L-fuzzy ideal theory on bounded semihoops

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Abstract. This article mainly focuses on the L-fuzzy ideal theory on bounded semihoops. Firstly, we propose two classes of L-fuzzy ideals on bounded semihoop and prove that each L-fuzzy strong ideal is an L-fuzzy ideal but an L-fuzzy ideal may not be an L-fuzzy strong ideal. We also get some properties and equivalent descriptions of L-fuzzy strong ideal. Secondly, we introduce the notion of L-fuzzy prime ideal and the second type of L-fuzzy prime ideal on bounded semihoops. Moreover, we discuss the relationship between these two types of L-fuzzy prime ideals. Finally, we present the concept of L-fuzzy maximal ideal on bounded semihoops and obtain some properties.

Keywords: bounded semihoop, L-fuzzy (strong) ideal, L-fuzzy prime ideal, L-fuzzy maximal ideal.

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1. Introduction

Classical logic can no longer fully adapt to people's reasoning and thinking activities in the development of today's era, and then non-classical logic came into being. Non-classical logic has become a useful tool for computers to deal with uncertain and fuzzy information. Various logical algebras have been introduced as the semantical systems of non-classical logic systems, for instance, MV-algebras [5], BL-algebras [8], MTL-algebras [16] and residuated lattices [**?**]. Semihoops [7] are generalization of hoops [1], which were originally proposed by Bosbach. Semihoops, as the basic residuated structures, contain all logical algebras that satisfy the residuated law. Recent years, many scholars have conducted research on semihoops and obtain some important conclusions. For example, Borzooei and Kologani [16] studied the relationships between various filters on semihoops in 2015. In 2019, Niu [13] proposed the tense operators on bounded semihoops and Zhang [17] introduced the derivations and differential filters on semihoops. In 2020, Niu and Xin [14] established the ideal theory on bounded semihoops. Since semihoops are the fundamental residuated structures, the study of semihoops is important for fuzzy logic and some corresponding algebras.

In 1965, Zadeh [18] proposed the concept of fuzzy subset of a nonempty set X as a function $f\colon X \to I$, where $I = [0,1]$ is the unit interval of real numbers. This marked the formation of fuzzy mathematics as a new discipline. The concept of fuzzy groups was introduced by Rosenfied [15] in 1971, fuzzy algebras have developed rapidly, especially fuzzy ideals on logical algebras. For example, in 2005, Liu and Li [11] proposed the definition of fuzzy filters on BL-algebras. In 2017, Liu [12] studied the ideal and fuzzy ideal in residuated lattices and obtained some important conclusions. In 2019, Borzooei [2] introduced the concept of fuzzy filters in pseudo hoops. However, we find that the current study of fuzzy ideals is limited to chain structures, but ignores that not all elements are comparable in some structures. For instance, there exists incomparable elements in lattice structures. Therefore, we try to associate semihoops with lattice structures and establish the L-fuzzy ideal theory.

This article is structured as follows: In Section 2, we summarize some fundamental definitions and conclusions on bounded semihoops, which will be used in the sequel chapters. In Section 3, we will propose two types of L-fuzzy ideals and discuss their relationship. We also study properties and equivalent characterizations of L-fuzzy strong ideal. In the remaining sections, we will introduce several special classes of L-fuzzy ideals on bounded semihoops, including L-fuzzy prime ideal, the second type of L-fuzzy prime ideal and L-fuzzy maximal ideal.

2. Preliminaries

In this section, we recall some definitions and conclusions, which will be used in the following sections.

Definition 2.1 ([7]). *An algebra* $(S, \odot, \to, \wedge, 1)$ *of type* $(2, 2, 2, 0)$ *is called a semihoop if it satisfies:*

$(S1)$ $(S, \wedge, 1)$ *is a* $\wedge$*-semilattice and it has a upper bound 1;*

$(S2)$ $(S, \odot, 1)$ *is a commutative monoid;*

$(S3)$ $(\alpha \odot \beta) \to \theta = \alpha \to (\beta \to \theta)$*, for any* $\alpha, \beta, \theta \in S$*.*

In a semihoop $(S, \odot, \to, \wedge, 1)$, we define $\alpha \leq \beta$ if and only if $\alpha \to \beta = 1$, for any $\alpha, \beta \in S$. It is easy to check that $\leq$ is a partial order relation on S and we get $\alpha \leq 1$, for all $\alpha \in S$.

Proposition 2.1 ([7]). *Let* S *be a semihoop. Then, the following properties hold:*

(1) $\alpha \odot \beta \leq \theta$ *if and only if* $\alpha \leq \beta \to \theta$*, for every* $\alpha, \beta, \theta \in S$*;*

(2) $\alpha \odot \beta \leq \alpha, \beta$*, for any* $\alpha, \beta \in S$*;*

(3) $1 \to \alpha = \alpha, \alpha \to 1 = 1$*, for all* $\alpha \in S$*;*

(4) $\alpha^n \leq \alpha$*, for every* $\alpha \in S$*,* $n \in \mathbb{N}^+$*;*

(5) $\alpha \odot (\alpha \to \beta) \leq \beta$*, for any* $\alpha, \beta \in S$*;*

(6) $\alpha \leq \beta$ *implies* $\alpha \odot \theta \leq \beta \odot \theta$*,* $\beta \to \theta \leq \alpha \to \theta$ *and* $\theta \to \alpha \leq \theta \to \beta$*, for every* $\alpha, \beta, \theta \in S$*;*

(7) $\alpha \leq (\alpha \to \beta) \to \beta$*, for any* $\alpha, \beta \in S$*;*

(8) $\alpha \to (\beta \to \theta) = \beta \to (\alpha \to \theta)$*, for every* $\alpha, \beta, \theta \in S$*.*

A semihoop $(S, \odot, \to, \wedge, 1)$ is called a bounded semihoop if there exists an element $0 \in S$ such that $0 \leq \alpha$, for all $\alpha \in S$. We denote a bounded semihoop $(S, \odot, \to, \wedge, 0, 1)$ by S.

Example 2.1 ([3]). Let $S = \{0, m, n, 1\}$ be a chain with $0 < m < n < 1$. We define $\odot$ and $\to$ on S as follows:

$\odot$	0	m	n	1
0	0	0	0	0
m	0	0	m	m
n	0	m	n	n
1	0	m	n	1

$\to$	0	m	n	1
0	1	1	1	1
m	m	1	1	1
n	0	m	1	1
1	0	m	n	1

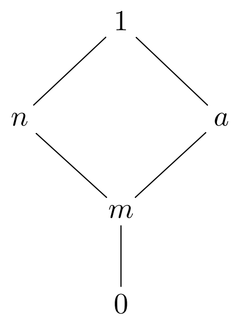
Then, $(S, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop.

Example 2.2 ([3]). Let $S = \{0, m, n, a, 1\}$. Define $\odot$ and $\to$ as follows:

$\boxtimes$	0	m	n	a	1
0	0	0	0	0	0
m	0	m	m	m	m
n	0	m	n	m	n
a	0	m	m	a	a
1	0	m	n	a	1

$\to$	0	m	n	a	1
0	1	1	1	1	1
m	0	1	1	1	1
n	0	a	1	a	1
a	0	n	n	1	1
1	0	m	n	a	1

It's Hasse diagram is as follows:



Then, $(S, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop.

In a bounded semihoop S, we define $\star$: $\alpha^\star = \alpha \to 0$, for any $\alpha \in S$. A bounded semihoop is said to have the Double Negation Property or (DNP) for short if it satisfies $\alpha^{\star\star} = \alpha$, for all $\alpha \in S$.

Proposition 2.2 ([3])**.** *Let S be a bounded semihoop. Then, we have the following statements hold: for any $\alpha, \beta \in S$,*

(1) $1^\star = 0$, $0^\star = 1$;

(2) $\alpha \leq \alpha^{\star\star}$;

(3) $\alpha^{\star\star\star} = \alpha^\star$;

(4) $\alpha \odot \alpha^\star = 0$;

(5) $\beta^\star \leq \beta \to \alpha$;

(6) $\alpha \leq \beta$ *implies* $\beta^\star \leq \alpha^\star$;

(7) *if S has (DNP), then* $\alpha \to \beta = \beta^\star \to \alpha^\star$;

(8) $\alpha \to \beta \leq \beta^\star \to \alpha^\star$;

(9) *if S has (DNP), then* $\alpha^\star \to \beta = \beta^\star \to \alpha$.

Definition 2.2 ([14])**.** *Assume that S is a bounded semihoop. The binary operation $\oplus$ is defined by $\alpha \oplus \beta = \alpha^\star \to \beta$, for any $\alpha, \beta \in S$.*

Proposition 2.3 ([14])**.** *Let S be a bounded semihoop. Then, the following properties hold:*

(1) $\alpha \leq \beta$ *implies* $\alpha \oplus \theta \leq \beta \oplus \theta$, *for every* $\alpha, \beta, \theta \in S$;

(2) $\alpha \leq \alpha \oplus \beta$, *for any* $\alpha, \beta \in S$;

(3) $\alpha \oplus \alpha^\star = 1$, *for all* $\alpha \in S$;

(4) $0 \oplus \alpha = \alpha$, $\alpha \oplus 0 = \alpha^{\star\star}$, *for all* $\alpha \in S$;

(5) $\alpha \oplus \beta = 1$ *if and only if* $\alpha^\star \leq \beta$, *for any* $\alpha, \beta \in S$;

(6) $\alpha^\star \odot \beta^\star = (\alpha \oplus \beta)^\star$ *if* S *has (DNP), for any* $\alpha, \beta \in S$;

(7) $\alpha^\star \oplus \beta^\star = (\alpha \odot \beta)^\star$ *if* S *has (DNP), for any* $\alpha, \beta \in S$.

Proposition 2.4 ([3]). *Let* S *be a bounded semihoop and for any* $\alpha, \beta \in S$, *we define:* $\alpha \vee \beta = [(\alpha \to \beta) \to \beta] \wedge [(\beta \to \alpha) \to \alpha]$. *Then, the following conditions are equivalent:*

(1) $\vee$ *is an associative operation on* S;

(2) $\alpha \leq \beta$ *implies* $\alpha \vee \theta \leq \beta \vee \theta$, *for all* $\alpha, \beta, \theta \in A$;

(3) $\alpha \vee (\beta \wedge \theta) \leq (\alpha \vee \beta) \wedge (\alpha \vee \theta)$, *for all* $\alpha, \beta, \theta \in A$;

(4) $\vee$ *is the join operation on* A.

Definition 2.3 ([3]). *A bounded semihoop is a bounded* $\vee$*-semihoop if it satisfies one of the equivalent conditions of Proposition* 2.9.

Definition 2.4 ([14]). *Let* S *be a bounded semihoop. A nonempty subset* D*of* S *is called an ideal if it satisfies:*

$(D1)$ *for any* $\alpha, \beta \in S$, $\alpha \leq \beta$ *and* $\beta \in D$*imply* $\alpha \in D$;

$(D2)$ *for any* $\alpha, \beta \in D$, $\alpha \oplus \beta \in D$.

Definition 2.5 ([3]). *Let* S *be a bounded semihoop. A nonempty subset* F*of* S *is called a filter if it satisfies:*

$(F1)$ *for any* $\alpha, \beta \in S$, $\alpha \leq \beta$ *and* $\alpha \in F$*imply* $\beta \in F$;

$(F2)$ *for any* $\alpha, \beta \in F$, $\alpha \odot \beta \in F$.

We denote the set of all ideals of S by $D(S)$.

Definition 2.6 ([14]). *Let* S *be a bounded semihoop. A proper ideal* D*of* S *is called a prime ideal if* $P \cap Q \subseteq D$ *implies* $P \subseteq D$ *or* $Q \subseteq D$, *for any* $P, Q \in D(S)$.

Proposition 2.5 ([14]). *Let* S *be a bounded* $\vee$*-semihoop with DNP. Then, every maximal ideal of* S *is prime ideal.*

3. L-fuzzy ideals

Definition 3.1. *Let S be a semihoop and $\rho:S \to [0,1]$ be a fuzzy subset on A. Then, ρ is called a fuzzy ideal of S, if for all $\alpha, \beta \in S$satifies:*

$(FI1)$ $\alpha \leq \beta$ *implies* $\rho(\alpha) \geq \rho(\beta)$*;*

$(FI2)$ $\rho(\alpha \oplus \beta) \geq min\{\rho(\alpha), \rho(\beta)\}$.

Let $(S, \odot, \to, \wedge, 0, 1)$ be a bounded semihoop and $(L, \sqcap, \sqcup, 0, 1)$ be a complete lattice. The map ρ: $S \to L$ is called an L-fuzzy subset of S. Let ρ and χ be two L-fuzzy subsets, then $\rho \wedge \chi$ and $\rho \vee \chi$ are L-fuzzy subsets, where $(\rho \wedge \chi)(\alpha) = \rho(\alpha) \sqcap \chi(\alpha)$ and $(\rho \vee \chi)(\alpha) = \rho(\alpha) \sqcup \chi(\alpha)$, for all $\alpha \in S$.

We can induce the partial order relation on $(L, \sqcap, \sqcup, 0, 1)$ with $\leq$. Define four types of level sets by $\rho_t^1 = \{\alpha \in S | \rho(\alpha) \geq t\}$, $\rho_t^2 = \{\alpha \in S | \rho(\alpha) \ngeq t\}$, $\rho_t^3 = \{\alpha \in S | \rho(\alpha) > t\}$, $\rho_t^4 = \{\alpha \in S | \rho(\alpha) \ngtr t\}$, for any $t \in L$.

Definition 3.2. *Let S be a bounded semihoop. The binary operation $\boxplus$ is defined by $\alpha \boxplus \beta = \alpha^\star \to \beta^{\star\star}$, for any $\alpha, \beta \in S$.*

If S is a bounded semihoop with DNP, then we have $\alpha \oplus \beta = \alpha \boxplus \beta$, for any $\alpha, \beta \in S$. The following example will illustrate that the two binary operations $\oplus$ and $\boxplus$ are different.

Example 3.1. Let $S = \{0, e, b, c, d, 1\}$ with $0 < e < 1$, $0 < b < c < d < 1$, where e and b are incomparable. Define $\boxtimes$ and $\to$as follows,

$\odot$	0	e	b	c	d	1
0	0	0	0	0	0	0
e	0	0	0	0	e	e
b	0	0	0	0	b	b
c	0	0	0	0	c	c
d	0	e	b	c	d	d
1	0	e	b	c	d	1

$\to$	0	e	b	c	d	1
0	1	1	1	1	1	1
e	c	1	c	1	1	1
b	c	c	1	1	1	1
c	c	c	c	1	1	1
d	0	e	b	c	1	1
1	0	e	b	c	d	1

Then, $(S, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop. In the bounded semihoop S, $\alpha \oplus \beta \neq \alpha \boxplus \beta$since $e \oplus b = e^\star \to b = (e \to 0) \to b = c \to b = c$, $e \boxplus b = e^\star \to b^{\star\star} = (e \to 0) \to ((b \to 0) \to 0) = c \to (c \to 0) = c \to c = 1$.

Since there are incomparable elements in lattice L, we will define two types of L-fuzzy ideals on bounded semihoops. The infimum and supremum of two elements $x, y \in L$ be denoted by $x \sqcap y$ and $x \sqcup y$.

Definition 3.3. *Let S be a bounded semihoop. An L-fuzzy subset ρ of S is called an L-fuzzy strong ideal if it satisfies: for any $\alpha, \beta \in S$,*

$(LFD1)$ $\alpha \leq \beta$ *implies* $\rho(\alpha) \geq \rho(\beta)$*;*

$(LFD2)$ $\rho(\alpha \boxplus \beta) \geq \rho(\alpha) \sqcap \rho(\beta)$.

Definition 3.4. *Let S be a bounded semihoop. An L-fuzzy subset ρ of S is called an L-fuzzy ideal if it satisfies: for any $\alpha, \beta \in S$,*

$(LFD1^{'})$ *$\alpha \leq \beta$ implies $\rho(\alpha) \geq \rho(\beta)$ or $\rho(\alpha)$ and $\rho(\beta)$ are incomparable;*

$(LFD2^{'})$ *$\rho(\alpha \boxplus \beta) \geq \rho(\alpha) \sqcap \rho(\beta)$.*

Obviously, each L-fuzzy strong ideal of S is an L-fuzzy ideal.

In the following we will explain the difference between fuzzy ideals and L-fuzzy ideals through definitions:

(1) Since all elements on [0,1] are comparable, $\alpha \leq \beta$ implies $\rho(\alpha) \geq \rho(\beta)$ in Definition 3.1($FD1$). However, not all elements in the lattice are comparable, so based on this feature we propose L-fuzzy strong ideals and L-fuzzy ideals.

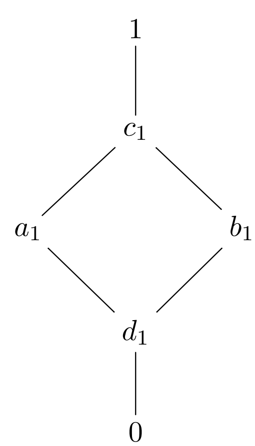
(2) Since all elements on [0,1] are comparable, $\rho(\alpha \oplus \beta) \geq min\{\rho(\alpha), \rho(\beta)\}$ in Definition 3.1($FD2$). However, not all elements in the lattice are comparable, but a lower bound exists for any two elements in the lattice. Thus, $\rho(\alpha \boxplus \beta) \geq \rho(\alpha) \sqcap \rho(\beta)$ is satisfied in Definitions 3.3($LFD2$) and 3.4($LFD2^{'}$), where $\rho(\alpha) \sqcap \rho(\beta)$ is the infimum of $\rho(\alpha)$ and $\rho(\beta)$.

Example 3.2 ([10])**.** Let $S = \{0, m, n, p, 1\}$ with $0 < m < p < 1, 0 < n < p < 1$, where m and n are incomparable. Define $\odot$ and $\rightarrow$ as follows,

$\odot$	0	m	n	p	1
0	0	0	0	0	0
m	0	m	0	m	m
n	0	0	n	n	n
p	0	m	n	p	p
1	0	m	n	p	1

$\rightarrow$	0	m	n	p	1
0	1	1	1	1	1
m	n	1	n	1	1
n	m	m	1	1	1
p	0	m	n	1	1
1	0	m	n	p	1

Then, $(S, \odot, \rightarrow, \wedge, 0, 1)$ is a bounded semihoop. Let $L = \{0, c_1, a_1, b_1, d_1, 1\}$ be a complete lattice. It's Hasse diagram is as follows:



We define an L-fuzzy subset ρ of $S$by

$$\rho(\alpha) = \begin{cases} c_1, & \text{if } \alpha = 0 \\ b_1, & \text{if } \alpha = m \\ d_1, & \text{if } \alpha = n, p, 1 \end{cases}$$

for all $\alpha \in S$. Then, ρ is an L-fuzzy strong ideal of S.

Remark 3.1. Let S be a bounded semihoop. Then, an L-fuzzy ideal of Smay not be an L-fuzzy strong ideal.

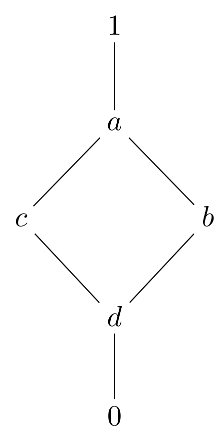
The following example will illustrate Remark 3.1.

Example 3.3 ([9])**.** Let $S = \{0, n, e, p, q, r, m, 1\}$ with $0 < n < e < 1$, $0 < p < q < m < 1$, $0 < r < m < 1$, where e and p are incomparable, q and r are incomparable. Define $\odot$ and $\to$ as follows,

$\odot$	0	n	e	p	q	r	m	1
0	0	0	0	0	0	0	0	0
n	0	0	0	0	0	0	0	n
e	0	0	e	0	e	0	0	e
p	0	0	0	0	0	p	p	p
q	0	0	e	0	e	p	q	q
r	0	0	0	p	p	r	r	r
m	0	0	e	p	q	r	m	m
1	0	n	e	p	q	r	m	1

$\to$	0	n	e	p	q	r	m	1
0	1	1	1	1	1	1	1	1
n	m	1	1	1	1	1	1	1
e	r	r	1	r	1	r	1	1
p	q	q	q	1	1	1	1	1
q	p	p	q	r	1	r	1	1
r	e	e	e	q	q	1	1	1
m	n	n	e	p	q	r	1	1
1	0	n	e	p	q	r	m	1

We can see that $(S, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop. Let $L = \{0, c, d, b, a, 1\}$ be a complete lattice. It's Hasse diagram is as follows:



We define an L-fuzzy subset ρ of S by

$$\rho(\alpha) = \begin{cases} a, & \text{if } \alpha = 0, n \\ b, & \text{if } \alpha = p \\ c, & \text{if } \alpha = e \\ d, & \text{if } \alpha = q, r, m, 1 \end{cases}$$

for all $\alpha \in S$. Then, ρ is an L-fuzzy ideal of Sbut it is not an L-fuzzy strong ideal since $e < q$but $\rho(e) = c$ and $\rho(q) = d$ are incomparable.

Definition 3.5. *Let S be a bounded semihoop. An L-fuzzy subset ρ of S is called an L-fuzzy strong filter if it satisfies: for each $\alpha, \beta \in S$,*

$(LFF1)$ *$\alpha \leq \beta$ implies $\rho(\alpha) \leq \rho(\beta)$;*

$(LFF2)$ *$\rho(\alpha \odot \beta) \geq \rho(\alpha) \sqcap \rho(\beta)$.*

Definition 3.6. *Let S be a bounded semihoop. An L-fuzzy subset ρ of S is called an L-fuzzy filter if it satisfies: for each $\alpha, \beta \in S$,*

$(LFF1')$ *$\alpha \leq \beta$ implies $\rho(\alpha) \leq \rho(\beta)$ or $\rho(\alpha)$ and $\rho(\beta)$ are incomparable;*

$(LFF2')$ *$\rho(\alpha \odot \beta) \geq \rho(\alpha) \sqcap \rho(\beta)$.*

Example 3.4. In Example 3.2, we define an L-fuzzy subset ρ of S by

$$\rho(\alpha) = \begin{cases} c_1, & \text{if } \alpha = p, 1 \\ b_1, & \text{if } \alpha = n \\ d_1, & \text{if } \alpha = 0, m \end{cases}$$

for all $\alpha \in S$. Then, ρ is an L-fuzzy strong filter of S.

Example 3.5. In Example 3.3, we define an L-fuzzy subset ρ of S by

$$\rho(\alpha) = \begin{cases} a, & \text{if } \alpha = 1, m \\ b, & \text{if } \alpha = r \\ c, & \text{if } \alpha = q, e \\ d, & \text{if } \alpha = p, n, 0 \end{cases}$$

for all $\alpha \in S$. Then, ρ is an L-fuzzy filter of Sbut it is not an L-fuzzy strong filter since $p < q$but $\rho(q) = c$ and $\rho(p) = d$ are incomparable.

Proposition 3.1. *Let S be a bounded semihoop with DNP.*

(1) *If ρ is an L-fuzzy strong ideal of S, then $\rho^\star$ is an L-fuzzy strong filter of S;*

(2) *If ρ is an L-fuzzy strong filter of S, then $\rho^\star$ is an L-fuzzy strong ideal of S;*

(3) *If ρ is an L-fuzzy ideal of S, then $\rho^\star$ is an L-fuzzy filter of S;*

(4) *If ρ is an L-fuzzy filter of S, then $\rho^\star$ is an L-fuzzy ideal of S;*

where $\rho^\star(\alpha) = \rho(\alpha^\star)$, for any $\alpha \in S$.

Proof. (1) Assume that ρ is an L-fuzzy strong ideal of S. Let $\alpha, \beta \in S$such that $\alpha \leq \beta$, then $\beta^\star \leq \alpha^\star$. By Definition 3.3($LFD1$), we get $\rho(\alpha^\star) \leq \rho(\beta^\star)$, so $\rho^\star(\alpha) \leq \rho^\star(\beta)$. Since S is a bounded semihoop with DNP, by Proposition 2.3(7), we have $\alpha^\star \boxplus \beta^\star = \alpha^\star \oplus \beta^\star = (\alpha \odot \beta)^\star$. By Definition 3.3($LFD2$), $\rho((\alpha \odot \beta)^\star) = \rho(\alpha^\star \boxplus \beta^\star) \geq \rho(\alpha^\star) \sqcap \rho(\beta^\star)$, then $\rho^\star(\alpha \odot \beta) \geq \rho^\star(\alpha) \sqcap \rho^\star(\beta)$. Therefore, $\rho^\star$ is an L-fuzzy strong filter.

(2) Let ρ be an L-fuzzy strong filter of S. Let $\alpha, \beta \in S$such that $\alpha \leq \beta$, then $\beta^\star \leq \alpha^\star$. By Definition 3.5($LFF1$), we get $\rho(\beta^\star) \leq \rho(\alpha^\star)$, so $\rho^\star(\beta) \leq \rho^\star(\alpha)$. Since S is a bounded semihoop with DNP, by Proposition 2.3(6), we have $\alpha^\star \odot \beta^\star = (\alpha \oplus \beta)^\star = (\alpha \boxplus \beta)^\star$. By Definition 3.5($LFF2$), $\rho((\alpha \boxplus \beta)^\star) = \rho(\alpha^\star \odot \beta^\star) \geq \rho(\alpha^\star) \sqcap \rho(\beta^\star)$, then $\rho^\star(\alpha \boxplus \beta) \geq \rho^\star(\alpha) \sqcap \rho^\star(\beta)$. Therefore, $\rho^\star$ is an L-fuzzy strong ideal.

(3)The proof of the conclusion is similar to (1).

(4)The proof of the conclusion is similar to (2). □

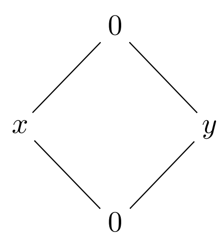
Proposition 3.2. *Assume that S is a bounded semihoop and ρ, χ are two L-fuzzy strong ideals of S. Then, $\rho \wedge \chi$ is an L-fuzzy strong ideal.*

Proof. The proof of this proposition is obvious. □

Remark 3.2. Assume that S is a bounded semihoop and ρ, χ are two L-fuzzy strong ideals of S. Then, $\rho \vee \chi$may not be an L-fuzzy strong ideal.

The following example will illustrate Remark 3.2.

Example 3.6. Let S be a bounded semihoop in Example 3.2 and $L = \{0, x, y, 1\}$ be a complete lattice. The Hasse diagram of L is as follows:



We define two L-fuzzy subsets by

$$\rho_1(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0 \\ x, & \text{if } \alpha = n \\ 0, & \text{if } \alpha = m, p, 1 \end{cases}$$

and

$$\rho_2(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0 \\ y, & \text{if } \alpha = m \\ 0, & \text{if } \alpha = n, p, 1 \end{cases}$$

for any $\alpha \in S$. Then, ρ_1 and ρ_2 are two L-fuzzy strong ideals of S. Moreover, we get an L-fuzzy subset $\rho_1 \vee \rho_2$ by

$$(\rho_1 \vee \rho_2)(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0 \\ y, & \text{if } \alpha = m \\ x, & \text{if } \alpha = n \\ 0, & \text{if } \alpha = p, 1 \end{cases}$$

for any $\alpha \in S$. Then, $\rho_1 \vee \rho_2$ is not an L-fuzzy strong ideal since $(\rho_1 \vee \rho_2)(n) = x$ and $(\rho_1 \vee \rho_2)(m) = y$ are incomparable.

Corollary 3.1. *Let S be a bounded semihoop and ρ, χ be two L-fuzzy strong ideals of S. If $\rho \subseteq \chi$, then $\rho \vee \chi$ is an L-fuzzy strong ideal.*

Proof. The proof is clearly. □

Proposition 3.3. *Given a bounded semihoop S.*

(1) *An L-fuzzy subset ρ of S is an L-fuzzy strong ideal if and only if the level set $\rho_t^1 = \{\alpha \in S | \rho(\alpha) \geq t\} (\neq \emptyset)$ is an ideal, for any $t \in L$.*

(2) *If Lsatisfies $y = \vee\{x \in L | x < y\}$, for any $y \in L$, then an L-fuzzy subset ρ of S is an L-fuzzy strong ideal if and only if the level set $\rho_t^3 = \{\alpha \in S | \rho(\alpha) > t\} (\neq \emptyset)$ is an ideal, for any $t \in L$.*

Proof. (1)Assume that ρ is an L-fuzzy strong ideal. Let $\alpha, \beta \in S$such that $\alpha \leq \beta$ and $\beta \in \rho_t^1$, then $\rho(\beta) \geq t$. By Definition 3.3($LFD1$), $\rho(\alpha) \geq \rho(\beta) \geq t$, so $\rho(\alpha) \geq t$, then $\alpha \in \rho_t^1$. Let $\alpha, \beta \in \rho_t^1$, that is $\rho(\alpha) \geq t$ and $\rho(\beta) \geq t$, so $\rho(\alpha) \sqcap \rho(\beta) \geq t$. By Definition 3.3($LFD2$), $\rho(\alpha \boxplus \beta) \geq \rho(\alpha) \sqcap \rho(\beta) \geq t$, so $\rho(\alpha \boxplus \beta) \geq t$, then $\alpha \boxplus \beta \in \rho_t^1$. Hence, $\rho_t^1 (\neq \emptyset)$ is an ideal of S.

Conversely, let $\rho_t^1 (\neq \emptyset)$ be an ideal. Taking $t = \rho(\alpha) \sqcap \rho(\beta)$, we get $\alpha \in \rho_t$ and $\beta \in \rho_t$, for any $\alpha, \beta \in S$. Since ρ_t^1 is an ideal of S, thus $\alpha \boxplus \beta \in \rho_t^1$, so $\rho(\alpha \boxplus \beta) \geq t = \rho(\alpha) \sqcap \rho(\beta)$. Let $\alpha, \beta \in S$such that $\alpha \leq \beta$. Taking $t = \rho(\beta)$, then $\beta \in \rho_{\rho(\beta)}^1$. Since $\rho_{\rho(\beta)}^1$ is an ideal of S, thus $\alpha \in \rho_{\rho(\beta)}^1$, that is $\rho(\alpha) \geq \rho(\beta)$. Therefore, ρ is an L-fuzzy strong ideal of S.

(2)The proof is similar to (1). □

By Proposition 3.3, we easily obtain that an L-fuzzy subset ρ of S is an L-fuzzy strong ideal if and only if the complement of $\rho_t^2 (\neq \emptyset)$ is an ideal. Similarly, an L-fuzzy subset ρ of S is an L-fuzzy strong ideal if and only if the complement of $\rho_t^4 (\neq \emptyset)$ is an ideal.

Proposition 3.4. *Let S be a bounded semihoop with DNP. An L-fuzzy subset ρ of S is an L-fuzzy strong ideal if and only if for any $\alpha, \beta \in S$, the following conditions hold:*

(1) $\rho(0) \geq \rho(\alpha)$;

(2) $\rho(\alpha) \sqcap \rho(\alpha^\star \odot \beta) \leq \rho(\beta)$.

Proof. For all $\alpha \in S$, we have $0 \leq \alpha$. Since ρ is an L-fuzzy strong ideal of S, by Definition 3.3($LFD1$), we get $\rho(0) \geq \rho(\alpha)$. Since $\beta \to (\alpha \boxplus (\alpha^\star \odot \beta)) = \beta \to (\alpha^\star \to (\alpha^\star \odot \beta)^{\star\star}) = (\beta \odot \alpha^\star) \to (\beta \odot \alpha^\star)^{\star\star} = (\beta \odot \alpha^\star) \to (\beta \odot \alpha^\star) = 1$, for any $\alpha, \beta \in S$, thus $\alpha \boxplus (\alpha^\star \odot \beta) \geq \beta$, then $\rho(\alpha \boxplus (\alpha^\star \odot \beta)) \leq \rho(\beta)$. By Definition 3.3($LFD2$), $\rho(\alpha) \sqcap \rho(\alpha^\star \odot \beta) \leq \rho(\alpha \boxplus (\alpha^\star \odot \beta)) \leq \rho(\beta)$. Hence, $\rho(\alpha) \sqcap \rho(\alpha^\star \odot \beta) \leq \rho(\beta)$.

Conversely, for any $\alpha, \beta \in S$such that $\alpha \leq \beta$, so $\beta^\star \leq \alpha^\star$, then $\alpha \odot \beta^\star \geq \alpha \odot \alpha^\star = 0$, so $\rho(\beta^\star \odot \alpha) \leq \rho(0)$. By (1), $\rho(0) \geq \rho(\beta^\star \odot \alpha)$, then $\rho(0) = \rho(\beta^\star \odot \alpha)$. By (2), $\rho(\alpha) \leq \rho(\beta) \sqcap \rho(\beta^\star \odot \alpha) = \rho(\beta) \sqcap \rho(0) = \rho(\beta)$, so $\rho(\alpha) \geq \rho(\beta)$. Since $\alpha^\star \odot (\alpha \boxplus \beta) = \alpha^\star \odot (\alpha^\star \to \beta^{\star\star}) = \alpha^\star \odot (\alpha^\star \to \beta) \leq \beta$, thus $\rho(\alpha^\star \odot (\alpha \boxplus \beta)) \geq \rho(\beta)$, then $\rho(\alpha \boxplus \beta) \geq \rho(\alpha) \sqcap \rho(\alpha^\star \odot (\alpha \boxplus \beta)) \geq \rho(\alpha) \sqcap \rho(\beta)$. Therefore, ρ is an L-fuzzy strong ideal of S. □

Proposition 3.5. *Let S be a bounded semihoop with DNP. An L-fuzzy subset ρ of S is an L-fuzzy strong ideal if and only if for any $\alpha, \beta \in S$, the following conditions hold:*

(1) $\rho(0) \geq \rho(\alpha)$;

(2) $\rho(\alpha) \sqcap \rho(\alpha^* \to \beta^\star)^\star \leq \rho(\beta)$.

Proof. By Proposition 2.3(6), $\alpha^\star \odot \beta = \alpha^\star \odot \beta^{\star\star} = (\alpha \boxplus \beta^\star)^\star = (\alpha^* \to \beta^{\star\star\star})^\star = (\alpha^\star \to \beta^\star)^\star$, for every $\alpha, \beta \in S$. Thus, by Proposition 3.4, the conclusion holds. □

Proposition 3.6. *Let S be a bounded semihoop. An L-fuzzy subset ρ of S is an L-fuzzy strong ideal if and only if for any $\alpha, \beta \in S$, the following conditions hold:*

(1) $\rho(\alpha \wedge \beta) \geq \rho(\alpha)$;

(2) $\rho(\alpha \boxplus \beta) \geq \rho(\alpha) \sqcap f(\beta)$.

Proof. The proof is clearly. □

Lemma 3.1. *Assume that S is a bounded semihoop and ρ is an L-fuzzy strong ideal of S. Then, $\rho(\alpha^{\star\star}) = \rho(\alpha)$, for each $\alpha \in S$.*

Proof. By Proposition 2.2(2), $\alpha \leq \alpha^{**}$. Since ρ is an L-fuzzy strong ideal, by Definition 3.3($LFD1$), we obtain $\rho(\alpha) \geq \rho(\alpha^{\star\star})$. By Definition 3.3($LFD2$), $\rho(\alpha^{\star\star}) = \rho(\alpha \boxplus 0) \geq \rho(\alpha) \sqcap \rho(0) = \rho(\alpha)$, so $\rho(\alpha^{\star\star}) \geq \rho(\alpha)$. Therefore, $\rho(\alpha^{\star\star}) = \rho(\alpha)$, for all $\alpha \in S$. □

Given a nonempty subset $D$of S and $x, y \in L$ such that $x > y$. Define an L-fuzzy set $\rho^D_{x,y}$ by

$$\rho^D_{x,y}(\alpha) = \begin{cases} x, & \text{if } \alpha \in D \\ y, & \textit{others} \end{cases}$$

for any $\alpha \in S$.

Proposition 3.7. *Let S be a bounded semihoop and D be a nonempty subset of S. Then, $\rho_{x,y}^D$ is an L-fuzzy strong ideal if and only if D is an ideal.*

Proof. Assume that $\rho_{x,y}^D$ is an L-fuzzy strong ideal of S. Let $\alpha, \beta \in D$, then $\rho_{x,y}^D(\alpha) = \rho_{x,y}^D(\beta) = x$, so $\rho_{x,y}^D(\alpha \boxplus \beta) \geq \rho_{x,y}^D(\alpha) \sqcap \rho_{x,y}^D(\beta) = x$, then $\alpha \boxplus \beta \in D$. Let $\alpha \leq \beta$ and $\beta \in D$, for any $\alpha, \beta \in S$, then $\rho_{x,y}^D(\alpha) \geq \rho_{x,y}^D(\beta)$ and $\rho_{x,y}^D(\beta) = x$, so $\rho_{x,y}^D(\alpha) \geq x$, then $\alpha \in D$. Therefore, D is an ideal of S.

Conversely, let D be an ideal of S.

Firstly, suppose $\alpha, \beta \in S$, then we discuss the following two situations.

Case (1). If $\alpha, \beta \in D$, then $\alpha \boxplus \beta \in D$ and $\rho_{x,y}^D(\alpha) = \rho_{x,y}^D(\beta) = x$, so $\rho_{x,y}^D(\alpha \boxplus \beta) = x = \rho_{x,y}^D(\alpha) \sqcap \rho_{x,y}^D(\beta)$.

Case (2). If $\alpha \notin D$ or $\beta \notin D$, then $\rho_{x,y}^D(\alpha) = y$ or $\rho_{x,y}^D(\beta) = y$, so $\rho_{x,y}^D(\alpha \boxplus \beta) = y = \rho_{x,y}^D(\alpha) \sqcap \rho_{x,y}^D(\beta)$.

Hence, $\rho_{x,y}^D(\alpha \boxplus \beta) \geq \rho_{x,y}^D(\alpha) \sqcap \rho_{x,y}^D(\beta)$, for any $\alpha, \beta \in S$.

Secondly, let $\alpha, \beta \in S$ and $\alpha \leq \beta$, then we also discuss the following two situations.

Case (1). If $\beta \in D$, so $\alpha \in D$ and $\rho_{x,y}^D(\beta) = x = \rho_{x,y}^D(\alpha)$.

Case (2). If $\beta \notin D$, then $\rho_{x,y}^D(\beta) = y$, so $\alpha \notin I$, then $\rho_{x,y}^D(\alpha) \geq \rho_{x,y}^D(\beta) = y$.

Hence, $\rho_{x,y}^D(\alpha) \geq \rho_{x,y}^D(\beta)$, for any $\alpha, \beta \in S$satisfying $\alpha \leq \beta$. Therefore, $\rho_{x,y}^D$ is an L-fuzzy strong ideal. □

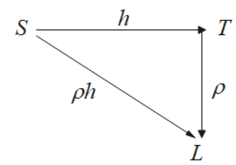
Let S and T be two bounded semihoops. The map h: $S \to T$ is said to be a homomorphism if $h(\alpha \to \beta) = h(\alpha) \to h(\beta)$, $h(\alpha \odot \beta) = h(\alpha) \odot h(\beta)$, $h(\alpha \wedge \beta) = h(\alpha) \wedge h(\beta)$, $h(0) = 0_L$, for any $\alpha, \beta \in S$. We also get $h(1) = 1_L$ and $h(\alpha^\star) = (h(\alpha))^\star$, for all $\alpha \in S$.

Let L_1 and L_2 be two complete lattices. The map h: $L_1 \to L_2$ is said to be a lattice-homomorphism if $h(\alpha \sqcap \beta) = h(\alpha) \sqcap h(\beta)$, $h(\alpha \sqcup \beta) = h(\alpha) \sqcup h(\beta)$, $h(0) = 0_{L_2}$, $h(1) = 1_{L_2}$, for any $\alpha, \beta \in S$.

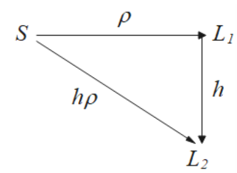
Proposition 3.8. *Let S and T be two bounded semihoops, ρ be an L-fuzzy strong ideal of T and h: $S \to T$ be a homomorphism. Then, ρh is also an L-fuzzy strong ideal of S.*

Proof. Since h is a homomorphism, thus $(\rho h)(0) = \rho(h(0)) = \rho(0) \geq \rho(h(\alpha)) = (\rho h)(\alpha)$, for all $\alpha \in S$. Since $(\rho h)(\alpha) \sqcap (\rho h)((\alpha^\star \to \beta^\star)^\star) = \rho(h(\alpha)) \sqcap \rho(h((\alpha^\star \to \beta^\star)^\star)) = \rho(h(\alpha)) \sqcap \rho((h(\alpha^\star \to \beta^\star))^\star) = \rho(h(\alpha)) \sqcap \rho((h(\alpha^\star) \to h(\beta^\star))^\star) = \rho(h(\alpha)) \sqcap \rho(((h(\alpha))^\star \to (h(\beta))^\star)^\star) \leq \rho(h(\beta)) = (\rho h)(\beta)$, for any $\alpha, \beta \in S$. Hence, by Proposition 3.3, ρh is an L-fuzzy strong ideal. □

Proposition 3.9. *Let L_1 and L_2 be two complete lattices, ρ be an L_1-fuzzy strong ideal of S and h: $L_1 \to L_2$ be a lattice-homomorphism. Then, $h\rho$ is also an L_2-fuzzy strong ideal of S.*



Proof. Since h is a lattice-homomorphism and ρ be an L_1-fuzzy strong ideal of S, thus $(h\rho)(\alpha \wedge \beta) = h(\rho(\alpha \wedge \beta)) \geq h(\rho(\alpha)) = (h\rho)(\alpha)$, for all $\alpha, \beta \in S$. Moreover, $(h\rho)(\alpha \boxplus \beta) = h(\rho(\alpha \boxplus \beta) \geq h(\rho(\alpha) \sqcap \rho(\beta)) = h(\rho(\alpha)) \sqcap h(\rho(\beta)) = (h\rho)(\alpha) \sqcap (h\rho)(\beta)$, for all $\alpha, \beta \in S$. Therefore, by Proposition 3.6, $h\rho$ is also an L_2-fuzzy strong ideal of S. □



Proposition 3.10. *Let S be a bounded semihoop with DNP, L be a complete lattice, ρ be an L-fuzzy strong ideal of S and H be an up-set sublattice of L. Then, $\rho^{-1}(H)$ is an ideal of S.*

Proof. We will prove the proposition in the following parts:

(i) Since H is a sublattice of L, thus there exists $\alpha \in S$such that $\rho(\alpha) = x \in H$, so $\alpha \in \rho^{-1}(H)$, then $\rho^{-1}(H)$ is a non-empty set of S.

(ii) Let $\alpha, \beta \in S$ with $\alpha \leq \beta$ and $\beta \in \rho^{-1}(H)$, then $\rho(\beta) \in H$ and $\rho(\alpha) \geq \rho(\beta)$. Since H is an up-set, thus $\rho(\alpha) \in H$, so $\alpha \in \rho^{-1}(H)$.

(iii) For any $\alpha, \beta \in \rho^{-1}(H)$, then $\rho(\alpha), \rho(\beta) \in H$. Since H is an up-set sublattice of L, thus $\rho(\alpha) \sqcap \rho(\beta) \in H$. From Definition 3.3($LFD2$) and S has with DNP, $\rho(\alpha \boxplus \beta) = \rho(\alpha \oplus \beta) \geq \rho(\alpha) \sqcap \rho(\beta)$, then $\rho(\alpha \oplus \beta) \in H$, so $\alpha \oplus \beta \in \rho^{-1}(H)$. Therefore, $\rho^{-1}(H)$ is an ideal of S. □

Definition 3.7. *Let S be a bounded semihoop and ρ be an L-fuzzy strong ideal of A. The smallest L-fuzzy strong ideal containing ρ is called the L-fuzzy strong ideal generalized by ρ, written $[\rho]$.*

Proposition 3.11. *Let S be a bounded semihoop and ρ be an L-fuzzy subset of S. Then, $[\rho](\alpha) = \sqcup\{\rho(\alpha_1) \sqcap \rho(\alpha_2) \sqcap \cdots \sqcap \rho(\alpha_n) | \alpha \leq \alpha_1 \boxplus \alpha_2 \boxplus \cdots \boxplus \alpha_n, \alpha_1, \alpha_2, \cdots, \alpha_n \in S\}$.*

Proof. Let $f(\alpha) = \sqcup\{\rho(\alpha_1) \sqcap \rho(\alpha_2) \sqcap \cdots \sqcap \rho(\alpha_n) | \alpha \leq \alpha_1 \boxplus \alpha_2 \boxplus \cdots \boxplus \alpha_n, \alpha_1, \alpha_2, \cdots, \alpha_n \in S\}$.

First, we prove that $f(\alpha)$ is an L-fuzzy strong ideal of S. Obviously, $f(0) \geq f(\alpha)$, for all $\alpha \in S$. Let $\alpha, \beta \in S$, if there are $a_1, \cdots, a_n, b_1, \cdots, b_m \in S$such that $\alpha \leq a_1 \boxplus \cdots \boxplus a_n$ and $\alpha^* \odot \beta \leq b_1 \boxplus \cdots \boxplus b_m$, then $\beta \leq \alpha \boxplus (\alpha^* \odot \beta) =$

$(a_1 \boxplus \cdots \boxplus a_n) \boxplus (b_1 \boxplus \cdots \boxplus b_m)$, so $f(\beta) \geq \rho(a_1) \sqcap \cdots \sqcap \rho(a_n) \sqcap \rho(b_1) \sqcap \cdots \sqcap \rho(b_m)$. Since $f(\alpha) \sqcap f(\alpha^* \odot \beta) = (\sqcup\{\rho(a_1) \sqcap \cdots \sqcap \rho(a_n) | \alpha \leq a_1 \boxplus \cdots \boxplus a_n, a_1, \cdots, a_n \in S\}) \sqcap (\sqcup\{\rho(b_1) \sqcap \cdots \sqcap \rho(b_m) | \alpha^* \odot \beta \leq b_1 \boxplus \cdots \boxplus b_m, b_1, \cdots, b_m \in S\}) = \sqcup\{\rho(a_1) \sqcap \cdots \sqcap \rho(a_n) \sqcap \rho(b_1) \sqcap \cdots \sqcap \rho(b_m) | \alpha \leq a_1 \boxplus \cdots \boxplus a_n, \alpha^* \odot \beta \leq b_1 \boxplus \cdots \boxplus b_m, a_1, \cdots, a_n, b_1, \cdots, b_m \in S\}$, thus $f(\alpha) \sqcap f(\alpha^* \odot \beta) \leq f(\beta)$. Therefore by Proposition 3.4, we have f is an L-fuzzy strong ideal of S.

Next, since $\alpha \leq \alpha \boxplus \alpha$, we have $f(\alpha) \geq \rho(\alpha) \sqcap \rho(\alpha) = \rho(\alpha)$. Thus, $f$contains ρ.

Finally, suppose ω is also an L-fuzzy strong ideal of Ssuch ωcontains ρ. Then, for any $\alpha \in S$, $f(\alpha) = \sqcup\{\rho(\alpha_1) \sqcap \cdots \sqcap \rho(\alpha_n) | \alpha \leq \alpha_1 \boxplus \cdots \boxplus \alpha_n, \alpha_1, \cdots, \alpha_n \in S\} \leq \sqcup\{\omega(\alpha_1) \sqcap \cdots \sqcap \omega(\alpha_n)\} | \alpha \leq \alpha_1 \boxplus \cdots \boxplus \alpha_n, \alpha_1, \cdots, \alpha_n \in S\} \leq \omega(\alpha)$. Therefore, f is an L-fuzzy strong ideal generated by ρ, that is $[\rho] = f$. □

4. L-fuzzy prime ideals

In this part, we will introduce the concept of L-fuzzy prime ideals on bounded semihoops and study some of their properties.

Definition 4.1. *Let S be a bounded semihoop. An L-fuzzy strong ideal ρ of S is called an L-fuzzy prime ideal if $\rho(\alpha \wedge \beta) \leq \rho(\alpha) \sqcup \rho(\beta)$, for any $\alpha, \beta \in S$.*

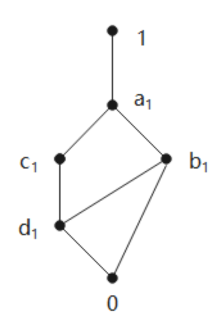
Example 4.1 ([14])**.** Let $S = \{0, r, m, n, 1\}$ be a chain with $0 < r < m < n < 1$. Define $\odot$ and $\rightarrow$ on S in the following:

$\boxtimes$	0	r	m	n	1
0	0	0	0	0	0
r	0	0	0	0	r
m	0	0	0	r	m
n	0	0	r	r	n
1	0	r	m	n	1

$\rightarrow$	0	r	m	n	1
0	1	1	1	1	1
r	r	1	1	1	1
m	m	n	1	1	1
n	r	n	n	1	1
1	0	r	m	n	1

Then, $(S, \odot, \rightarrow, \wedge, 0, 1)$ is a bounded semihoop.

Let $L = \{0, a_1, b_1, c_1, d_1, 1\}$ be a lattice. It's Hasse diagram is as follows:



Define an L-fuzzy subset ρ of S by

$$\rho(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0 \\ a_1, & \text{if } \alpha = r \\ c_1, & \text{if } \alpha = m \\ d_1, & \text{if } \alpha = n \\ 0, & \text{if } \alpha = 1 \end{cases}$$

for $\alpha \in S$. We can see that ρ is an L-fuzzy prime ideal of S.

Proposition 4.1. *Suppose that S is a bounded semihoop. An L-fuzzy strong ideal ρ of S is an L-fuzzy prime ideal if and only if $\rho(\alpha \wedge \beta) = \rho(\alpha)$ or $\rho(\alpha \wedge \beta) = \rho(\beta)$, for any $\alpha, \beta \in S$.*

Proof. Let ρ be a L-fuzzy prime ideal of S, then $\rho(\alpha) \sqcup \rho(\beta) \geq \rho(\alpha \wedge \beta)$, for every $\alpha, \beta \in S$, so $\rho(\alpha) \geq \rho(\alpha \wedge \beta)$ or $\rho(\beta) \geq \rho(\alpha \wedge \beta)$. Since $\alpha \wedge \beta \leq \alpha$ and $\alpha \wedge \beta \leq \beta$, thus $\rho(\alpha \wedge \beta) \geq \rho(\alpha)$, $\rho(\alpha \wedge \beta) \geq \rho(\beta)$. Therefore, $\rho(\alpha \wedge \beta) = \rho(\alpha)$ or $\rho(\alpha \wedge \beta) = \rho(\beta)$.

Conversely, the proof is obviously. □

Proposition 4.2. *Suppose that S is a bounded semihoop.*

(1) *An L-fuzzy strong ideal ρ of S is an L-fuzzy prime ideal if and only if the level set $\rho_t^1 = \{\alpha \in S | \rho(\alpha) \geq t\} (\neq \emptyset)$ is a prime ideal, for any $t \in L$.*

(2) *If Lsatisfies $y = \vee\{x \in L | x < y\}$, for any $y \in L$, then an L-fuzzy strong ideal ρ of S is an L-fuzzy prime ideal if and only if the level set $\rho_t^3 = \{\alpha \in S | \rho(\alpha) > t\} (\neq \emptyset)$ is a prime ideal, for any $t \in L$.*

Proof. (1) Let ρ be an L-fuzzy prime ideal of S. By Proposition 3.3(1), we get that ρ_t^1 is an ideal of S. For any $\alpha, \beta \in S$satisfying $\alpha \wedge \beta \in \rho_t^1$, then $\rho(\alpha \wedge \beta) \geq t$, so $t \leq \rho(\alpha \wedge \beta) \leq \rho(\alpha) \sqcup \rho(\beta)$, so $\rho(\alpha) \sqcup \rho(\beta) \geq t$. Since $\rho(\alpha) \sqcup \rho(\beta) = \rho(\alpha)$ or $\rho(\alpha) \sqcup \rho(\beta) = \rho(\beta)$, thus $\rho(\alpha) \geq t$ or $\rho(\beta) \geq t$, that is $\alpha \in \rho_t^1$ or $\beta \in \rho_t^1$. Therefore, by the definition of prime ideal, we get that ρ_t^1 is a prime ideal.

Conversely, let ρ_t^1 be a prime ideal. By Proposition 3.3(1), we get ρ is an L-fuzzy strong ideal. Taking $t = \rho(\alpha \wedge \beta)$, so $\alpha \wedge \beta \in \rho^1_{\rho(\alpha \wedge \beta)}$, for $\alpha, \beta \in S$. So $\alpha \in \rho^1_{\rho(\alpha \wedge \beta)}$ and $\beta \in \rho^1_{\rho(\alpha \wedge \beta)}$, then $\rho(\alpha) \geq \rho(\alpha \wedge \beta)$ and $\rho(\beta) \geq \rho(\alpha \wedge \beta)$, so $\rho(\alpha) \sqcup \rho(\beta) \geq \rho(\alpha \wedge \beta)$. Hence, ρ is an L-fuzzy prime ideal.

(2) The proof is similar to part (1). □

By Proposition 4.2, we obtain that an L-fuzzy strong ideal ρ of S is an L-fuzzy prime ideal if and only if the complement of $\rho_t^2 (\neq \emptyset)$ is a prime ideal. Similarly, an L-fuzzy strong ideal ρ of S is an L-fuzzy prime ideal if and only if the complement of $\rho_t^4 (\neq \emptyset)$ is a prime ideal.

Proposition 4.3. *Assume that S is a bounded semihoop and ρ is an L-fuzzy strong ideal of S. Then, the following conditions are equivalent:*

(1) *ρ is an L-fuzzy prime ideal of S;*

(2) *$\rho(\alpha \wedge \beta) = \rho(0)$ implies $\rho(\alpha) = \rho(0)$ or $\rho(\beta) = \rho(0)$, for any $\alpha, \beta \in S$.*

Proof. (1) $\Rightarrow$ (2) Let ρ be an L-fuzzy prime ideal of S, then $\rho(\alpha \wedge \beta) \leq \rho(\alpha) \sqcup \rho(\beta)$, for any $\alpha, \beta \in S$. Suppose $\alpha, \beta \in S$such that $\rho(\alpha \wedge \beta) = \rho(0)$, so $\rho(0) \geq \rho(\alpha) \sqcup \rho(\beta)$. Since ρ is an L-fuzzy strong ideal, thus $\rho(0) \leq \rho(\alpha) \sqcup \rho(\beta)$, then $\rho(\alpha) \sqcup \rho(\beta) = \rho(0)$. Hence, $\rho(\alpha) = \rho(0)$ or $\rho(\beta) = \rho(0)$.

(2) $\Rightarrow$ (1) For any $\alpha, \beta \in S$satisfying $\alpha \wedge \beta \in \rho_t^1$, then $\rho(\alpha \wedge \beta) \geq t$, taking $t = \rho(0)$. Since ρ is an L-fuzzy strong ideal, thus $\rho(\alpha \wedge \beta) \leq t = \rho(0)$, then $\rho(\alpha \wedge \beta) = \rho(0)$, so $\rho(\alpha) = \rho(0) \geq t$ or $\rho(\beta) = \rho(0) \geq t$, that is $\alpha \in \rho_t^1$ or $\beta \in \rho_t^1$, then ρ_t^1 is a prime ideal. By Proposition 4.2, we obtain that ρ is an L-fuzzy prime ideal. □

Proposition 4.4. *Suppose that S is a bounded semihoop and D is an ideal of S and ρ is an L-fuzzy strong ideal of S. Then, $\rho_{x,y}^D$ is an L-fuzzy prime ideal if and only if D is a prime ideal.*

Proof. Given an L-fuzzy prime ideal $\rho_{x,y}^D$. By Proposition 4.1, $\rho_{x,y}^D(\alpha \wedge \beta) = \rho_{x,y}^D(\alpha)$ or $\rho_{x,y}^D(\alpha \wedge \beta) = \rho_{x,y}^D(\beta)$. Let for any $\alpha \wedge \beta \in D$, that is $\rho_{x,y}^D(\alpha \wedge \beta) = x$, then $\rho_{x,y}^D(\alpha) = x$ or $\rho_{x,y}^D(\beta) = x$, so $\alpha \in D$ or $\beta \in D$. Therefore, D is a prime ideal.

Conversely, let D be a prime ideal of S. For any $\alpha, \beta \in S$, if $\alpha \wedge \beta \in D$, then $\alpha \in D$ and $\beta \in D$, in other words, $\rho_{x,y}^D(\alpha) = x$ or $\rho_{x,y}^D(\beta) = x$, so $\rho_{x,y}^D(\alpha \wedge \beta) = x = \rho_{x,y}^D(\alpha) \sqcup \rho_{x,y}^D(\beta)$. If $\alpha \wedge \beta \notin D$, then $\alpha \notin D$ and $\beta \notin D$, that is $\rho_{x,y}^D(\alpha) = y$ and $\rho_{x,y}^D(\beta) = y$, so $\rho_{x,y}^D(\alpha \wedge \beta) = y = \rho_{x,y}^D(\alpha) \sqcup \rho_{x,y}^D(\beta)$ and so $\alpha \wedge \beta \notin D$. Therefore, $\rho_{x,y}^D$ is an L-fuzzy prime ideal. □

Proposition 4.5. *Suppose that S is a bounded semihoop and ρ is an L-fuzzy subset of S. Define a map $\rho^{\Box} : S \to L$by $\rho^{\Box}(\alpha) = \rho(\alpha) \sqcup w$, for any $\alpha \in S$, $w \in L$satisfying $w < \rho(0)$. Then, ρ is an L-fuzzy prime ideal if and only if $\rho^{\Box}$ is an L-fuzzy prime ideal.*

Proof. Let ρ be an L-fuzzy prime ideal of S, then $\rho(0) \geq \rho(\alpha)$, for every $\alpha \in S$, so $\rho^{\Box}(\alpha) = \rho(\alpha) \sqcup w \leq \rho(0) \sqcup w = \rho^{\Box}(0)$. Since $\rho^{\Box}(\alpha) \sqcap \rho^{\Box}(\alpha^{\star} \odot \beta) = (\rho(\alpha) \sqcup w) \sqcap (\rho(\alpha^{\star} \odot \beta) \sqcup w) = (\rho(\alpha) \sqcap \rho(\alpha^{\star} \odot \beta)) \sqcup w \leq \rho(\beta) \sqcup w = \rho^{\Box}(\beta)$, for any $\alpha, \beta \in S$. So by Proposition 3.4, $\rho^{\Box}$ is an L-fuzzy strong ideal. Since ρ is an L-fuzzy prime ideal, thus $\rho(\alpha) \sqcup \rho(\beta) \geq \rho(\alpha \wedge \beta)$, then $\rho^{\Box}(\alpha \wedge \beta) = \rho(\alpha \wedge \beta) \sqcup w \leq (\rho(\alpha) \sqcup \rho(\beta)) \sqcup w = (\rho(\alpha) \sqcup w) \sqcup (\rho(\beta) \sqcup w) = \rho^{\Box}(\alpha) \sqcup \rho^{\Box}(\beta)$. Therefore, $\rho^{\Box}$ is an L-fuzzy prime ideal of S.

Conversely, given an L-fuzzy prime ideal $\rho^{\Box}$, so $\rho^{\Box}(0) \geq \rho^{\Box}(\alpha)$, so $\rho(0) \sqcup w \geq \rho(\alpha) \sqcup w$, then $\rho(0) \geq \rho(\alpha)$. Since $\rho^{\Box}(\alpha) \sqcap \rho^{\Box}(\alpha^{\star} \odot \beta) \leq \rho^{\Box}(\beta)$, thus $(\rho(\alpha) \sqcup w) \sqcap (\rho(\alpha^{\star} \odot \beta) \sqcup w) \leq (\rho(\beta) \sqcup w)$, then $\rho(\alpha) \sqcap \rho(\alpha^{\star} \odot \beta) \leq \rho(\beta)$, so ρ is an L-fuzzy strong ideal. Since $\rho^{\Box}(\alpha \wedge \beta) \leq \rho^{\Box}(\alpha) \sqcup \rho^{\Box}(\beta)$, thus $\rho(\alpha \wedge \beta) \sqcup w \leq (\rho(\alpha) \sqcup w) \sqcup (\rho(\beta) \sqcup w) = (\rho(\alpha) \sqcup \rho(\beta)) \sqcup w$, so $\rho(\alpha \wedge \beta) \leq \rho(\alpha) \sqcup \rho(\beta)$. Therefore, ρ is an L-fuzzy prime ideal of S. □

5. The second type of L-fuzzy prime ideals

Definition 5.1. *Let S be a bounded semihoop. An L-fuzzy strong ideal ρ is called the second type of L-fuzzy prime if ρ is non-constant and $\rho((\alpha \to \beta)^\star) = \rho(0)$ or $\rho((\beta \to \alpha)^\star) = \rho(0)$, for any $\alpha, \beta \in S$.*

Example 5.1. Let S be a bounded semihoop in Example 3.5 and L be a lattice in Example 3.2. Define two L-fuzzy subsets ρ and $\chi$by

$$\rho(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0, m \\ x, & \text{if } \alpha = n, p, 1 \end{cases}$$

and

$$\chi(\alpha) = \begin{cases} x, & \text{if } \alpha = 0, n \\ 0, & \text{if } \alpha = m, p, 1 \end{cases}$$

for any $\alpha \in S$. Through verification, we can see that ρ and χ are the second type of L-fuzzy prime ideals.

Lemma 5.1. *Given a bounded semihoop S. Then, $(\alpha \wedge \beta) \boxplus (\alpha \to \beta)^\star \geq \alpha$, $(\alpha \wedge \beta) \boxplus (\beta \to \alpha)^\star \geq \beta$, for any $\alpha, \beta \in S$.*

Proof. Since $(\alpha \odot \beta)^\star = (\alpha \odot \beta) \to 0 = \alpha \to (\beta \to 0) = \alpha \to \beta^\star$, for every $\alpha, \beta \in S$, thus $(\alpha \wedge \beta) \boxplus (\alpha \to \beta)^\star = (\alpha \wedge \beta)^\star \to (\alpha \to \beta)^{\star\star\star} = (\alpha \wedge \beta)^\star \to (\alpha \to \beta)^\star = ((\alpha \wedge \beta)^\star \odot (\alpha \to \beta))^\star = ((\alpha \wedge \beta)^\star \odot (\alpha \to \beta)) \to 0 = ((\alpha \to \beta) \odot (\alpha \wedge \beta)^\star) \to 0 = (\alpha \to \beta) \to ((\alpha \wedge \beta)^\star \to 0) = (\alpha \to \beta) \to (\alpha \wedge \beta)^{\star\star}$. Since $(\alpha \wedge \beta)^{\star\star} \geq \alpha \wedge \beta$, thus $(\alpha \wedge \beta) \boxplus (\alpha \to \beta)^\star = (\alpha \to \beta) \to (\alpha \wedge \beta)^{\star\star} \geq (\alpha \to \beta) \to (\alpha \wedge \beta) = ((\alpha \to \beta) \to \alpha) \wedge ((\alpha \to \beta) \to \beta) \geq \alpha \wedge \alpha = \alpha$, then $(\alpha \wedge \beta) \boxplus (\alpha \to \beta)^\star \geq \alpha$. Similarly, $(\alpha \wedge \beta) \boxplus (\beta \to \alpha)^\star \geq \beta$. □

Proposition 5.1. *Let S be a bounded semihoop. Then, the second type of L-fuzzy prime ideal ρ of S is an L-fuzzy prime ideal.*

Proof. Suppose that ρ is the second type of L-fuzzy prime ideal of S, then $\rho((\alpha \to \beta)^\star) = \rho(0)$ or $\rho((\beta \to \alpha)^\star) = \rho(0)$, for any $\alpha, \beta \in S$. Since ρ is an L-fuzzy strong ideal and by Lemma 5.1, thus $\rho((\alpha \wedge \beta)) \sqcap \rho((\alpha \to \beta)^\star) \leq \rho((\alpha \wedge \beta) \boxplus (\alpha \to \beta)^\star) \leq \rho(\alpha)$, so $\rho((\alpha \wedge \beta)) \sqcap \rho((\alpha \to \beta)^\star) = \rho((\alpha \wedge \beta)) \sqcap \rho(0) = \rho((\alpha \wedge \beta)) \leq \rho(\alpha)$, then $\rho((\alpha \wedge \beta)) \leq \rho(\alpha)$. The same to be, $\rho((\alpha \wedge \beta)) \leq \rho(\beta)$. So $\rho((\alpha \wedge \beta)) \leq \rho(\alpha) \sqcup \rho(\beta)$. Therefore, the conclusion holds. □

Definition 5.2. *A bounded semihoop S is called a bounded prelinearity semihoop if it satisfies $(\alpha \to \beta) \vee (\beta \to \alpha) = 1$, for any $\alpha, \beta \in S$.*

Proposition 5.2. *Let S be a bounded prelinearity semihoop. Then, an L-fuzzy prime ideal ρ of S is the second type of L-fuzzy prime ideal.*

Proof. Since S be a bounded prelinearity semihoop, thus $(\alpha \to \beta) \vee (\beta \to \alpha) = 1$, then $(\alpha \to \beta) = 1$ or $(\beta \to \alpha) = 1$, then $(\alpha \to \beta)^\star = 1^\star = 0$ and $(\beta \to \alpha)^\star = 1^\star = 0$, so $(\alpha \to \beta)^\star \wedge (\beta \to \alpha)^\star = 0$. Since ρ is an L-fuzzy prime ideal of S, thus $\rho(0) = \rho(0 \wedge 0) = \rho((\alpha \to \beta)^\star \wedge (\beta \to \alpha)^\star) \leq \rho((\alpha \to \beta)^\star) \sqcup \rho((\beta \to \alpha)^\star)$, so $\rho(0) \leq \rho((\alpha \to \beta)^\star)$ or $\rho(0) \leq \rho((\beta \to \alpha)^\star)$, by Proposition 3.4(1), we get $\rho(0) \geq \rho((\alpha \to \beta)^\star)$ and $\rho(0) \geq \rho((\beta \to \alpha)^\star)$, then $\rho(0) = \rho((\alpha \to \beta)^\star)$ or $\rho(0) = \rho((\beta \to \alpha)^\star)$. Therefore, the conclusion holds. □

If there exist $\alpha, \beta \in S$such that $(\alpha \to \beta) \vee (\beta \to \alpha) \neq 1$, then an L-fuzzy prime ideal may not be the second type of L-fuzzy prime ideal, by the following example will illustrate.

Example 5.2 ([10])**.** Let $S = \{0, m, n, r, p, q, 1\}$ with $0 < m < n < q < 1$, $0 < r < p < q < 1$ and L be a complete lattice in Example 3.6. Define $\odot$ and $\to$as follows,

$\odot$	0	m	n	r	p	q	1
0	0	0	0	0	0	0	0
m	0	m	m	0	0	m	m
n	0	m	m	0	0	m	n
r	0	0	0	r	r	r	r
p	0	0	0	r	r	r	p
q	0	m	m	r	r	q	q
1	0	m	n	r	p	q	1

$\to$	0	m	n	r	p	q	1
0	1	1	1	1	1	1	1
m	p	1	1	p	p	1	1
n	p	q	1	p	p	1	1
r	n	n	n	1	1	1	1
p	n	n	n	q	1	1	1
q	0	n	n	p	p	1	1
1	0	m	n	r	p	q	1

We can see that $(S, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop but S is not satisfy $(\alpha \to \beta) \vee (\beta \to \alpha) = 1$, for any $\alpha, \beta \in S$since $(r \to n) \vee (n \to r) = n \vee p \neq 1$. We define an L-fuzzy subset by

$$\rho(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0 \\ x, & \text{if } \alpha = m, n \\ y, & \text{if } \alpha = r, p \\ 0, & \text{if } \alpha = q, 1 \end{cases}$$

for all $\alpha \in S$. Then, ρ is an L-fuzzy prime ideal but it is not the second type of L-fuzzy prime since $\rho((m \to r)^\star) = \rho(p^\star) = \rho(n) = x \neq 1 = \rho(0)$, $\rho((r \to m)^\star) = \rho(n^\star) = \rho(p) = y \neq 1 = \rho(0)$.

In a bounded semihoop S, we denote that $FD(S)$ is the L-fuzzy strong ideal set of S. A partial order relation $\preceq$ is defined by $\rho \preceq \chi$if $\rho(\alpha) \leq \chi(\alpha)$, for all $\alpha \in S$, $\rho, \chi \in FD(S)$.

Proposition 5.3. *Assume that S is a bounded semihoop and ρ, χ are L-fuzzy strong ideals of S and satisfying $\rho \preceq \chi$ and $\rho(0) = \chi(0)$. If ρ is the second type of L-fuzzy prime ideal of S, then χ is also the second type of L-fuzzy prime ideal.*

Proof. Let ρ be the second type of L-fuzzy prime ideal, then $\rho((\alpha \to \beta)^\star) = \rho(0)$ or $\rho((\beta \to \alpha)^\star) = \rho(0)$, for any $\alpha, \beta \in S$. So $\chi(0) = \rho(0) \leq \rho((\alpha \to \beta)^\star) \leq \chi((\alpha \to \beta)^\star)$ or $\chi(0) = \rho(0) \leq \rho((\beta \to \alpha)^\star) \leq \chi((\beta \to \alpha)^\star)$, then $\chi(0) \leq \chi((\alpha \to \beta)^\star)$ or $\chi(0) \leq \chi((\beta \to \alpha)^\star)$. Since χ is an L-fuzzy strong ideal, thus $\chi(0) \geq \chi((\alpha \to \beta)^\star)$ or $\chi(0) \geq \chi((\beta \to \alpha)^\star)$, then $\chi(0) = \chi((\alpha \to \beta)^\star)$ or $\chi(0) = \chi((\beta \to \alpha)^\star)$. Therefore, χ is the second type of L-fuzzy prime ideal. □

Proposition 5.4. *Assume that S is a bounded semihoop and ρ is the second type of L-fuzzy prime ideal of S. If $w < \rho(0)$, for any $w \in L$, then $\rho^\square$ is the second type of L-fuzzy prime ideal.*

Proof. By Proposition 4.5, we get that $\rho^\square$ is an L-fuzzy strong ideal. Since $\rho(\alpha) \leq \rho(\alpha) \sqcup w = \rho^\square(\alpha)$, for all $\alpha \in S$, thus $\rho \preceq \rho^\square$. Since $\rho(0) = \rho(0) \sqcup w = \rho^\square(0)$. Therefore, by Proposition 5.3, $\rho^\square$ is the second type of L-fuzzy prime ideal. □

6. L-fuzzy maximal ideals

Definition 6.1. *Let S be a bounded semihoop. A proper L-fuzzy strong ideal ρ of S is called an L-fuzzy maximal ideal if ρ_t^1 is non-trivial implies ρ_t^1 is a maximal ideal, for any $t \in L$.*

Example 6.1 ([10])**.** Let $S = \{0, m, n, a, p, q, 1\}$ with $0 < m < n < 1$, $0 < a < p < q < 1$, where n and a are incomparable. Define $\odot$ and $\to$as bellow:

$\odot$	0	m	n	a	p	q	1
0	0	0	0	0	0	0	0
m	0	m	m	m	m	m	m
n	0	m	m	m	m	m	n
a	0	m	m	a	a	a	a
p	0	m	m	a	a	a	p
q	0	m	m	a	a	q	q
1	0	m	n	a	p	q	1

$\to$	0	m	n	a	p	q	1
0	1	1	1	1	1	1	1
m	0	1	1	1	1	1	1
n	0	q	1	q	1	1	1
a	0	n	n	1	1	1	1
p	0	n	n	q	1	1	1
q	0	n	n	p	p	1	1
1	0	m	n	a	p	q	1

Then, $(S, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop. Let $L = \{0, x, y, 1\}$ be a complete lattice with $0 < x < y < 1$. We define an L-fuzzy subset ρ by

$$\rho(\alpha) = \begin{cases} x, & \text{if } \alpha = 0, \\ 0, & \text{if } \alpha \neq 0, \end{cases}$$

for any $\alpha \in S$. Then, ρ is an L-fuzzy maximal ideal of S.

Proposition 6.1. *Assume that S is a bounded semihoop and ρ is an L-fuzzy maximal ideal of S. If $\rho(\alpha) < \rho(\beta)$ and $\rho^1_{\rho(\beta)} \neq \rho^1_{\rho(\alpha)}$, then $\rho^1_{\rho(\beta)} = \{0\}$ or $\rho^1_{\rho(\alpha)} = S$, for any $\alpha, \beta \in S$.*

Proof. Since $\rho(\alpha) < \rho(\beta)$, thus $\rho^1_{\rho(\beta)} \subset \rho^1_{\rho(\alpha)}$, for any $\alpha, \beta \in S$. If $\rho^1_{\rho(\beta)} \neq \{0\}$, since ρ is an L-fuzzy maximal ideal, then $\rho^1_{\rho(\beta)}$ is a maximal ideal, so $\rho^1_{\rho(\alpha)} = S$. Therefore, $\rho^1_{\rho(\beta)} = \{0\}$ or $\rho^1_{\rho(\alpha)} = S$, for any $\alpha, \beta \in S$. □

Proposition 6.2. *Let S be a bounded semihoop, L be a complete lattice and $\rho : S \to L$ is a non-constant L-fuzzy strong ideal of S. Then, the following statements are equivalent:*

(1) *ρ is an L-fuzzy maximal ideal of S;*

(2) *$\rho^1_{\rho(0)}$ is a maximal ideal of S;*

(3)

$$\rho(\alpha) = \begin{cases} \rho(0), & \text{if } \alpha \in \rho^1_{\rho(0)} \\ \rho(\alpha_1), & \text{if } \alpha \notin \rho^1_{\rho(0)} \end{cases}$$

for some $\alpha_1 \in S$ with $\rho(\alpha_1) < \rho(0)$.

Proof. (1) ⇒ (2) Let ρ is an L-fuzzy maximal ideal of S. By Proposition 3.4(1), we have $\rho(0) \geq \rho(\alpha)$, for any $\alpha \in S$. Since ρ is not constant, thus there exists $\alpha_1 \neq \rho(0)$, so $\rho(\alpha_1) < \rho(0)$, then $\alpha_1 \notin \rho^1_{\rho(0)}$. Since $0 \in \rho^1_{\rho(0)}$, thus $\rho^1_{\rho(0)} \neq \emptyset$ and $\rho^1_{\rho(0)} \neq S$, so $\rho^1_{\rho(0)}$ is a maximal ideal of S.

(2) ⇒ (3) Let $\rho^1_{\rho(0)}$ be a maximal ideal of S. Since ρ is an L-fuzzy strong ideal of S, for any $\alpha \in \rho^1_{\rho(0)}$, we have $\rho(\alpha) \geq \rho(0)$ and $\rho(\alpha) \leq \rho(0)$by Proposition 3.4(1), then $\rho(\alpha) = \rho(0)$. Since ρ is not constant, thus there is $\alpha_1 \in S$such that $\rho(\alpha_1) \neq \rho(0)$, so $\rho(\alpha_1) < \rho(0)$. Suppose that there exists $\alpha_2 \in S$such that $\rho(\alpha_2) \neq \rho(0)$ and $\rho(\alpha_2) \neq \rho(\alpha_1)$. We will discuss the following cases:

(i) If $\rho(\alpha_1) < \rho(\alpha_2) < \rho(0)$ or $\rho(\alpha_2) < \rho(\alpha_1) < \rho(0)$, then $\rho^1_{\rho(0)} \subset \rho^1_{\rho(\alpha_2)} \subset \rho^1_{\rho(\alpha_1)}$ or $\rho^1_{\rho(0)} \subset \rho^1_{\rho(\alpha_1)} \subset \rho^1_{\rho(\alpha_2)}$. From Proposition 3.3(1), $\rho^1_{\rho(\alpha_1)}$ and $\rho^1_{\rho(\alpha_2)}$ are ideals, which contradicts $\rho^1_{\rho(0)}$ be a maximal ideal of S.

(ii) If $\rho(\alpha_1)$ and $\rho(\alpha_2) < \rho(0)$ are incomparable, then $\alpha_1 \notin \rho^1_{\rho(\alpha_2)}$ and $\alpha_2 \notin \rho^1_{\rho(\alpha_1)}$, so $\rho^1_{\rho(0)} \subset \rho^1_{\rho(\alpha_1)} \subset S$ and $\rho^1_{\rho(0)} \subset \rho^1_{\rho(\alpha_2)} \subset S$, which contradicts $\rho^1_{\rho(0)}$ be a maximal ideal of S.

Then, $\rho(\alpha_2) = \rho(0)$ or $\rho(\alpha_2) = \rho(\alpha_1)$. Therefore, the conclusion holds.

(3) ⇒ (1) Suppose

$$\rho(\alpha) = \begin{cases} \rho(0), & \text{if } \alpha \in \rho^1_{\rho(0)} \\ \rho(\alpha_1), & \text{if } \alpha \notin \rho^1_{\rho(0)} \end{cases}$$

for some $\alpha_1 \in S$ with $\rho(\alpha_1) < \rho(0)$. Then, $\rho^1_t \in \{\rho^1_{\rho(0)}, S, \emptyset\}$, for any $t \in L$, so ρ is an L-fuzzy maximal ideal of S. □

Corollary 6.1. *Let S be a bounded semihoop and $\rho : S \to [0,1]$ be a fuzzy maximal ideal of S. Then, ρ has exactly two values.*

Proposition 6.3. *Let S be a bounded $\vee$-semihoop with DNP and ρ be an L-fuzzy strong ideal on S. If ρ is an L-fuzzy maximal ideal of S, then ρ is an L-fuzzy prime ideal of S.*

Proof. Let ρ be an L-fuzzy maximal ideal of S. Then, for any $t \in L$such that ρ_t^1 is non-trivial implies ρ_t^1 is a maximal ideal of S. From Proposition 2.5, so ρ_t^1 is a prime ideal of S. Therefore, from Proposition 4.4(1), we have that ρ is an L-fuzzy prime ideal of S. □

Proposition 6.4. *Suppose that S is a bounded semihoop and ρ is a proper L-fuzzy strong ideal of S. If s and t are incomparable, for any $s, t \in L$, then the following conditions hold:*

(1) *ρ_s^1 and ρ_t^1 are proper ideals;*

(2) *if ρ is an L-fuzzy maximal ideal, then ρ_s^1 and ρ_t^1 are maximal ideals.*

Proof. (1) Let $\alpha, \beta \in S$such that $\rho(\alpha) = s$ and $\rho(\beta) = t$. From Proposition 3.3(1), we have ρ_s^1 and ρ_t^1 are two ideals. Since $\rho(\alpha) = s$ and $\rho(\beta) = t$ are incomparable, thus $\rho(\alpha) \neq \rho(0)$ and $\rho(\beta) \neq \rho(0)$, so $\{0\} \subset \rho_s^1$ and $\{0\} \subset \rho_t^1$. Moreover, $\alpha \notin \rho_t^1$ and $\beta \notin \rho_s^1$, then $\rho_s^1 \subset S$ and $\rho_t^1 \subset S$. Therefore, ρ_s^1 and ρ_t^1 are proper ideals.

(2) Let ρ be an L-fuzzy maximal ideal of S. From (1), ρ_s^1 and ρ_t^1 are proper ideals, so ρ_s^1 and ρ_t^1 are non-trivial. Thus, ρ_s^1 and ρ_t^1 are maximal ideals by Definition 6.1. □

7. Conclusion

In this paper, we associate bounded semihoops with lattice structures and establish L-fuzzy ideals theory on bounded semihoop. In particular, we obtain several important conclusions. (1)Let S be a bounded semihoop and L be a complete lattice. Then, each L-fuzzy strong ideal is an L-fuzzy ideal but an L-fuzzy ideal may not be an L-fuzzy strong ideal. (2)Let S be a bounded semihoop, L be a complete lattice and $\rho : S \to L$ be an L-fuzzy set of S. (i)If ρ is an L-fuzzy strong ideal(filter) of S, then $\rho^\star$ is an L-fuzzy strong filter(ideal). (ii)If ρ is an L-fuzzy ideal(filter) of S, then $\rho^\star$ is an L-fuzzy filter(ideal), where $\rho^\star(\alpha) = \rho(\alpha^\star)$, for any $\alpha \in S$. (3)We establish equivalence descriptions between L-fuzzy strong ideals and ideals using four types of level sets. (4)Let S be a bounded semihoop, ρ be an L-fuzzy strong ideal of S and H be an up-set sublattice of L. Then, $\rho^{-1}(H)$ is an ideal of S. (5)Let S be a bounded semihoop. Then, each the second type of L-fuzzy prime ideal is an L-fuzzy prime ideal but an L-fuzzy prime ideal may not be the second type of L-fuzzy prime ideal unless S is a bounded prelinearity semihoop. (6)Let S be a bounded semihoop and $\rho : S \to [0, 1]$ be an L-fuzzy maximal ideal of S. Then, ρ has exactly two values. (7) Let S be a bounded $\vee$-semihoop with DNP and ρ be an L-fuzzy strong ideal on S. If ρ is an L-fuzzy maximal ideal of S, then ρ is an L-fuzzy prime ideal of S.

Since semihoops are the fundamental residuated structures, these properties and conclusions in this article can be applied to other residuated structures.

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