On one-sided MPCEP-inverse for matrices of an arbitrary index

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Abstract. One-sided MPCEP-inverse for matrices was introduced in this paper. The MPCEP-inverse can be described by using the core part A_1 in Core-EP decomposition of A and the Moore-Penrose inverse of A. The MPCEP-inverse of A coincides with the $(A^{\dagger}A^k, (A^k)^*)$-inverse of A. In addition, the CE matrix was introduced, a necessary and sufficient condition such that a matrix A to be a CE matrix is the MPCEP-inverse of A commutes with A.

Keywords: MPCEP-inverse, Core-EP decomposition, CE matrix.

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1. Introduction

Let $\mathbb{C}$ be the complex filed. The set $\mathbb{C}^{m\times n}$ denotes the set of all $m\times n$ matrices over $\mathbb{C}$. Let $A\in\mathbb{C}^{m\times n}$. The symbol A^* denotes the conjugate transpose of A. Notations $\mathcal{R}(A)=\{y\in\mathbb{C}^m : y=Ax, x\in\mathbb{C}^n\}$, $\mathcal{N}(A)=\{x\in\mathbb{C}^n : Ax=0\}$ and $\mathbb{C}_n^{CM}=\{A\in\mathbb{C}^{n\times n}|\operatorname{rank}(A)=\operatorname{rank}(A^2)\}$ will be used in the sequel. The smallest positive integer k such that $\operatorname{rank}(A^k)=\operatorname{rank}(A^{k+1})$ is called the index of $A\in\mathbb{C}^{n\times n}$ and denoted by $\operatorname{ind}(A)$.

Let $A\in\mathbb{C}^{m\times n}$. If a matrix $X\in\mathbb{C}^{n\times m}$ satisfies

$$AXA=A,\ XAX=X,\ (AX)^*=AX \text{ and } (XA)^*=XA,$$

then X is called the Moore-Penrose inverse of A [11, 15] and denoted by $X=A^\dagger$.We call X is an inner inverse of A, if we have $AXA=A$. The set $A\{1\}$ denotes the set of all inner inverse of A. We call X is a $\{1,4\}$ inverse of A, if we have $AXA=A$ and $(XA)^*=XA$. The set $A\{1,4\}$ denotes the set of all $\{1,4\}$ inverse of A. The Moore-Penrose can be used to represent orthogonal projectors $P_A\triangleq AA^\dagger$ and $Q_A\triangleq A^\dagger A$ onto $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively. Let $A,X\in\mathbb{C}^{n\times n}$ with $\operatorname{ind}(A)=k$. Then, algebraic definition of the Drazin inverse as follows: if

$$X=XAX,\ XA^{k+1}=A^k \text{ and } AX=XA,$$

then X is called a Drazin inverse of A. It is unique and denoted by A^D [4]. Note that, for a square complex matrix, the algebraic definition of the Drazin inverse is equivalent to the functional definition of the Drazin inverse. If $\operatorname{ind}(A)=1$, the Drazin inverse is called the group inverse of A and denoted by $A^\#$. The core inverse and the dual core inverse for a complex matrix were introduced by Baksalary and Trenkler [2]. Let $A\in\mathbb{C}^{n\times n}$. A matrix $X\in\mathbb{C}^{n\times n}$ is called a core inverse of A, if it satisfies $AX=P_A$ and $\mathcal{R}(X)\subseteq\mathcal{R}(A)$, where P_A is the orthogonal projector onto $\mathcal{R}(A)$. And if such a matrix exists, then it is unique (and denoted by $A^{\text{#}\!\!\!\bigcirc}$). Baksalary and Trenkler gave several characterizations of the core inverse by using the decomposition of Hartwig and Spindelböck [7]. Let $A\in\mathbb{C}^{n\times n}$, the DMP inverse of A was introduced by using the Drazin and the Moore-Penrose inverses of A in [14], and the formula of the DMP inverse of A is $A^{D,\dagger}=A^DAA^\dagger$ [14, Theorem 2.2]. The CMP inverse of $A\in\mathbb{C}^{n\times n}$ was introduced by Mehdipour and Salemi in [13], who using the core part in core-nilpotent decomposition of A and the Moore-Penrose inverse of A, the CMP inverse of A was denoted by $A^{c,\dagger}$. Manjunatha Prasad and Mohana [12] introduced the core-EP inverse of matrix [12, Definition 3.1]. Let $A\in\mathbb{C}^{n\times n}$. If there exists $X\in\mathbb{C}^{n\times n}$ such that $XAX=X,\mathcal{R}(X)=\mathcal{R}(X^*)=\mathcal{R}(A^k)$, then X is called the core-EP inverse of A. If such inverse exists, then it is unique and denoted by $A^{\textcircled{\dagger}}$. The concept of the MPCEP-inverse of a Hilbert space operators was initially introduced by Chen, Mosić and Xu [3] and this concept was expanded on quaternion matrices by Kyrchei, Mosić and Stanimirović [8, 9]. Let $A\in\mathbb{C}^{n\times n}$ with $\operatorname{ind}(A)=k$. If there exists a matrix $X\in\mathbb{C}^{n\times n}$ such that

$$XAX=X,\ AX=AA^{\textcircled{\dagger}} \text{ and } XA=A^\dagger AA^{\textcircled{\dagger}}A$$

then X is called the MPCEP-inverse of A and denoted by $A^{\dagger,\textcircled{\dagger}}$.

In [18, Theorem 2.1], Wang introduced a new matrix decomposition, namely the Core-EP decomposition of $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Given a matrix $A \in \mathbb{C}^{n\times n}$, then A can be written as the sum of matrices $A_1 \in \mathbb{C}^{n\times n}$ and $A_2 \in \mathbb{C}^{n\times n}$, that is $A = A_1 + A_2$, where $A_1 \in \mathbb{C}_n^{CM}$, $A_2^k = 0$ and $A_1^* A_2 = A_2 A_1 = 0$. In [18, Theorem 2.3 and Theorem 2.4], Wang proved this matrix decomposition is unique and there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A_1 = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* \quad \text{and} \quad A_2 = U \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} U^*, \tag{1}$$

where $T \in \mathbb{C}^{r\times r}$ is nonsingular and $N \in \mathbb{C}^{(n-r)\times(n-r)}$ is nilpotent with $\operatorname{rank}(A^k) = r$.

Let $A, B, C \in \mathbb{C}^{n\times n}$. We say that $Y \in \mathbb{C}^{n\times n}$ is a (B, C)-inverse of A if we have

$$YAB = B,\ CAY = C,\ \mathcal{N}(C) \subseteq \mathcal{N}(Y) \text{ and } \mathcal{R}(Y) \subseteq \mathcal{R}(B).$$

If such Y exists, then it is unique (see [1, Definition 4.1] and [16, Definition 1.2]). Note that, the (B, C)-inverse was introduced in the setting of semigroups [5].

In [6, Definition 1.2] and [10, Definition 2.1], the authors introduced the one-sided (b, c)-inverse in rings. In [1, Definition 2.7], the authors introduced the one-sided (B, C)-inverse for complex matrices. Let $A, B, C \in \mathbb{C}^{n\times n}$. We call that $X \in \mathbb{C}^{n\times n}$ is a left (B, C)-inverse of A if we have $\mathcal{N}(C) \subseteq \mathcal{N}(X)$ and $XAB = B$. We call that $Y \in \mathbb{C}^{n\times n}$ is a right (B, C)-inverse of A if we have $\mathcal{R}(Y) \subseteq \mathcal{R}(B)$ and $CAY = C$.

In fact, there is an important generalized inverse was introduced in [17] by Rao and Mitra. Let $A \in \mathbb{C}^{n\times n}$. In [16], Rakić showed that Rao and Mitra's constrained inverse of A coincides with the (B, C)-inverse of A, where $B, C \in \mathbb{C}^{n\times n}$.

In 1972, Rao and Mitra introduced two different types of constraints in order to extend the concept of Bott-Duffin inverse and define a new constrained inverse $Y \in \mathbb{C}^{n\times n}$ of a matrix $A \in \mathbb{C}^{n\times n}$ in [17]. Let $B, C \in \mathbb{C}^{n\times n}$.

Constraints of type 1 :
$\mathfrak{c}$: Y maps vectors of $\mathbb{C}^m$ into $\mathcal{R}(B)$;
$\mathfrak{r}$: Y^* maps vectors of $\mathbb{C}^n$ into $\mathcal{R}(C^*)$;
Constraints of type 2 :
$\mathfrak{C}$: YA is an identity on $\mathcal{R}(B)$;
$\mathfrak{R}$: $(AY)^*$ is an identity on $\mathcal{R}(C^*)$.

Note that, Rao and Mitra denoted their inverse by $A_{\mathfrak{cr}\mathfrak{C}\mathfrak{R}}$. In fact, they defined this inverse in a broader context, where A is an $m \times n$ matrix mapping vectors of $\mathbb{C}^n$ to $\mathbb{C}^m$, where $\mathbb{C}^n$ denotes an n dimensional vector space with an inner product.

Let $A, B, C \in \mathbb{C}^{n\times n}$. A matrix $Y \in \mathbb{C}^{n\times n}$ is a $\mathfrak{cr}^{\mathfrak{CR}}$ *constrained inverse* of A if it satisfies constraints $\mathfrak{c}, \mathfrak{r}, \mathfrak{C}$ and $\mathfrak{R}$. Here the $\mathfrak{cr}^{\mathfrak{CR}}$ constrained inverse of A will be denoted by $A^{\|(B,C)}$. In the sequel, one can see that the $\mathfrak{cr}^{\mathfrak{CR}}$ constrained inverse of A coincides with the (B, C)-inverse of A, thus, we use the symbol of the (B, C)-inverse to denoted the $\mathfrak{cr}^{\mathfrak{CR}}$ constrained inverse of A.

In order to rewrite the constraints $\mathfrak{c}, \mathfrak{r}, \mathfrak{C}$ and $\mathfrak{R}$ in purely multiplicative language, we need the following fact: the condition $\mathcal{R}(Y) \subseteq \mathcal{R}(B)$ if and only if $Y = BK$, for some $K \in \mathbb{C}^{n\times n}$; the condition $\mathcal{R}(Y^*) \subseteq \mathcal{R}(C^*)$ if and only if $\mathcal{N}(C) \subseteq \mathcal{R}(Y)$ if and only if $Y = LC$, for some $L \in \mathbb{C}^{n\times n}$; the constraint C is clearly equivalent to $YAB = B$ and the constraint R is equivalent to $CAY = C$. Therefore, these constraints can be rewritten as follows:

Constraints of type 1 :
$\mathfrak{c}: \ \mathcal{R}(Y) \subseteq \mathcal{R}(B);$
$\mathfrak{r}: \ \mathcal{R}(Y^*) \subseteq \mathcal{R}(C^*);$
Constraints of type 2 :
$\mathfrak{C}: \ YAB = B;$
$\mathfrak{R}: \ CAY = C.$

Let $A \in \mathbb{C}^{m\times n}$ with $\operatorname{rank}(A) = r$. Let T, S be two subspaces of $\mathbb{C}^n$ with $\dim(T) = s \leqslant r$ and $\dim(S) = n - r$. Recall that the out inverse $A^{(2)}_{T,S}$ with prescribed the column space T and null space S is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying $AT \oplus S = \mathbb{C}^n$. It is well-known fact that the following ten kinds of generalized inverse are all special cases of the out inverse $A^{(2)}_{T,S}$ with prescribed the column space T and null space S: the Moore-Penrose inverse $A^\dagger$ [11, 15], the Drazin inverse A^D [4], the group inverse $A^\#$ [4], the core inverse $A^{\oplus}$ [2], the DMP-inverse $A^{D,\dagger}$ [14] and the core-EP inverse $A^{\textcircled{\dagger}}$ [12]. Thus, all the results related the the out inverse $A^{(2)}_{T,S}$ with prescribed the column space T and null space S are applicable to these generalized inverses.

2. Existence criteria and expressions of one sided MPCEP-inverse

In [18, Theorem 2.3], Wang proved that A_1 can be described by using the Moore-Penrose inverse of A^k. The explicit expressions of A_1 can be found in the follows lemma.

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If $A = A_1 + A_2$ is the Core-EP decomposition of A, then $A_1 = A^k(A^k)^\dagger A$ and $A_2 = A - A^k(A^k)^\dagger A$.*

Motivated by the ideal of one-sided (B, C)-inverse of A, one-sided MPCEP-inverse was introduced.

Definition 2.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. We call that $X \in \mathbb{C}^{n\times n}$ is a left MPCEP-inverse of A if we have*

$$\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(X) \ \text{and} \ XA^k = A^\dagger A^k. \tag{2}$$

We call that $Y \in \mathbb{C}^{n\times n}$ is a right MPCEP-inverse of A if we have

$$\mathcal{R}(Y) \subseteq \mathcal{R}(A^\dagger A^k) \text{ and } (AY)^* A^k = A^k. \tag{3}$$

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then, $A^\dagger A^k (A^k)^\dagger$ is a left MPCEP-inverse of A.*

Proof. Let X be a left MPCEP-inverse of A. Then, by Definition 2.1, we have

$$\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(X) \text{ and } XA^k = A^\dagger A^k. \tag{4}$$

Then

$$\begin{aligned} X &= U(A^k)^* \text{ for some } U \in \mathbb{C}^{n\times n} \\ &= U(A^k)^*[(A^k)^*]^\dagger (A^k)^* = X[(A^k)^*]^\dagger (A^k)^* \\ &= X[A^k(A^k)^\dagger]^* = XA^k(A^k)^\dagger = A^\dagger A^k (A^k)^\dagger \end{aligned} \tag{5}$$

by (4). Thus, $A^\dagger A^k (A^k)^\dagger$ is a left MPCEP-inverse of A by (5). □

In the following theorem, a general expression of the left MPCEP-inverse of A was given.

Theorem 2.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then, a general solution of the left MPCEP-inverse of A is*

$$A^\dagger A^k (A^k)^\dagger + V \left[I_n - (A^k)^* A^k (A^k)^- \left((A^k)^* \right)^- \right] (A^k)^*,$$

for any $V \in \mathbb{C}^{n\times n}$, any $((A^k)^*)^- \in (A^k)^*\{1\}$ and some $(A^k)^- \in A^k\{1\}$.

Proof. Let X be a left MPCEP-inverse of A. Then, by Definition 2.1, we have

$$\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(X) \text{ and } XA^k = A^\dagger A^k. \tag{6}$$

Then

$$X = U(A^k)^* \text{ for some } U \in \mathbb{C}^{n\times n}. \tag{7}$$

Hence

$$A^\dagger A^k = XA^k = U(A^k)^* A^k \tag{8}$$

by (6) and (7). That is $A^\dagger A^k = U(A^k)^* A^k$.

Since $\operatorname{rank}\left((A^k)^* A^k\right) = \operatorname{rank}(A^k)$, so one can check that $\left((A^k)^* A^k\right)^- = (A^k)^- \left((A^k)^*\right)^-$, for any $((A^k)^*)^- \in (A^k)^*\{1\}$ and some $(A^k)^- \in A^k\{1\}$ as follows:

The condition $\operatorname{rank}\left((A^k)^* A^k\right) = \operatorname{rank}(A^k)$ implies $\mathcal{N}\left((A^k)^* A^k\right) = \mathcal{N}(A^k)$. We have the equality $(A^k)^* A^k [I_n - \left((A^k)^* A^k\right)^- (A^k)^* A^k] = 0$ in view of the equality $(A^k)^*A^k\left((A^k)^*A^k\right)^-(A^k)^*A^k=(A^k)^*A^k$, so $I_n-\left((A^k)^*A^k\right)^-(A^k)^*A^k\in\mathcal{N}((A^k)^*A^k)\subseteq\mathcal{N}(A^k)$, thus $A^k[I_n-\left((A^k)^*A^k\right)^-(A^k)^*A^k]=0$, that is

$$A^k=A^k\left((A^k)^*A^k\right)^-(A^k)^*A^k,$$

gives $\left((A^k)^*A^k\right)^-(A^k)^*$ is an inner inverse of A^k.

Since $\left((A^k)^*A^k\right)^-(A^k)^*\in A^k\{1\}$, so let $(A^k)^-=\left((A^k)^*A^k\right)^-(A^k)^*$, then

$$\begin{aligned}(A^k)^*A^k(A^k)^-((A^k)^*)^-(A^k)^*A^k&=(A^k)^*A^k[((A^k)^*A^k)^-(A^k)^*]((A^k)^*)^-(A^k)^*A^k\\&=(A^k)^*A^k((A^k)^*A^k)^-((A^k)^*((A^k)^*)^-(A^k)^*)A^k\\&=(A^k)^*A^k((A^k)^*A^k)^-(A^k)^*A^k\\&=(A^k)^*A^k.\end{aligned}$$

That is, for any $\left((A^k)^*\right)^-\in(A^k)^*\{1\}$ and some $(A^k)^-\in A^k\{1\}$, the equality $\left((A^k)^*A^k\right)^-=(A^k)^-\left((A^k)^*\right)^-$ holds.

Since

$$\begin{aligned}&\left\{A^\dagger\left((A^k)^\dagger\right)^*+V\left[I_n-(A^k)^*A^k\left((A^k)^*A^k\right)^-\right]\right\}(A^k)^*A^k\\&=A^\dagger\left((A^k)^\dagger\right)^*(A^k)^*A^k+V\left[I_n-(A^k)^*A^k\left((A^k)^*A^k\right)^-\right](A^k)^*A^k\\&=A^\dagger\left((A^k)^\dagger\right)^*(A^k)^*A^k=A^\dagger\left(A^k(A^k)^\dagger\right)^*A^k\\&=A^\dagger A^k,\end{aligned}$$

hence a general solution of $A^\dagger A^k=U(A^k)^*A^k$ is

$$A^\dagger\left((A^k)^\dagger\right)^*+V\left[I_n-(A^k)^*A^k\left((A^k)^*A^k\right)^-\right]$$

can be written as

$$A^\dagger\left((A^k)^\dagger\right)^*+V\left[I_n-(A^k)^*A^k(A^k)^-\left((A^k)^*\right)^-\right],$$

for any $V\in\mathbb{C}^{n\times n}$, any $((A^k)^*)^-\in(A^k)^*\{1\}$ and some $(A^k)^-\in A^k\{1\}$. Let $\widetilde{X}=A^\dagger A^k(A^k)^\dagger+V\left[I_n-(A^k)^*A^k(A^k)^-((A^k)^*)^-\right](A^k)^*$. One can check $\widetilde{X}$ is a left MPCEP-inverse of A in what follows.

$$\begin{aligned}\widetilde{X}A^k&=A^\dagger A^k(A^k)^\dagger A^k+V\left[I_n-(A^k)^*A^k(A^k)^-\left((A^k)^*\right)^-\right](A^k)^*A^k\\&=A^\dagger A^k(A^k)^\dagger A^k+V\left[I_n-(A^k)^*A^k\left((A^k)^*A^k\right)^-\right](A^k)^*A^k\\&=A^\dagger A^k+V\left[I_n(A^k)^*A^k-(A^k)^*A^k\left((A^k)^*A^k\right)^-(A^k)^*A^k\right]\\&=A^\dagger A^k.\end{aligned}\tag{9}$$

Since

$$
\begin{aligned}
\widetilde{X} &= A^\dagger A^k (A^k)^\dagger + V\left[I_n - (A^k)^* A^k (A^k)^- \left((A^k)^*\right)^-\right](A^k)^* \\
&= A^\dagger \left[A^k (A^k)^\dagger\right]^* + V\left[I_n - (A^k)^* A^k (A^k)^- \left((A^k)^*\right)^-\right](A^k)^* \\
&= A^\dagger \left[(A^k)^\dagger\right]^* (A^k)^* + V\left[I_n - (A^k)^* A^k (A^k)^- \left((A^k)^*\right)^-\right](A^k)^* \\
&= Q(A^k)^*,
\end{aligned} \tag{10}
$$

where $Q = A^\dagger[(A^k)^\dagger]^* + V[I_n - (A^k)^* A^k (A^k)^-((A^k)^*)^-]$. Hence, (10) gives

$$\widetilde{X} = Q(A^k)^*. \tag{11}$$

The equality in (11) is equivalent to $\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(\widetilde{X})$. Thus, $\widetilde{X}$ is a left MPCEP-inverse of A by $\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(\widetilde{X})$ and $\widetilde{X}A^k = A^\dagger A^k$ in (9). □

Theorem 2.3. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then, $A^\dagger A^k (A^k)^\dagger$ is a right MPCEP-inverse of A.*

Proof. Let Y be a right MPCEP-inverse of A. Then, by Definition 2.1, we have

$$\mathcal{R}(Y) \subseteq \mathcal{R}(A^\dagger A^k) \text{ and } (AY)^* A^k = A^k. \tag{12}$$

Then

$$
\begin{aligned}
Y &= A^\dagger A^k V \text{ for some } V \in \mathbb{C}^{n\times n} \\
&= A^\dagger A^k (A^k)^\dagger A^k V = A^\dagger \left[A^k (A^k)^\dagger\right]^* A^k V = A^\dagger \left[(A^k)^\dagger\right]^* (A^k)^* A^k V \\
&= A^\dagger \left[(A^k)^\dagger\right]^* (A^{k-1})^* A^* A^k V = A^\dagger \left[(A^k)^\dagger\right]^* (A^{k-1})^* (AA^\dagger A)^* A^k V \\
&= A^\dagger \left[(A^k)^\dagger\right]^* (A^{k-1})^* A^* (AA^\dagger)^* A^k V = A^\dagger \left[(A^k)^\dagger\right]^* (A^{k-1})^* A^* AA^\dagger A^k V \\
&= A^\dagger \left[(A^k)^\dagger\right]^* (A^k)^* AA^\dagger A^k V = A^\dagger \left[(A^k)^\dagger\right]^* (A^k)^* AY \\
&= A^\dagger \left[(A^k)^\dagger\right]^* \left[(AY)^* A^k\right]^* = A^\dagger \left[(A^k)^\dagger\right]^* (A^k)^* \\
&= A^\dagger A^k (A^k)^\dagger
\end{aligned} \tag{13}
$$

by (12). Thus, $A^\dagger A^k (A^k)^\dagger$ is a right MPCEP-inverse of A by (13). □

Theorem 2.4. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then, a general solution of the right MPCEP-inverse of A is*

$$A^\dagger A^k (A^k)^\dagger + A^\dagger A^k \left[I_n - (A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k\right] T,$$

for any $T \in \mathbb{C}^{n\times n}$, any $((A^k)^*)^- \in (A^k)^*\{1\}$ and some $(A^k)^- \in A^k\{1\}$.

Proof. Let Y be a right MPCEP-inverse of A. Then, by Definition 2.1, we have

$$\mathcal{R}(Y) \subseteq \mathcal{R}(A^\dagger A^k) \text{ and } (AY)^* A^k = A^k. \tag{14}$$

Then

$$Y = A^\dagger A^k S \text{ for some } S \in \mathbb{C}^{n\times n}. \tag{15}$$

Hence

$$(A^k)^* = (A^k)^* AY = (A^k)^* AA^\dagger A^k S = (A^k)^* AY = (A^k)^* A^k S \tag{16}$$

by (14) and (15). That is $(A^k)^* = (A^k)^* A^k S$.

Since $\operatorname{rank}\left((A^k)^* A^k\right) = \operatorname{rank}(A^k)$, so $\left((A^k)^* A^k\right)^- = (A^k)^- \left((A^k)^*\right)^-$, for any $((A^k)^*)^- \in (A^k)^*\{1\}$ and some $(A^k)^- \in A^k\{1\}$ by the proof Theorem 2.2.

Since

$$\begin{aligned}
&(A^k)^* A^k \left\{(A^k)^\dagger + \left[I_n - \left((A^k)^* A^k\right)^- (A^k)^* A^k\right] T\right\}\\
&= (A^k)^* A^k (A^k)^\dagger + (A^k)^* A^k \left[I_n - \left((A^k)^* A^k\right)^- (A^k)^* A^k\right] T\\
&= (A^k)^* A^k (A^k)^\dagger = (A^k)^* [A^k (A^k)^\dagger]^*\\
&= (A^k)^*,
\end{aligned}$$

hence a general solution of $(A^k)^* = (A^k)^* A^k S$ is

$$(A^k)^\dagger + \left[I_n - \left((A^k)^* A^k\right)^- (A^k)^* A^k\right] T$$

can be written as

$$(A^k)^\dagger + \left[I_n - (A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k\right] T,$$

for any $T \in \mathbb{C}^{n\times n}$, any $((A^k)^*)^- \in (A^k)^*\{1\}$ and some $(A^k)^- \in A^k\{1\}$. Let $\widetilde{Y} = A^\dagger A^k (A^k)^\dagger + A^\dagger A^k [I_n - (A^k)^- ((A^k)^*)^- (A^k)^* A^k] T$. One can check $\widetilde{Y}$ is a right MPCEP-inverse of A in what follows.

$$\begin{aligned}
\widetilde{Y} &= A^\dagger A^k (A^k)^\dagger + A^\dagger A^k \left[I_n - (A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k\right] T\\
&= A^\dagger A^k \left\{(A^k)^\dagger + \left[I_n - (A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k\right] T\right\}\\
&= A^\dagger A^k P,
\end{aligned} \tag{17}$$

where $P = (A^k)^\dagger + [I_n - (A^k)^- ((A^k)^*)^- (A^k)^* A^k] T$. Hence, (17) gives

$$\widetilde{Y} = A^\dagger A^k P. \tag{18}$$

The following equality will be used in the sequel.

$$\begin{aligned}
A^k &= A^k(A^k)^\dagger A^k = \left[A^k(A^k)^\dagger\right]^* A^k = \left[(A^k)^\dagger\right]^* (A^k)^* A^k \\
&= \left[(A^k)^\dagger\right]^* (A^k)^* A^k \left[(A^k)^* A^k\right]^- (A^k)^* A^k \\
&= A^k \left[(A^k)^* A^k\right]^- (A^k)^* A^k \\
&= A^k (A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k
\end{aligned} \tag{19}$$

by $\left((A^k)^* A^k\right)^- = (A^k)^- \left((A^k)^*\right)^-$, for any $((A^k)^*)^- \in (A^k)^*\{1\}$ and some $(A^k)^- \in A^k\{1\}$.

Since

$$\begin{aligned}
(A\widetilde{Y})^* A^k &= \left\{AA^\dagger A^k (A^k)^\dagger + AA^\dagger A^k \left[I_n - (A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k\right] T\right\}^* A^k \\
&= \left\{A^k (A^k)^\dagger + A^k \left[I_n - (A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k\right] T\right\}^* A^k \\
&= \left\{A^k (A^k)^\dagger + \left[A^k - A^k (A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k\right] T\right\}^* A^k \\
&= \left[A^k (A^k)^\dagger\right]^* A^k \\
&= A^k
\end{aligned} \tag{20}$$

by (19). The equality in (18) is equivalent to $\mathcal{R}(\widetilde{Y}) \subseteq \mathcal{R}(A^\dagger A^k)$. Thus, $\widetilde{Y}$ is a right MPCEP-inverse of A by $\mathcal{R}(\widetilde{Y}) \subseteq \mathcal{R}(A^\dagger A^k)$ and $(A\widetilde{Y})^* A^k = A^k$ in (20). □

In the following theorem, we will use the core part A_1 of the Core-EP decomposition to describe the left MPCEP-inverse of A.

Theorem 2.5. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *Then, $X \in \mathbb{C}^{n\times n}$ is a left MPCEP-inverse of A if and only if $\mathcal{N}(A_1 A^\dagger) \subseteq \mathcal{N}(X)$ and $XAA^\dagger A_1 = A^\dagger A_1$ hold.*

Proof. Firstly, we will prove $\mathcal{N}(A_1 A^\dagger) = \mathcal{R}(A^k)^\perp$. Let $u \in \mathcal{N}\left((A^k)^* AA^\dagger\right)$, then

$$\begin{aligned}
A_1 A^\dagger u &= A^k (A^k)^\dagger AA^\dagger u = \left[A^k (A^k)^\dagger\right]^* AA^\dagger u \\
&= \left[(A^k)^\dagger\right]^* (A^k)^* AA^\dagger u = 0
\end{aligned} \tag{21}$$

by Lemma 2.1. Let $v \in \mathcal{N}(A_1 A^\dagger)$, then

$$\begin{aligned}
(A^k)^* AA^\dagger v &= (A^k)^* \left[(A^k)^*\right]^\dagger (A^k)^* AA^\dagger v = (A^k)^* \left[A^k (A^k)^\dagger\right]^* AA^\dagger v \\
&= (A^k)^* A^k (A^k)^\dagger AA^\dagger v = (A^k)^* A_1 A^\dagger v = 0
\end{aligned} \tag{22}$$

by Lemma 2.1. So, by (21) and (22) we have

$$\mathcal{N}(A_1A^\dagger) = \mathcal{N}\Big((A^k)^*AA^\dagger\Big) \tag{23}$$

Note that

$$\mathcal{R}(A^k)^\perp = \mathcal{N}\Big((A^k)^*\Big) = \mathcal{N}\Big((AA^\dagger A^k)^*\Big) = \mathcal{N}\Big((A^k)^*AA^\dagger\Big). \tag{24}$$

The equality $\mathcal{N}(A_1A^\dagger) = \mathcal{N}((A^k)^*AA^\dagger)$ in (23) gives $\mathcal{N}(A_1A^\dagger) = \mathcal{R}(A^k)^\perp$ by (24). Hence, $\mathcal{N}(A_1A^\dagger) \subseteq \mathcal{N}(X)$ if and only if $\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(X)$ by $\mathcal{N}(A_1A^\dagger) = \mathcal{R}(A^k)^\perp$.

Next, we will prove $XAA^\dagger A_1 = A^\dagger A_1$ if and only if $XA^k = A^\dagger A^k$. The condition $XAA^\dagger A_1 = A^\dagger A_1$ can be written as

$$XAA^\dagger A^k(A^k)^\dagger A = A^\dagger A^k(A^k)^\dagger A \tag{25}$$

by Lemma 2.1, (25) can be written as

$$XA^k(A^k)^\dagger A = A^\dagger A^k(A^k)^\dagger A \tag{26}$$

by Lemma $AA^\dagger A = A$. Post-multiplying by A^{k-1} on (26) gives

$$XA^k(A^k)^\dagger AA^{k-1} = A^\dagger A^k(A^k)^\dagger AA^{k-1},$$

that is $XA^k = A^\dagger A^k$. □

In the following theorem, we will use the core part A_1 of the Core-EP decomposition to describe the right MPCEP-inverse of A.

Theorem 2.6. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then, $Y \in \mathbb{C}^{n\times n}$ is a right MPCEP-inverse of A if and only if $\mathcal{R}(Y) \subseteq \mathcal{R}(A^\dagger A_1)$ and $A_1A^\dagger AY = A_1A^\dagger$ hold.*

Proof. Firstly, we will proof $\mathcal{R}(A^\dagger A^k) = \mathcal{R}(A^\dagger A_1)$. Since, we have

$$A^\dagger A_1 = A^\dagger A^k(A^k)^\dagger A \tag{27}$$

and

$$A^\dagger A^k = A^\dagger A^k(A^k)^\dagger A^k = A^\dagger A^k(A^k)^\dagger AA^{k-1} = A^\dagger A_1A^{k-1} \tag{28}$$

by Lemma 2.1. The conditions in (27) and (28) imply $\mathcal{R}(A^\dagger A^k) = \mathcal{R}(A^\dagger A_1)$.

Since

$$\begin{aligned} & A_1A^\dagger AY = A_1A^\dagger \\ & \Leftrightarrow A^k(A^k)^\dagger AA^\dagger AY = A^k(A^k)^\dagger AA^\dagger \\ & \Leftrightarrow A^k(A^k)^\dagger AY = A^k(A^k)^\dagger AA^\dagger \\ & \Leftrightarrow (A^k)^\dagger AY = (A^k)^\dagger AA^\dagger \\ & \Leftrightarrow (A^k)^* AY = (A^k)^* AA^\dagger \\ & \Leftrightarrow (A^k)^* AY = (A^k)^*(AA^\dagger)^* \\ & \Leftrightarrow (A^k)^* AY = (AA^\dagger A^k)^* \\ & \Leftrightarrow (AY)^*A^k = A^k \end{aligned} \tag{29}$$

by Lemma 2.1. □

Theorem 2.7. *Let $A \in \mathbb{C}^{n\times n}$. If A is both left and right MPCEP-invertible, then the left MPCEP-inverse of A and the right MPCEP-inverse of A are unique. Moreover, the left MPCEP-inverse of A coincides with the right MPCEP-inverse of A.*

Proof. Let X be a left MPCEP-inverse of A and Y be a right MPCEP-inverse of A. Then

$$\mathcal{R}(A^k)^{\perp} \subseteq \mathcal{N}(X) \text{ and } XA^k = A^{\dagger}A^k. \tag{30}$$

and

$$\mathcal{R}(Y) \subseteq \mathcal{R}(A^{\dagger}A^k) \text{ and } (AY)^*A^k = A^k \tag{31}$$

hold. Thus, $X = U(A^k)^*$ and $Y = A^{\dagger}A^kV$, for some $U, V \in \mathbb{C}^{n\times n}$ by (30) and (31). Therefore,

$$\begin{aligned} X &= U(A^k)^* = U(A^k)^*AY = XAY, \\ Y &= A^{\dagger}A^kV = XA^kV = XAA^{\dagger}A^kV = XAY \end{aligned} \tag{32}$$

by (30) and (31). Hence, $X = Y$ by (32). If Z is a another right MPCEP-inverse of A, one can prove $X = Z$ in a similar way. Then, $Y = Z$ by $X = Y$ and $X = Z$, which says the right MPCEP-inverse of A is unique. One also can prove the left MPCEP-inverse of A is unique by a similar proof of the uniqueness of the right MPCEP-inverse of A. By the above proof, we can get that the left MPCEP-inverse of A coincides with the right MPCEP-inverse of A. □

The concept of the MPCEP-inverse of A will be introduced by using left MPCEP-inverse of A and right MPCEP-inverse of A. The concept of the MPCEP-inverse of a Hilbert space operators was introduced by Chen, Mosić and Xu in [3].

Definition 2.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. We call that $X \in \mathbb{C}^{n\times n}$ is the MPCEP-inverse of A if A is both left MPCEP-invertible and right MPCEP-invertible. That is,*

$$\mathcal{R}(A^k)^{\perp} \subseteq \mathcal{N}(X), \mathcal{R}(X) \subseteq \mathcal{R}(A^{\dagger}A^k),\ XA^k = A^{\dagger}A^k \text{ and } (AX)^*A^k = A^k. \tag{33}$$

And X is denoted by the symbol $A^{\dagger,\textcircled{\dagger}}$, that is $A^{\dagger,\textcircled{\dagger}} = X$.

By Theorem 2.7 and Definition 2.2, we have the uniqueness of the MPCEP-inverse of A in what follows:

We have $A^{\dagger,\textcircled{\dagger}}=A^{\dagger}AA^{\textcircled{\dagger}} = A^{\dagger}AA^DA^k(A^k)^{\dagger} = A^{\dagger}A^DA^{k+1}(A^k)^{\dagger} = A^{\dagger}A^k(A^k)^{\dagger}$ by $A^{\textcircled{\dagger}} = A^DA^k(A^k)^{\dagger}$. So, the MPCEP-inverse defined in Definition 2.2 coincides with ones introduced in [3] that was expanded to matrices in [8, 9].

Theorem 2.8. *Let $A \in \mathbb{C}^{n\times n}$. Then, the MPCEP-inverse of A is unique.*

The formula of the MPCEP-inverse of a complex matrix was given in the following theorem.

Theorem 2.9. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then, $A^\dagger A^k (A^k)^\dagger$ is the MPCEP-inverse of A.*

Proof. By Definition 2.2, a MPCEP-invertible matrix, is both left MPCEP-invertible and right MPCEP-invertible. Then, By Theorem 2.1, we have $A^\dagger A^k (A^k)^\dagger$ is a left MPCEP-inverse of A. And by Theorem 2.3, we have $A^\dagger A^k (A^k)^\dagger$ is a right MPCEP-inverse of A. The proof is finished by Theorem 2.7. □

3. Existence criteria and expressions of the MPCEP-inverse

The CMP inverse of $A \in \mathbb{C}^{n\times n}$ was introduced by Mehdipour and Salemi in [13], who using the core part in core-nilpotent decomposition of A and the Moore-Penrose inverse of A. Motivated by the above method, we have a natural question as follows: Using the core part A_1 in Core-EP decomposition of A and the Moore-Penrose inverse of A to introduce a matrix $X = A^\dagger A_1 A^\dagger$.

Question What is X ?

In the following theorem, we answer this question, we proved that $X = A^\dagger A_1 A^\dagger$ is a formula of the MPCEP-inverse.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and $A = A_1 + A_2$ is the Core-EP decomposition of A. Then, the formula of the MPCEP-inverse is $X = A^\dagger A_1 A^\dagger$.*

Proof. Let $X = A^\dagger A_1 A^\dagger$. Then, by Lemma 2.1, we have

$$\begin{aligned} X &= A^\dagger A_1 A^\dagger = A^\dagger A^k (A^k)^\dagger A A^\dagger \\ &= A^\dagger [A^k (A^k)^\dagger]^* (AA^\dagger)^* = A^\dagger [AA^\dagger A^k (A^k)^\dagger]^* \\ &= A^\dagger [A^k (A^k)^\dagger]^* = A^\dagger [(A^k)^\dagger]^* (A^k)^*. \end{aligned} \tag{34}$$

The condition $\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(X)$ holds by (34). Since

$$X = A^\dagger A_1 A^\dagger = A^\dagger A^k (A^k)^\dagger A A^\dagger \tag{35}$$

so, the condition $\mathcal{R}(X) \subseteq \mathcal{R}(A^\dagger A^k)$ holds by (35). Since

$$XA^k = A^\dagger A^k (A^k)^\dagger AA^\dagger A^k = A^\dagger A^k (A^k)^\dagger A^k = A^\dagger A^k \tag{36}$$

so, the condition $XA^k = A^\dagger A^k$ holds by (36). Since

$$(AX)^* A^k = [AA^\dagger A^k (A^k)^\dagger AA^\dagger]^* A^k = AA^\dagger A^k (A^k)^\dagger AA^\dagger A^k = A^k \tag{37}$$

so, the condition $(AX)^* A^k = A^k$ holds by (37). Thus, the proof is finished by Definition 2.2. □

The following exmaple shows that the core part in core-nilpotent decomposition of A is different from the core part in Core-EP decomposition of A. Moreover, this example also shows that the MPCEP-inverse is different from the CMP inverse.

Example 3.1. Let $A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{C}^{4\times 4}$. Then, the core part in core-nilpotent decomposition of A is $AA^DA = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and the core part in Core-EP decomposition of A is $AA^{\textcircled{\dagger}}A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Thus, $A^{c,\dagger} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ by $A^{c,\dagger} = A^{\dagger}AA^DAA^{\dagger}$ and $A^{\dagger,\textcircled{\dagger}} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ by $A^{\dagger,\textcircled{\dagger}} = A^{\dagger}AA^{\textcircled{\dagger}}AA^{\dagger}$.

The following example shows that the MPCEP-inverse can equal to the CMP inverse.

Example 3.2. Let $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -1 & 4 \end{bmatrix} \in \mathbb{C}^{4\times 4}$. It is easy to check that the index of A is 2. By [18, Corollary 3.3], we have

$$A^{\textcircled{\dagger}}=A^2(A^3)^{\textcircled{\#}}=A^2(A^2)^{\textcircled{\#}}=A^2(A^2)^{\#}A^2(A^2)^{\dagger}=A^2(A^2)^{\dagger}=\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A^D,$$

which gives the core part in core-nilpotent decomposition of A equals to the core part in Core-EP decomposition of A. Moreover, the MPCEP-inverse of A equals to the CMP inverse of A.

In [18, Theorem 3.4], Wang proved that A_1 can be described by using the Core-EP inverse of A. The explicit expressions of A_1 can be found in the follows lemma.

Lemma 3.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. If $A = A_1 + A_2$ is the Core-EP decomposition of A, then $A_1 = AA^{\textcircled{\dagger}}A$ and $A_2 = A - AA^{\textcircled{\dagger}}A$.*

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$. Then, the MPCEP-inverse of A is an outer inverse of A.*

Proof. Let $A = A_1 + A_2$ is the Core-EP decomposition of A and $X \in \mathbb{C}^{n\times n}$ be the MPCEP-inverse of A. Then, $X = A^\dagger A_1 A^\dagger$ by Theorem 3.1, thus

$$
\begin{aligned}
XAX &= A^\dagger A_1 A^\dagger A A^\dagger A_1 A^\dagger = A^\dagger A_1 A^\dagger A_1 A^\dagger \\
&= A^\dagger A^k (A^k)^\dagger A A^\dagger A^k (A^k)^\dagger A A^\dagger \\
&= A^\dagger A^k (A^k)^\dagger A^k (A^k)^\dagger A A^\dagger \\
&= A^\dagger A^k (A^k)^\dagger A A^\dagger \\
&= A^\dagger A_1 A^\dagger \\
&= X
\end{aligned} \tag{38}
$$

by Lemma 2.1. □

Let $A \in \mathbb{C}^{n\times n}$ and $i, m \in \mathbb{N}$. A matrix $X \in \mathbb{C}^{n\times n}$ is called an $\langle i, m\rangle$-core inverse of A, if it satisfies

$$X = A^D A X \quad \text{and} \quad A^m X = A^i (A^i)^\dagger. \tag{39}$$

The $\langle i, m\rangle$-core inverse of A is unique and denoted by $A^{\oplus}_{i,m}$.

Proposition 3.1 ([19, Proposition 1])**.** *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If $i \geqslant k$, then $A^m A^{\oplus}_{i,m}$ is the orthogonal projector onto $\mathcal{R}(A^i)$ along $\mathcal{R}(A^i)^\perp$.*

Theorem 3.3. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and $i, m \in \mathbb{N}$. If $i \geqslant k$, then $AA^{\dagger,\textcircled{\dagger}}$ is the orthogonal projector onto $\mathcal{R}(A^i)$ along $\mathcal{R}(A^i)^\perp$. Moreover, we have*

$$AA^{\dagger,\textcircled{\dagger}} = A_1 A^\dagger = AA^{\textcircled{\dagger}} = A^m A^{\oplus}_{i,m} = A^k (A^k)^\dagger = A^i (A^i)^\dagger, \tag{40}$$

where A_1 is the core part A_1 in Core-EP decomposition of A and $A^{\textcircled{\dagger}}$ is the Core-EP inverse of A.

Proof. By Theorem 2.9, we have $A^{\dagger,\textcircled{\dagger}} = A^\dagger A^k (A^k)^\dagger$. Then

$$AA^{\dagger,\textcircled{\dagger}} = AA^\dagger A^k (A^k)^\dagger = A^k (A^k)^\dagger. \tag{41}$$

The equality $AA^{\textcircled{\dagger}} = A^k (A^k)^\dagger$ can be got [18, Corollary 3.3]. The equality $A^m A^{\oplus}_{i,m} = A^k (A^k)^\dagger = A^i (A^i)^\dagger$ is hold by Lemma 3.1. By Lemma 2.1, we have $A_1 = A^k (A^k)^\dagger A$, then

$$
\begin{aligned}
A_1 A^\dagger &= A^k (A^k)^\dagger A A^\dagger = [A^k (A^k)^\dagger]^* (AA^\dagger)^* \\
&= [AA^\dagger A^k (A^k)^\dagger]^* = [A^k (A^k)^\dagger]^* \\
&= A^k (A^k)^\dagger.
\end{aligned}
$$

Thus, the proof is finished by (41). □

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$ be the MPCEP-inverse of A. Then, X can be written as the $\mathfrak{cr}^{\mathfrak{CR}}$ constrained inverse of A, where*

$$\begin{aligned}
&\textbf{Constraints of type 1}:\\
&\mathfrak{c}:\ \mathcal{R}(X) \subseteq \mathcal{R}(A^\dagger A_1);\\
&\mathfrak{r}:\ \mathcal{R}(X^*) \subseteq \mathcal{R}((A_1A^\dagger)^*);\\
&\textbf{Constraints of type 2}:\\
&\mathfrak{C}:\ XAA^\dagger A_1 = A^\dagger A_1;\\
&\mathfrak{R}:\ A_1A^\dagger AX = A_1A^\dagger.
\end{aligned}$$

Where A_1 is the core part of the Core-EP decomposition of A.

Proof. The proof of Constraints of type 1:

Let $X \in \mathbb{C}^{n\times n}$ be the MPCEP-inverse of A. Then, $X = A^\dagger A_1 A^\dagger$ by Theorem 3.1, which gives the condition $\mathfrak{c} : \mathcal{R}(X) \subseteq \mathcal{R}(A^\dagger A_1)$. Let $u \in \mathcal{N}(A_1A^\dagger)$, then $Xu = A^\dagger A_1 A^\dagger u = 0$, which implies $\mathcal{N}(A_1A^\dagger) \subseteq \mathcal{N}(X)$. The condition $\mathfrak{r} : \mathcal{R}(X^*) \subseteq \mathcal{R}((A_1A^\dagger)^*)$ is satisfied by $\mathcal{R}(X^*) \subseteq \mathcal{R}((A_1A^\dagger)^*)$ if and only if $\mathcal{N}(A_1A^\dagger) \subseteq \mathcal{N}(X)$.

The proof of Constraints of type 2:

By Lemma 2.1, we have $A_1 = A^k(A^k)^\dagger A$. Then

$$\begin{aligned}
XAA^\dagger A_1 &= XAA^\dagger A_1 = A^\dagger A_1 A^\dagger AA^\dagger A_1\\
&= A^\dagger A_1 A^\dagger A_1 = A^\dagger A^k(A^k)^\dagger AA^\dagger A^k(A^k)^\dagger A\\
&= A^\dagger A^k(A^k)^\dagger A^k(A^k)^\dagger A\\
&= A^\dagger A^k(A^k)^\dagger A\\
&= A^\dagger A_1,\\
A_1A^\dagger AX &= A_1A^\dagger AA^\dagger A_1A^\dagger\\
&= A_1A^\dagger A_1A^\dagger = A^k(A^k)^\dagger AA^\dagger A^k(A^k)^\dagger AA^\dagger\\
&= A^k(A^k)^\dagger A^k(A^k)^\dagger AA^\dagger\\
&= A^k(A^k)^\dagger AA^\dagger\\
&= A_1A^\dagger.
\end{aligned} \tag{42}$$

The condition $\mathfrak{C}$ and $\mathfrak{R}$ are satisfied by (42). □

If we let $B = A^\dagger A_1$ and $C = A_1A^\dagger$, then by the proof of Theorem 3.4, we have that the MPCEP-inverse of A coincides with the $(A^\dagger A_1, A_1A^\dagger)$-inverse of A. That is, we have the following theorem.

Theorem 3.5. *Let $A \in \mathbb{C}^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$ be the MPCEP-inverse of A. Then, X is the $(A^\dagger A_1, A_1A^\dagger)$-inverse of A, where A_1 is the core part of the Core-EP decomposition of A.*

Theorem 3.6. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. The MPCEP-inverse of A coincides with the $(A^\dagger A^k, (A^k)^*)$-inverse of A.*

Proof. One can prove this theorem by using Theorem 2.5, Theorem 2.6 and Theorem 2.7. □

The MPCEP-inverse of A can be got by using the "S" part of the Core-EP inverse and the "T" part of the CMP inverse by Theorem 3.6.

4. The CE matrix based on the Core-EP decomposition

We introduced CE matrix by mimicking the concept of EP matrix. The notation $[A, B] = AB - BA$ will be used in the sequel.

Definition 4.1. *Let $A \in \mathbb{C}^{n\times n}$ with $A = A_1 + A_2$ be the Core-EP decomposition of A as in (1). If $A^\dagger A_1 = A_1 A^\dagger$, then we call A is a CE matrix.*

Let $A \in \mathbb{C}^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$ be the MPCEP-inverse of A. If A is a CE matrix, then X is the $(A^\dagger A_1, A_1 A^\dagger)$-inverse by Theorem 3.5.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$. Then, A is a CE matrix if and only if $[A^{\dagger,\textcircled{\dagger}}, A] = 0$.*

Proof. By Theorem 3.3, we have $AA^{\dagger,\textcircled{\dagger}} = A_1 A^\dagger$. By Theorem 2.9, we have $A^{\dagger,\textcircled{\dagger}} = A^\dagger A^k (A^k)^\dagger$. Then, $A^{\dagger,\textcircled{\dagger}} A = A^\dagger A^k (A^k)^\dagger A = A^\dagger [A^k (A^k)^\dagger A] = A^\dagger A_1$. Thus

$$A^{\dagger,\textcircled{\dagger}} A - AA^{\dagger,\textcircled{\dagger}} = A^\dagger A_1 - A_1 A^\dagger = 0$$

by the definition of the CE matrix. □

Proposition 4.1. *Let $A \in \mathbb{C}^{n\times n}$ is a CE matrix with $\operatorname{ind}(A) = k$. Then, $A^\dagger A^{k+1} = A^k$.*

Proof. By the definition of the CE matrix, we have $A^\dagger A_1 = A_1 A^\dagger$, which is equivalent to

$$A^\dagger A^k (A^k)^\dagger A = A^k (A^k)^\dagger A A^\dagger \tag{43}$$

by Lemma 2.1. Post-multiplying by A^k on (43) gives

$$\begin{aligned} & A^\dagger A^k (A^k)^\dagger A A^k = A^k (A^k)^\dagger A A^\dagger A^k \\ \Leftrightarrow\ & A^\dagger A^k (A^k)^\dagger A^k A = A^k (A^k)^\dagger A^k \\ \Leftrightarrow\ & A^\dagger A^{k+1} = A^k. \end{aligned} \tag{44}$$

Thus, $A^\dagger A^k (A^k)^\dagger A A^k = A^k (A^k)^\dagger A A^\dagger A^k$ if and only if $A^\dagger A^{k+1} = A^k$. The proof is finished by $A^\dagger A_1 = A_1 A^\dagger$ implies $A^\dagger A^k (A^k)^\dagger A A^k = A^k (A^k)^\dagger A A^\dagger A^k$. □

Proposition 4.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If $A^\dagger A^{k+1} = A^k$, then $A^\dagger A^2 \in A^{\dagger,\textcircled{\dagger}}\{1, 4\}$.*

Proof. By the hypothesis of the proposition, we have $A^\dagger A^{k+1} = A^k$. From Theorem 3.3, we have $AA^{\dagger,\textcircled{\dagger}} = A_1 A^\dagger = A^k(A^k)^\dagger$. In view of Lemma 2.1, we have $A_1 = A^k(A^k)^\dagger A$. Then

$$
\begin{aligned}
AA^{\dagger,\textcircled{\dagger}} &= A_1A^\dagger = A^k(A^k)^\dagger AA^\dagger = A^\dagger A^{k+1}(A^k)^\dagger AA^\dagger \\
&= A^\dagger AA^k(A^k)^\dagger AA^\dagger = A^\dagger A[A^k(A^k)^\dagger]^*(AA^\dagger)^* \\
&= A^\dagger A[AA^\dagger A^k(A^k)^\dagger]^* = A^\dagger A[A^k(A^k)^\dagger]^* \\
&= A^\dagger AA^k(A^k)^\dagger = A^\dagger AAA^{\dagger,\textcircled{\dagger}} \\
&= A^\dagger A^2A^{\dagger,\textcircled{\dagger}}.
\end{aligned}
\tag{45}
$$

The equality (45) gives $AA^{\dagger,\textcircled{\dagger}} = A^\dagger A^2A^{\dagger,\textcircled{\dagger}}$. By Theorem 3.2, we have the MPCEP-inverse of A is an outer inverse of A. Pre-multiplying by $A^{\dagger,\textcircled{\dagger}}$ on $AA^{\dagger,\textcircled{\dagger}} = A^\dagger A^2A^{\dagger,\textcircled{\dagger}}$ gives $A^{\dagger,\textcircled{\dagger}} = A^{\dagger,\textcircled{\dagger}}AA^{\dagger,\textcircled{\dagger}} = A^{\dagger,\textcircled{\dagger}}A^\dagger A^2A^{\dagger,\textcircled{\dagger}}$, that is $A^\dagger A^2$ is an inner inverse of $A^{\dagger,\textcircled{\dagger}}$. Since $A^\dagger A^2A^{\dagger,\textcircled{\dagger}} = A^\dagger A^2A^\dagger A^k(A^k)^\dagger = A^\dagger AA^k(A^k)^\dagger = A^\dagger A^{k+1}(A^k)^\dagger = A^k(A^k)^\dagger$, then $A^\dagger A^2 \in A^{\dagger,\textcircled{\dagger}}\{4\}$ by $A^k(A^k)^\dagger = [A^k(A^k)^\dagger]^*$. □

5. Conclusions

One-sided MPCEP-inverse for matrices was introduced in this paper. The MPCEP-inverse can be described by using the core part A_1 in Core-EP decomposition of A and the Moore-Penrose inverse of A. The MPCEP-inverse of A coincides with the $(A^\dagger A^k, (A^k)^*)$-inverse of A, that is, the MPCEP-inverse of A is $A^{(2)}_{\mathcal{R}(A^\dagger A^k),\mathcal{N}((A^k)^*)}$. In addition, the CE matrix was introduced, a necessary and sufficient condition such that a matrix A to be a CE matrix is the MPCEP-inverse of A commutes with A, that is $[A^{\dagger,\textcircled{\dagger}}, A] = 0$, where $A^{\dagger,\textcircled{\dagger}}$ is the MPCEP-inverse of A. The future perspectives for research are proposed:

Part 1. The reverse order law of the MPCEP-inverse.

Part 2. The rank properties of the MPCEP-inverse.

Part 3. The weighted MPCEP-inverse of matrices.

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