^2\alpha - b_3^2\beta + b_4^2\alpha\beta))/m, \\ x_2 &= (Wc_2 - b_2c_1 + b_2Y + b_3c_4\beta - b_4c_3\beta - tb_2(W - t))/m, \\ x_3 &= (Wc_3 - b_2c_4\alpha - b_3c_1 + b_3Y + b_4c_2\alpha - tb_3(W - t))/m, \\ x_4 &= (Wc_4 - b_2c_3 + b_3c_2 + b_4c_1 + b_4Y - tb_4(W - t))/m. \end{aligned}$$

We obtain the solution as

$$x = x_1 + x_2e_1 + x_3e_2 + x_4e_3.$$

□

4. Numerical applications and examples

For the implementation of numerical applications, let's consider the general case of $\mathbb{H}(\alpha, \beta)$, $b = b_1 + b_2 \cdot e_1 + b_3 \cdot e_2 + b_4 \cdot e_3$ and $c = c_1 + c_2 \cdot e_1 + c_3 \cdot e_2 + c_4 \cdot e_3$. Using Proposition 4.1, we present the algorithm from the table 1. The algorithm described has been implemented in Scilab 6.1.1. To verify our computations, we apply all the formulas, on some remarkable examples.

Steps		
1.		Input α, β, b, c
2.		Compute C, A, B
3.		Identify case
4.	If case 1: $C = 0, A \geq 4B$	Compute $W = 0$, $Y = (A \pm \sqrt{A^2 - 4B})/2$
	If case 2: $C = 0, A^2 < 4B$	Compute $W = \pm\sqrt{2\sqrt{B} - A}$, $Y = \sqrt{B}$
	If case 3: $C \neq 0$	Solve the polynomial equations $z^3 + 2Az^2 + (A^2 - 4B)z - C^2 = 0$ and find the positive root.
5.		Compute solutions using formula (9).

Table 1: Algorithm for computing the solutions of the quadratic equation.

Example 4.1 ([10], Example 2.12)**.** Consider the quadratic equation $x^2 + xe_1 + (1 + e_2) = 0$, i.e., $b = e_1$ and $c = 1 + e_2$. This belongs to Case 4 in Theorem 2.3. Then $b' = e_1$ and $c' = 1 + e_2$. Moreover, $A = 3$, $B = 2$, $C = 0$. It is Subcase 1 in Case 4. Hence, $W = 0$ and $Y = 2$ or $Y = 1$. Consequently, the two solutions are $x_1 = -e_1 + e_3$ and $x_2 = e_3$. For $\alpha = -1, \beta = -1$, the solution is:

$$C = 0.000000,$$
$$A = 3.000000,$$
$$B = 2.000000,$$
$$Y_1 = 2.000000,$$
$$Y_2 = 1.000000,$$

$$x_1 = -0.000000 - 1.000000e_1 - 0.000000e_2 + 1.000000e_3,$$
$$x_2 = -0.000000 - 0.000000e_1 - 0.000000e_2 + 1.000000e_3.$$

Example 4.2. ([10], Example 2.13) Consider the quadratic equation $x^2+xe_1+e_2=0$, i.e., $b=e_1$ and $c=e_2$. This belongs to Case 4 in Theorem 2.3. Then $b'=e_1$ and $c'=e_2$. Moreover, $A=1$, $B=1$, $C=0$. It is Subcase 2 in Case 4. Hence, $W=+1$ or -1 and $Y=1$. Consequently, the two solutions are $x_1=(e_1+1)^{-1}(1-e_2)=(1/2)(1-e_1-e_2+e_3)$ and $x_2=(e_1-1)^{-1}(1-e_2)=(1/2)(-1-e_1+e_2+e_3)$. For $\alpha=-1, \beta=-1$, the solution of the program:

$$C = 0.000000,$$
$$A = 1.000000,$$
$$B = 1.000000,$$
$$x_1 = 0.500000 - 0.500000e_1 - 0.500000e_2 + 0.500000e_3,$$
$$x_2 = -0.500000 - 0.500000e_1 + 0.500000e_2 + 0.500000e_3.$$

Example 4.3. ([10], Example 2.14) Consider the quadratic equation $x^2+xe_1+(1+e_1+e_2)=0$, i.e., $b=e_1$ and $c=1+e_1+e_2$. This belongs to Case 4 in Theorem 2.3. Then $b'=e_1$ and $c'=1+e_1+e_2$. Moreover, $A=3$, $B=3$, $C=2$. It is Subcase 3 in Case 4. Now the unique positive roots of z^3+6z^2-3z-4 is 1, and hence,$W=1$ and $Y=3$ or $W=-1$ and $Y=1$. Consequently, the two solutions are $x_1=(1/2)(1-3e_1-e_2+e_3)$ and $x_2=(1/2)(-1+e_1+e_2+e_3)$. For $\alpha=-1, \beta=-1$, the solution of the program:

$$C = 2.000000,$$
$$A = 3.000000,$$
$$B = 3.000000,$$
$$x_1 = 0.500000 - 1.500000e_1 - 0.500000e_2 + 0.500000e_3,$$
$$x_2 = -0.500000 + 0.500000e_1 + 0.500000e_2 + 0.500000e_3.$$

The results obtained in Examples 5.1-5.3 are exactly the ones obtain by direct computation by the authors in [10].

In the following, we will present a few examples using the results presented above and also calculate the solutions of the equations using the described algorithm, for different values of α and β.

Example 4.4. Next, we aim to find the solution of the equation $x^2+bx+c=0$ in the case where b and c are quaternions:

$$b = 5 \cdot 1 + 6 \cdot e_1 + 7 \cdot e_2 + 8 \cdot e_3$$

and

$$c = 2 \cdot 1 + 3 \cdot e_1 + 4 \cdot e_2 + 5 \cdot e_3.$$

For $\alpha = -1, \beta = -1$, we can compute $b' = b - Re(b) = 6e_1 + 7e_2 + 8e_3$ and

$$c' = c - \frac{1}{2}Re(b)\left(b - \frac{1}{2}Re(b)\right),$$
$$c' = \left(2 - \frac{25}{2} + \frac{25}{4}\right)1 + (3-15)\,e_1 + \left(4 - \frac{35}{2}\right)e_2 + (5-20)\,e_3.$$

Then

$$c' = -\frac{17}{4} - 12e_1 - \frac{27}{2}e_2 - 15e_3.$$

Consequently,

$$A = |b'|^2 + 2Re(c') = 6^2 + 7^2 + 8^2 + 2\left(-\frac{17}{4}\right) = 140,5,$$
$$B = |c'|^2 = \left(\frac{-17}{4}\right)^2 + 12^2 + \left(\frac{27}{2}\right)^2 + (15)^2 = 569,3125,$$
$$C = 2Re(\overline{b'}c') = -573.$$

We can check that $A^2 \geq 4B$, so we can use case 4. Using the formulas in case 4, the next step is to find the values of (W, Y) using one of the three situations described in the formula from case 4. Since $C \neq 0$, we will use situation 3 $z^3 + 2Az^2 + (A^2 - 4B)z - C^2 = 0$.

To find the unique positive solution z, we will use the Newton-Raphson method. In this case, we have:

$$f(z) = z^3 + 2Az^2 + (A^2 - 4B)z - C^2,$$
$$f'(z) = 3z^2 + 4Az + (A^2 - 4B).$$

The analytical method to find the solutions of the equation is given by choosing $z_0 = 1$ and applying the Newton-Raphson formula. We can obtain successive values for z as the fixed number given by:

$$z_1 = z_0 - \frac{f(z_0)}{f'(z_0)} = 1 - \frac{f(1)}{f'(1)},$$
$$z_2 = z_1 - \frac{f(z_1)}{f'(z_1)},$$
$$z_3 = z_2 - \frac{f(z_2)}{f'(z_2)},$$
$$z_4 = z_3 - \frac{f(z_3)}{f'(z_3)}.$$

Computing by this formula we use decimal fractions with many decimals, therefore we used the Scilab solver:

$$p = -328329 + 17463x + 281x^2 + x^3.$$

By using of the solver in Scilab, we obtain: $W_1 = \pm 3.871934$, and using a numerical application, we obtain:

$$\begin{aligned}
&C = -573.000000,\\
&A = 140.500000,\\
&B = 569.312500,\\
&x_1 = -0.564033 + 0.008853e_1 + 0.306465e_2 - 0.017904e_3,\\
&x_2 = -4.435967 - 5.972266e_1 - 6.647896e_2 - 7.945509e_3.
\end{aligned}$$

For $\alpha = -2, \beta = -3$, the solution is

$$\begin{aligned}
&C = -2295.000000,\\
&A = 594.500000,\\
&B = 2202.812500,\\
&W = \pm 3.813764,\\
&x_1 = -0.593118 + 0.012038e_1 + 0.168839e_2 - 0.004699e_3,\\
&x_2 = -4.406882 - 5.982890e_1 - 6.819067e_2 - 7.985585e_3.
\end{aligned}$$

Example 4.5 ([7]). We aim to solve the following equation: $x^2 + (2 + 3e_1 + 4e_2 + 5e_3)x + (4 - 5e_1 - 6e_2 - 7e_3) = 0$. For $\alpha = -1, \beta = -1$, we write:

$$(a+be_1+ce_2+de_3)^2+(2+3e_1+4e_2+5e_3)(a+be_1+ce_2+de_3)+(4-5e_1-6e_2-7e_3) = 0.$$

We expand this equation and group the terms based on the quaternionic units:

$$\begin{aligned}
&(a^2 - b^2 - c^2 - d^2 + 2a - 3b - 4c - 5d + 4) + (2ab + 3a + 2b - 5c + 4d - 5)e_1\\
&+ (2ac + 4a + 5b + 2c - 3d - 6)e_2 + (2ad + 5a - 4b + 3c + 2d - 7)e_3 = 0.
\end{aligned}$$

Thus, we can obtain a system of linear equations with 4 equations and 4 unknowns:

$$\begin{cases}
a^2 - b^2 - c^2 - d^2 + 2a - 3b - 4c - 5d + 4 = 0,\\
2ab + 3a + 2b - 5c + 4d - 5 = 0,\\
2ac + 4a + 5b + 2c - 3d - 6 = 0,\\
2ad + 5a - 4b + 3c + 2d - 7 = 0.
\end{cases}$$

Solving this system of equations can provide us with the quaternionic solutions to the initial equation. Unfortunately, this system does not seem to have a simple and analytical solution, but we can try to solve it numerically or look for a specialized method for solving quaternionic equations.

Using the algorithm, we found the following results:

$$\begin{aligned}
&C = -248.000000,\\
&A = 56.000000,\\
&B = 317.000000,\\
&x_1 = 0.988335 + 0.435138e_1 - 0.199557e_2 + 0.624407e_3,\\
&x_2 = -2.988335 - 3.374360e_1 - 5.198324e_2 - 5.563629e_3.
\end{aligned}$$

For $\alpha = -2.35, \beta = -100$, the solution of the equations is

$$x_1 = 1.416406 + 0.030602e_1 - 0.009466e_2 + 0.006083e_3,$$
$$x_2 = -3.416406 - 2.977407e_1 - 4.019286e_2 - 5.005551e_3.$$

Moreover, $C = -36312.800000$, $A = 7502.150000$, $B = 43999.400000$.

Example 4.6. Next, we aim to find the solution of the equation in the case where b and c are quaternions: $b = 1.25 + 0.2e_1 - 0.31e_2 - 0.69e_3$ and $c = -1 + 0.56e_1 - 2.35e_2 - 4.56e_2$. Then, the equations is

$$x^2 + (1.25 + 0.2e_1 - 0.31e_2 - 0.69e_3)x - 1 + 0.56e_1 - 2.35e_2 - 4.56e_2 = 0.$$

Using the program, for $\alpha = -1, \beta = -1$, we found the following results:

$$C = 7.208550,$$
$$A = -2.169050,$$
$$B = 23.819054,$$
$$W = \pm 3.485216,$$
$$x_1 = 1.117608 + -0.251329e_1 + 0.667362e_2 + 1.505501e_3,$$
$$x_2 = -2.367608 + 0.018740e_1 - 0.560890e_2 - 0.861963e_3.$$

For $\alpha = -6, \beta = -8.5$, the solution is

$$C = 302.988862,$$
$$A = 22.556700,$$
$$B = 911.964612,$$
$$W = \pm 7.155732,$$
$$x_1 = 2.952866 - 0.219073e_1 + 0.340546e_2 + 0.917961e_3,$$
$$x_2 = -4.202866 - 0.027102e_1 - 0.234104e_2 - 0.235706e_3.$$

Example 4.7. Next, we aim to calculate by using of the program an example where $C = 0$:

Find the solutions of the equation: $x^2 + (e_1 + e_2 + e_3)x + (-3e_1 - 4e_2 + 7e_3) = 0$.

We can see that $b = e_1 + e_2 + e_3 \notin \mathbb{R}$, so we need to use the formula from case 4. Firstly, we will calculate the values of b', c', A, B, and C:

$$b' = b - \operatorname{Re}(b) = e_1 + e_2 + e_3,$$
$$c' = c - \frac{\operatorname{Re}(b)}{2}(b - \frac{\operatorname{Re}(b)}{2}) = -3e_1 - 4e_2 + 7e_3,$$
$$A = |b'|^2 + 2\operatorname{Re}(c') = 3,$$
$$B = |c'|^2 = 74,$$
$$C = 2\operatorname{Re}(\overline{b'}c') = 0.$$

The next step is to find the values of (W, Y) using one of the three situations described in the formula from case 4. Since $C = 0$ and $A^2 < 4B$. Now we can calculate (W, Y): $W = \pm\sqrt{2\sqrt{B} - A} = \pm 3,7689057476$ and $Y = \sqrt{B} = 8,602325267$.

By using of the program, we have found the following results:

$$
\begin{aligned}
&C = 0.000000,\\
&A = 3.000000,\\
&B = 74.000000,\\
&W = \pm 3.768906,\\
&Y = 8.602325,\\
&x_1 = 1.884453 + 0.796552e_1 + 0.608748e_2 - 2.091566e_3,\\
&x_2 = -1.884453 - 0.517828e_1 - 1.143758e_2 + 0.975319e_3.
\end{aligned}
$$

The same equation can be solved for $\alpha = -6$ and $\beta = -9$. In this case, $C \neq 0$. We get

$$
\begin{aligned}
&C = 2088.000000,\\
&A = 528.000000,\\
&B = 2844.000000,\\
&W = \pm 3.919010,\\
&x_1 = 1.959505 - 0.537980e_1 - 1.973780e_2 - 3.017625e_3,\\
&x_2 = -1.959505 + 0.399290e_1 - 0.070390e_2 + 0.024986e_3.
\end{aligned}
$$

Example 4.8. Next, we intend to use the program to calculate an example where C=0:

Let's find the solutions of the equation: $x^2 + (e_1 + e_2 + e_3)x + (-e_1 + e_3) = 0$.

We can see that $b = e_1 + e_2 + e_3 \notin \mathbb{R}$, so we need to use the formula from case 4.

Firstly, we will calculate the values of b', c', A, B and C:

$$
\begin{aligned}
&b' = b - \mathrm{Re}(b) = e_1 + e_2 + e_3,\\
&c' = c - \frac{\mathrm{Re}(b)}{2}(b - \frac{\mathrm{Re}(b)}{2}) = -e_1 + e_3,\\
&A = |b'|^2 + 2\,\mathrm{Re}(c') = 3,\\
&B = |c'|^2 = 2,\\
&C = 2\,\mathrm{Re}(\overline{b'}c') = 0.
\end{aligned}
$$

The next step is to find the values of (W, Y) using one of the three situations described in the formula of case 4. Since $C = 0$ and $A^2 \geq 4B$, we will use situation 1, $W = 0$, $Y = (A \pm \sqrt{A^2 - 4B})/2$ result $Y_1 = 2$,$Y_2 = 1$.

Calculating with the numerical application, we get:

$$C = 0.000000,$$
$$A = 3.000000,$$
$$B = 2.000000,$$
$$Y_1 = 2.000000,$$
$$Y_2 = 1.000000,$$
$$x_1 = -0.000000 - 0.333333e_1 - 0.666667e_2 - 0.333333e_3,$$
$$x_2 = -0.000000 - 0.000000e_1 - 0.333333e_2 - 0.000000e_3.$$

For $\alpha = -100, \beta = -100$, we get $C \neq 0$, like in the other example, and the solution is

$$C = 19800.000000,$$
$$A = 10200.000000,$$
$$B = 10100.000000,$$
$$W = \pm 1.940836,$$
$$x_1 = 0.970418 - 0.989912e_1 - 0.999903e_2 - 0.999995e_3,$$
$$x_2 = -0.970418 + 0.009513e_1 - 0.000097e_2 + 0.000191e_3.$$

Example 4.9. ([7]) Let f_n be the Fibonacci sequence define as $f_0 = 0, f_1 = 1$ and $f_k = f_{k-1} + f_{k-2}$. We define the quaternion $F_n = f_n + f_{n+1}e_1 + f_{n+2}e_2 + f_{n+3}e_3$.

Consider the monic quadratic equation $x^2 + F_n x + F_m = 0$. We use the same algorithm for solving the equation.

For $n = 3, m = 3$, case discussed in ([7]), we obtain $F_3 = 2 + 3e_1 + 5e_2 + 8e_3$ and the equation $x^2 + (2 + 3e_1 + 5e_2 + 8e_3)x + (2 + 3e_1 + 5e_2 + 8e_3) = 0$.

Solving the equations for $\alpha = -1, \beta = -1$, and we get

$$C = 0.000000,$$
$$A = 100.000000,$$
$$B = 1.000000,$$
$$Y_1 = 99.989999,$$
$$Y_2 = 0.010001,$$
$$x_1 = -1.000000 - 3.030306e_1 - 4.560714e_2 - 8.080816e_3,$$
$$x_2 = -1.000000 + 0.030306e_1 + 0.540306e_2 + 0.080816e_3.$$

Solving the equations for $\alpha = -6.3, \beta = -5.25$, and we get

$$C = 0.000000,$$
$$A = 2306.750000,$$

$$B = 1.000000,$$
$$Y_1 = 2306.749566,$$
$$Y_2 = 0.000434,$$
$$x_1 = -1.000000 + -3.001301e_1 - 4.870961e_2 - 8.003470e_3,$$
$$x_2 = -1.000000 + 0.001301e_1 + 0.133376e_2 + 0.003470e_3.$$

For $n = 5, m = 10$ we obtain $F_5 = 5 + 8e_1 + 13e_2 + 21e_3$, and $F_{10} = 55 + 89e_1 + 144e_2 + 233e_3$. Thus, the equation in this case is $x^2 + F_5 x + F_{10} = 0$. Then, the solution for $\alpha = -1, \beta = -1$ found by the algorithm is

$$C = 11584.000000,$$
$$A = 771.500000,$$
$$B = 52150.062500,$$
$$W = \pm 13.722364,$$
$$x_1 = 4.361182 - 9.008123e_1 - 10.308573e_2 - 23.657396e_3,$$
$$x_2 = -9.361182 + 1.019720e_1 + 5.966780e_2 + 2.645800e_3.$$

For $\alpha = -6.3, \beta = -5.25$ the solution provided by the algorithm is

$$C = -272916.525000,$$
$$A = 15974.025000,$$
$$B = 1175231.943750,$$
$$W = \pm 16.934907,$$
$$x_1 = 5.967453 - 8.058866e_1 - 11.642625e_2 - 21.158442e_3,$$
$$x_2 = -10.967453 + 0.062114e_1 + 1.552659e_2 + 0.157823e_3.$$

Example 4.10. Let p_n be the Pell sequence define as $p_0 = 0, p_1 = 1$ and $p_k = 2p_{k-1} + p_{k-2}$. Consider the quaternions $P_n = p_n + p_{n+1}e_1 + p_{n+2}e_2 + p_{n+3}e_3$. We solve the monic quadratic equation $x^2 + P_n x + P_m = 0$. For $n = 3, m = 3$, we get $P_3 = 3 + 7e_1 + 17e_2 + 41e_3$ and the equation is $x^2 + (3 + 7e_1 + 17e_2 + 41e_3)x + 3 + 7e_1 + 17e_2 + 41e_3 = 0$.

Solving the equations for $\alpha = -1, \beta = -1$ using the algorithm we obtain

$$C = -2019.000000,$$
$$A = 2020.500000,$$
$$B = 505.312500,$$
$$W = \pm 0.999011,$$
$$x_1 = -1.000494 + 0.003464e_1 + 0.292570e_2 + 0.020287e_3,$$
$$x_2 = -1.999506 - 7.003464e_1 - 16.724253e_2 - 41.020287e_3.$$

For $\alpha = -7, \beta = -6$ the solutions are

$$C = -72679.000000,$$
$$A = 72680.500000,$$
$$B = 18170.312500,$$
$$W = \pm 0.999972,$$
$$x_1 = -1.000014 + 0.000096e_1 + 0.055517e_2 + 0.000564e_3,$$
$$x_2 = -1.999986 - 7.000096e_1 - 16.944950e_2 - 41.000564e_3.$$

For $n = 12, m = 19$, the quaternions are $P_{12} = 8119 + 19601e_1 + 47321e_2 + 114243e_3$ and $P_{19} = 3880899 + 9369319e_1 + 22619537e_2 + 54608393e_3$. The equations is $x^2 + P_{12}x + P_{19} = 0$. Solving for $\alpha = -1, \beta = -1$, we get

$$C = -112279524556439.000000,$$
$$A = 15649742008.500000,$$
$$B = 201223166914529952.000000,$$
$$W = \pm 7162.787683,$$
$$x_1 = -478.106158 + 0.284778e_1 + 136.813987e_2 + 1.659808e_3,$$
$$x_2 = -7640.893842 - 19601.284778e_1 - 47185.561043e_2 - 114244.659808e_3.$$

For $\alpha = -7, \beta = -6$ the solutions are

$$C = -4041981872234103.000000,$$
$$A = 564261307428.500000,$$
$$B = 7238333535486963712.000000,$$
$$W = \pm 7162.990016,$$
$$x_1 = -478.004992 + 0.007936e_1 + 26.572949e_2 + 0.046254e_3,$$
$$x_2 = -7640.995008 - 19601.007936e_1 - 47294.465369e_2 - 114243.046254e_3.$$

Example 4.11. Consider now the Lucas number sequences define as $l_0 = 2, l_1 = 1$ and $l_n = l_{n-1} + l_{n-2}$. We define the quaternion $L_n = l_n + l_{n+1}e_1 + l_{n+2}e_2 + l_{n+3}e_3$. We solve the monic quadratic equation $x^2 + L_nx + L_m = 0$. For $n = 3, m = 8$, the quaternions are $L_3 = 4 + 7e_1 + 11e_2 + 18e_3$ and $L_8 = 47 + 76e_1 + 123e_2 + 199e_3$.

Solving the equation $x^2 + L_3x + L_8 = 0$ for $\alpha = -1, \beta = -1$, we get

$$C = 8958.000000,$$
$$A = 580.000000,$$
$$B = 42463.000000,$$
$$W = \pm 13.777285,$$
$$x_1 = 4.888642 - 8.113676e_1 - 8.726917e_2 - 20.805123e_3,$$
$$x_2 = -8.888642 + 1.040556e_1 + 5.802217e_2 + 2.878243e_3.$$

On the other hand, for $\alpha = -3, \beta = -10$, the solution are

$$\begin{aligned}
C &= 200864.000000,\\
A &= 11163.000000,\\
B &= 912461.000000,\\
W &= \pm 17.747518,\\
x_1 &= 6.873759 - 7.095766e_1 - 10.394477e_2 - 18.193193e_3,\\
x_2 &= -10.873759 + 0.051875e_1 + 0.848658e_2 + 0.197582e_3.
\end{aligned}$$

For $n = 11, m = 14$, $L_{11} = 199+322e_1+521e_2+843e_3$ and $L_{14} = 843+1364e_1+2207e_2+3571e_3$.

Solving the equation $x^2 + L_{11}x + L_{14} = 0$ for $\alpha = -1, \beta = -1$, we get

$$\begin{aligned}
C &= -4638388100.000000,\\
A &= 24326817.500000,\\
B &= 221017587902.562500,\\
W &= \pm 190.527592,\\
x_1 &= -4.236204 + 0.000233e_1 + 0.283353e_2 + 0.000622e_3,\\
x_2 &= -194.763796 - 322.000241e_1 - 520.717414e_2 - 843.000621e_3.
\end{aligned}$$

Finally, we solve the same equation for $\alpha = -1.236, \beta = -10.023$, the solution are

$$\begin{aligned}
C &= -2220150838.889460,\\
A &= 11634516.036772,\\
B &= 105832184609.751312,\\
W &= \pm 190.527281,\\
x_1 &= -4.236359 + 0.000487e_1 + 0.243975e_2 + 0.001297e_3,\\
x_2 &= -194.763641 - 322.000504e_1 - 520.757626e_2 - 843.001295e_3.
\end{aligned}$$

Conclusion

In this article, we have provided an algorithm in Scilab which allows us to find solutions for the monic quadratic equation $x^2 + bx + c = 0$, with $b, c \in \mathbb{H}(\alpha, \beta)$.

In Theorem 2.3, the authors offer solutions for all cases of the monic equation $x^2 + bx + c = 0$. We are interested only in cases 3 and 4 of the theorem. The article presents several equations solved using the algorithm, implemented in Scilab. By assigning specific values to the two quaternions, b and c, in the form of $b = b_1 + b_2e_1 + b_3e_2 + b_4e_3$ and $c = c_1 + c_2e_1 + c_3e_2 + c_4e_3$, and utilizing the formulas provided in the article, we perform the following calculations: Compute the values of A, B, and C: A is determined by evaluating the expression $A = |b'|^2 + 2Re(c')$, where $b' = b - Re(b)$ and $c' = c - (Re(b)/2)(b - (Re(b))/2)$. B is computed as $B = |c'|^2$. C is obtained by calculating $C = 2Re(\overline{b'}c')$. Identify the case we are in, based on the four cases specified in the theorem. Proceeding with the determined case, we find the two solutions of the monic quadratic equation, $x^2 + bx + c = 0$, using the appropriate formulas presented in the article. That this detailed procedure allows us to obtain precise and accurate solutions for the given quadratic equation in the context of the algebra of real quaternions.

The algorithm can solve monic quadratic equations for any base that respects the multiplication table of quaternions.

References

[1] M. Abrate, *Quadratic formulas for generalized quaternions*, Journal of Algebra and Its Applications, 8 (2009), 289–306.

[2] W. Cao, J.R. Parker, X. Wang, *On the classification of quaternionic Möbius transformations*, Math. Proc. Camb. Phil. Soc., 137 (2004), 349–361.

[3] W. Cao, *Quadratic equation in split quaternions*, Axioms, 11 (2022), 188.

[4] C. Flaut, D. Savin, *Quaternion algebras and generalized Fibonacci-Lucas quaternions*, Adv. Appl. Clifford Algebras, 25 (2015), 853-862.

[5] C. Flaut, D. Savin, *Some remarks regarding* (a, b, x_0, x_1)*-numbers and* (a, b, x_0, x_1)*-quaternions*, https://arxiv.org/pdf/1705.00361.pdf.

[6] C. Flaut, V. Shpakivskyi, *An efficient method for solving equations in generalized quaternion and octonion algebras*, Adv. Appl. Clifford Algebras, 25 (2015), 337-350.

[7] C. Flaut, G. Zaharia, *Remarks regarding computational aspects in algebras obtained by Cayley–Dickson process and some of their applications*, Mathematics, 10 (2022), 1141.

[8] A.F. Horadam, *A generalized Fibonacci sequence*, Amer. Math. Monthly, 68 (1961), 455-459.

[9] A.F. Horadam, *Complex Fibonacci numbers and Fibonacci quaternions*, Amer. Math. Monthly, 70 (1963), 289-291.

[10] L. Huang, W. So, *Quadratic formulas for quaternions*, Applied Mathematics Letters, 15 (2002), 533-540,.

[11] L. Huang, W. Thus, *Quadratic formulas for quaternions*, Appl. Math. Lett., 15 (2002), 533–540.

[12] Z. Li, D.F. Scharler, H.P. Schröcker, *Factorization results for left polynomials in some associative real algebras: state of the art, applications, and open questions*, J. Comput. Appl. Math., 349 (2019), 508–522.

[13] R. Porter, *Quaternionic linear and quadratic equations*, J. Nat. Geom., 11 (1997), 101–106.

[14] J.R. Parker, I. Short, *Conjugacy classification of quaternionic Möbius transformations*, Comput. Methods Funct. Theory, 9 (2009), 13–25.

[15] D.F. Scharler, J. Siegele, H.P. Schröcker, *Quadratic split quaternion polynomials: factorization and geometry*, Adv. Appl. Clifford Algebras, 30 (2020), 11.

[16] https://mathworld.wolfram.com/NewtonsMethod.html (accessed: 11.04.2023)

[17] https://engcourses-uofa.ca/books/numericalanalysis/finding-roots-of-equations/open-methods/newton-raphson-method/ (accessed: 11.04.2023)

Accepted: August 10, 2023