Degree sum exponent distance energy of non-commuting graph for dihedral groups

Mamika Ujianita Romdhini*
Department of Mathematics
Faculty of Mathematics and Natural Science
Universitas Mataram
Mataram 83125
Indonesia
mamika@unram.ac.id

Athirah Nawawi
Department of Mathematics and Statistics
Faculty of Science
Universiti Putra Malaysia
43400 Serdang, Selangor
Malaysia
athirah@upm.edu.my

Abstract. The non-commuting graph is defined on a finite group G, denoted by Γ_G, with $G\backslash Z(G)$ is the vertex set of Γ_G and $v_p \neq v_q \in G\backslash Z(G)$ are adjacent whenever they do not commute in G. In this paper, we focus on Γ_G for dihedral groups of order $2n$, D_{2n}, where $n \geq 3$. We show the spectrum, spectral radius and energy of the graph corresponding to the degree sum exponent distance matrix and analyze the hyperenergetic property. Moreover, we then present the correlation between the obtained energy and the adjacency energy.

Keywords: non-commuting graph, the energy of a graph, dihedral group, degree sum exponent distance matrix.

1. Introduction

Let G be a group and $Z(G)$ be a center of G. The non-commuting graph of G, denoted by Γ_G, has vertex set $G\backslash Z(G)$ and two distinct vertices v_p, v_q in Γ_G are connected by an edge whenever $v_p v_q \neq v_q v_p$ ([1]).

The non-commuting graphs have been studied by many authors for various kinds of groups. Abdollahi et al. [1] discussed Γ_G for a non-abelian group G and stated that it is always connected with diameter 2. Consequently, the distance between two vertices in Γ_G is well defined, and it is the length of the shortest path between v_p and v_q. Moreover, this discussion continues by examining the isomorphic properties of two non-commuting graphs related to the isomorphic properties of the corresponding groups. Darafsheh [6] proved the conjecture

*. Corresponding author

that two non-commuting graphs which are isomorphic imply that the groups are also isomorphic as well. Likewise, Abdollahi and Shahverdi [2] stated that if Γ_G is isomorphic to Γ_G of the alternating group A_n, then $G \cong A_n$. Besides, they presented this conjecture as verified for Γ_G with the simple groups of Lie type.

Afterward, Tolue et al. [28] extended the study of Γ_G and introduced the new concept of $g-$non-commuting graph of finite groups that involve the commutator between two members of the group. If two groups are isoclinic and the numbers of their center are the same, then their associated $g-$non-commuting graphs are isomorphic. Moreover, Khasraw, et al. [15] presented the mean distance of Γ_G for the dihedral groups.

Moreover, Γ_G on n vertices can be interpreted with the adjacency matrix of Γ_G. It is $A(\Gamma_G) = [a_{pq}]$ of size $n \times n$ whose entries $a_{pq} = 1$ for adjacent v_p and v_q; otherwise, $a_{pq} = 0$. For the identity matrix of order n, I_n, the characteristic polynomial of Γ_G is defined as $P_{A(\Gamma_G)}(\lambda) = det\left(\lambda I_n - A(\Gamma_G)\right)$, and its roots are $\lambda_1, \lambda_2, \ldots, \lambda_n$ as the eigenvalues of Γ_G. The spectrum of Γ_G is $Spec(\Gamma_G) = \left\{\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_m^{k_m}\right\}$, with $k_1, k_2, \ldots, k_m$ are the respective multiplicities of $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Energy of Γ_G is calculated by adding all the absolute values of $\lambda_1, \lambda_2, \ldots, \lambda_n$. Gutman [10] pioneered this definition in 1978. The graph energy on n vertices with a value more than $E_A(K_n)$ can be stated as hyperenergetic, or it can be said that $E(\Gamma_G) > 2(n-1)$ [16]. In addition, the adjacency energy bounds of the graph can be found at [7] and graphs with self-loops can be seen at [11]. Additionally, Sun et al. have demonstrated that the clique path has the maximum distance of eigenvalues and energy in their work [27]. It has been shown that the adjacency energy is not equal to an odd integer [4] and is never equal to its square root [18].

In 2008, Indulal et al. [12] introduced the graph matrix whose entries depend on the distance between two vertices. They showed the distance energy of graphs. For the degree product distance energy, the readers can refer [13]. Moreover, the discussion of the degree sum exponent distance of graphs can be found in [14].

In this work, the set of vertex for Γ_G is the non-abelian dihedral group of order $2n$, D_{2n} where $n \geq 3$ which denoted by $D_{2n} = \left\langle a, b : a^n = b^2 = e, bab = a^{-1}\right\rangle$ [3]. The center of D_{2n} and the centralizer of v, where $v \in D_{2n}$ are denoted by $Z\left(D_{2n}\right)$ and $C_{D_{2n}}(v)$, respectively. Therefore, we have

$$Z\left(D_{2n}\right) = \begin{cases} \{e\}, & \text{if } n \text{ is odd} \\ \left\{e, a^{\frac{n}{2}}\right\}, & \text{if } n \text{ is even,} \end{cases} \quad C_{D_{2n}}(a^i) = \{a^j : 1 \leq j \leq n\}, \text{ and}$$

$$C_{D_{2n}}(a^i b) = \begin{cases} \{e, a^i b\}, & \text{if } n \text{ is odd} \\ \left\{e, a^{\frac{n}{2}}, a^i b, a^{\frac{n}{2}+i} b\right\}, & \text{if } n \text{ is even.} \end{cases}$$

Energy studies of the commuting and non-commuting graphs involving D_{2n} as the set of vertex have been carried out by several authors. Romdhini and Nawawi [21, 22] and Romdhini et al. [23] formulated the energy of Γ_G by considering the eigenvalues of the degree sum, degree subtraction, and neighbors degree sum matrices, meanwhile, [17] presented the adjacency energy. The degree exponent sum, maximum and minimum degree energies were shown in [24, 25].

In studies of correlations between molecules containing heteroatoms and their total electron energy, Gowtham and Swamy [9] reports a correlation coefficient of 0.952 between Sombor energy values and total electron energy. The authors of Redzepovic and Gutman [20] also developed a numerical approach to compare a graph's Sombor energy with its adjacency energy, and it remains an open problem for mathematical verification. Based on these two papers, the authors take the initiative to apply it to Γ_G. Then, this paper is dedicated to formulating the energy based on the degree sum exponent distance matrix $DSED$ for Γ_G on D_{2n} and comparing the results obtained and the adjacency energy.

2. Preliminaries

In this part, we begin with the definition of $DSED$-matrix. Suppose that d_{pq} is the distance between vertex v_p and v_q in Γ_G and d_{v_p} is the degree of vertex v_p.

Definition 2.1 ([14]). *The degree sum exponent distance matrix of Γ_G is an $n \times n$ matrix $DSED(\Gamma_G) = [dsed_{pq}]$ whose (p, q)-th entry is*

$$dsed_{pq} = \begin{cases} \left(d_{v_p} + d_{v_q}\right)^{d_{pq}}, & \text{if } v_p \neq v_q \\ 0, & \text{if } v_p = v_q. \end{cases}$$

The $DSED-$energy of Γ_G is given by

$$E_{DSED}(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i|,$$

with $\lambda_1, \lambda_2, \ldots, \lambda_n$ represent the eigenvalues (not necessarily distinct) of $DSED(\Gamma_G)$.

The degree sum exponent distance spectral radius of Γ_G is

$$(1) \qquad \rho_{DSED}(\Gamma_G) = max\{|\lambda| : \lambda \in Spec(\Gamma_G)\}.$$

From the fact that Γ_G has $2n-1$ and $2n-2$ vertices for odd and even n, respectively, then Γ_G can be classified as hyperenergetic whenever the $DSED-$energy fulfil the following terms:

$$(2) \qquad E_{DSED}(\Gamma_G) > \begin{cases} 4(n-1), & \text{for odd } n \\ 4(n-1)-2, & \text{for even } n, \end{cases}.$$

We now supply some previous results in support of the theorems derived in Section 3. Obtaining the graph energy requires formulating the characteristic polynomial of Γ_G. Here is an essential result that assists in formulating the characteristic polynomial of Γ_G.

Theorem 2.1 ([8]). *If* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *is a square matrix with four block matrices and* $|A| \neq 0$*, then*

$$|M| = \begin{vmatrix} A & B \\ O & D - CA^{-1}B \end{vmatrix} = |A| \left| D - CA^{-1}B \right|.$$

Lemma 2.1 ([5]). *If* K_n *is the complete graph on* n *vertices, then its adjacency matrix is* $(J-I)_n$*, and the spectrum is* $\{(n-1)^{(1)}, (-1)^{(n-1)}\}$.

This article concerned on D_{2n} of order $2n$, D_{2n}, where $n \geq 3$. Let $G_1 = \{a^i : 1 \leq i \leq n\} \backslash Z(D_{2n})$ and $G_2 = \{a^i b : 1 \leq i \leq n\}$. Now, the degree of every vertex of Γ_G for $G = G_1 \cup G_2$ is determined as follows:

Theorem 2.2 ([15]). *Let* Γ_G *be the non-commuting graph on* G*, where* $G = G_1 \cup G_2$*. Then*

1. $d_{a^i} = n$*, and*
2. $d_{a^i b} = \begin{cases} 2(n-1), & \text{if } n \text{ is odd} \\ 2(n-2), & \text{if } n \text{ is even.} \end{cases}$

Thus, we can see the isomorphism between Γ_G and some common graph types in the theorem as given below:

Theorem 2.3 ([15]). *Let* Γ_G *be a non-commuting graph for* G.

1. *If* $G = G_1$*, then* $\Gamma_G \cong \bar{K}_s$*, for* $s = |G_1|$.
2. *If* $G = G_2$*, then* $\Gamma_G \cong \begin{cases} K_n, & \text{if } n \text{ is odd} \\ K_n - \frac{n}{2}K_2, & \text{if } n \text{ is even.} \end{cases}$,

where $\frac{n}{2}K_2$ *denotes* $\frac{n}{2}$ *copies of* K_2.

In order to compare the $DSED$ and adjacency energies of Γ_G for D_{2n}, here we write the adjacency energy from Mahmoud et al. [17] as given below:

Theorem 2.4 ([17]). *The adjacency energy of* Γ_G*, where* $G = G_1 \cup G_2$*,* $E_A(\Gamma_G)$ *is*

1. *for odd* n*,* $E_A(\Gamma_G) = (n-1) + \sqrt{5n^2 - 6n + 1}$*, and*
2. *for even* n*,* $E_A(\Gamma_G) = \begin{cases} 8, & \text{if } n = 4 \\ (n-2) + \sqrt{5n^2 - 12n + 4}, & \text{if } n > 4 \end{cases}$.

To define the elements of $DSED-$matrix, we need to determine the distance for every pair of vertices in Γ_G, for $G = G_1 \cup G_2$. The discussion is in Theorem 2.5 below:

Theorem 2.5 ([26]). *For two distinct vertices v_p, v_q in Γ_G, where $G = G_1 \cup G_2$, the distance between v_p and v_q is*

1. *for the odd n,* $d_{pq} = \begin{cases} 2, & \text{if } v_p, v_q \in G_1 \\ 1, & \text{otherwise,} \end{cases}$ *and*

2. *for the even n,* $d_{pq} = \begin{cases} 2, & \text{if } (v_p, v_q \in G_1) \text{ or } \left(v_p \in G_2, v_q \in \left\{a^{\frac{n}{2}+i}b\right\},\right. \\ & \left.\text{or vice versa}\right) \\ 1, & \text{otherwise.} \end{cases}$.

3. Characteristic polynomial of some matrices

Several properties need to be performed in order to provide $DSED-$energy of Γ_G, for $G = G_1 \cup G_2$ in Section 4. In this section, we derive three theorems of the solution of the determinant of a particular matrix.

Lemma 3.1 ([19]). *If $a, b, c,$ and d are real numbers, and J_n is an $n \times n$ matrix whose all entries are equal to one, then the determinant of*

$$\begin{vmatrix} (\lambda + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\lambda + b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

can be simplified as

$$(\lambda + a)^{n_1 - 1}(\lambda + b)^{n_2 - 1}\left((\lambda - (n_1 - 1)\,a)\,(\lambda - (n_2 - 1)\,b) - n_1 n_2 cd\right),$$

where $1 \leq n_1, n_2 \leq n$ and $n_1 + n_2 = n$.

Theorem 3.1. *For real numbers a, b, the characteristic polynomial of an $n \times n$ matrix*

$$M = \begin{bmatrix} a & b & \dots & b \\ b & a & \dots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \dots & a \end{bmatrix}$$

can be simplified as

$$P_M(\lambda) = (\lambda - a - (n-1)b)(\lambda - a + b)^{n-1}.$$

Proof. Let a, b are real numbers and M is a square matrix of order n as

$$M = \left[(a - b)I_n + bJ_n\right].$$

Then, we get the characteristic polynomial of M as

$$P_M(\lambda) = |\lambda I_n - M| = |(\lambda - a + b)I_n - bJ_n|\,. \tag{3}$$

The first step, we apply $R_i' = R_i - R_1$, for $2 \leq i \leq n$. Consequently, Equation 3 is as the following:

$$P_M(\lambda) = \begin{vmatrix} \lambda - a & -bJ_{1\times(n-1)} \\ -(\lambda - a + b)J_{(n-1)\times 1} & (\lambda - a + b)I_{(n-1)} \end{vmatrix}. \tag{4}$$

The next step is replacing C_1 by $C_1' = C_1 + C_2 + C_3 + \ldots + C_n$, then Equation 4 can be written as

$$P_M(\lambda) = \begin{vmatrix} \lambda - a - (n-1)b & -bJ_{1\times(n-1)} \\ 0_{(n-1)\times 1} & (\lambda - a + b)I_{(n-1)} \end{vmatrix}. \tag{5}$$

It is obvious from Equation 5, $P_M(\lambda)$ is an upper triangle matrix. Thus, it can be simplified as given below:

$$P_M(\lambda) = (\lambda - a - (n-1)b)(\lambda - a + b)^{n-1},$$

and we complete the proof. □

Theorem 3.2. *For real numbers a, b, the characteristic polynomial of an $n \times n$ matrix*

$$M = \begin{bmatrix} U & V \\ V & U \end{bmatrix},$$

where $U = [b(J-I)_{\frac{n}{2}}]$ and $V = [b(J-I)_{\frac{n}{2}} + aI_{\frac{n}{2}}]$, can be simplified as

$$P_M(\lambda) = (\lambda - a + 2b)^{\frac{n}{2}-1}\left(\lambda - a - (n-2)\,b\right)(\lambda + a)^{\frac{n}{2}}.$$

Proof. For real numbers s, t, suppose that M is an $n \times n$ matrix

$$M = \begin{bmatrix} U & V \\ V & U \end{bmatrix} = \begin{bmatrix} 0 & \ldots & b & a & \ldots & b \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b & \ldots & 0 & b & \ldots & a \\ a & \ldots & b & 0 & \ldots & b \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b & \ldots & a & b & \ldots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} b(J-I)_{\frac{n}{2}} & b(J-I)_{\frac{n}{2}} + aI_{\frac{n}{2}} \\ b(J-I)_{\frac{n}{2}} + aI_{\frac{n}{2}} & b(J-I)_{\frac{n}{2}} \end{bmatrix}.$$

Then, equation $P_M(\lambda) = |\lambda I_n - M|$ can be written as follows:

$$P_M(\lambda) = \begin{vmatrix} (\lambda + b)I_{\frac{n}{2}} - bJ_{\frac{n}{2}} & -aI_{\frac{n}{2}} - b(J-I)_{\frac{n}{2}} \\ -bI_{\frac{n}{2}} - b(J-I)_{\frac{n}{2}} & (\lambda + b)I_{\frac{n}{2}} - bJ_{\frac{n}{2}} \end{vmatrix}. \tag{6}$$

To solve the determinant in Equation 6, it is necessary to perform row and column operations. The first step is replacing $R_{\frac{n}{2}+i}$ by $R'_{\frac{n}{2}+i} = R_{\frac{n}{2}+i} - R_i$, where $1 \leq i \leq \frac{n}{2}$. Consequently, Equation 6 is as the following:

$$P_M(\lambda) = \begin{vmatrix} (\lambda+b)I_{\frac{n}{2}} - bJ_{\frac{n}{2}} & -aI_{\frac{n}{2}} - b(J-I)_{\frac{n}{2}} \\ -(\lambda+a)I_{\frac{n}{2}} & (\lambda+a)I_{\frac{n}{2}} \end{vmatrix}. \tag{7}$$

Next, the second step is replacing C_i by $C'_i = C_i + C_{\frac{n}{2}+i}$, where $1 \leq i \leq \frac{n}{2}$. Hence, Equation 7 can be written as follows:

$$P_M(\lambda) = \begin{vmatrix} (\lambda-a+2b)I_{\frac{n}{2}} - 2bJ_{\frac{n}{2}} & -aI_{\frac{n}{2}} - b(J-I)_{\frac{n}{2}} \\ 0_{\frac{n}{2}} & (\lambda+a)I_{\frac{n}{2}} \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}. \tag{8}$$

Bearing in mind Theorem 2.1 and since $C = 0$, it implies Equation 8 can be simplified to

$$P_M(\lambda) = |A|\,|D|\,. \tag{9}$$

We first consider $|A|$ using Theorem 3.1 as follows:

$$|A| = (\lambda - a + 2b)^{\frac{n}{2}-1}\,(\lambda - a - (n-2)\,b)\,. \tag{10}$$

Meanwhile, as a result of D as a diagonal matrix, as a consequence, we derive:

$$|D| = (\lambda + a)^{\frac{n}{2}}. \tag{11}$$

Therefore, by substituting Equations 10 and 11 to Equation 9, we obtain

$$P_M(\lambda) = (\lambda - a + 2b)^{\frac{n}{2}-1}\,(\lambda - a - (n-2)\,b)\,(\lambda + a)^{\frac{n}{2}}. \qquad \square$$

Theorem 3.3. *For real numbers a, b, c, d, the characteristic polynomial of a $(2n-2) \times (2n-2)$ matrix:*

$$M = \begin{bmatrix} a(J-I)_{n-2} & cJ_{(n-2)\times\frac{n}{2}} & cJ_{(n-2)\times\frac{n}{2}} \\ cJ_{\frac{n}{2}\times(n-2)} & d(J-I)_{\frac{n}{2}} & d(J-I)_{\frac{n}{2}} + bI_{\frac{n}{2}} \\ cJ_{\frac{n}{2}\times(n-2)} & d(J-I)_{\frac{n}{2}} + bI_{\frac{n}{2}} & d(J-I)_{\frac{n}{2}} \end{bmatrix},$$

can be simplified as

$$P_M(\lambda) = (\lambda + a)^{n-3}\,(\lambda - b + 2d)^{\frac{n}{2}-1}\,(\lambda + b)^{\frac{n}{2}}$$
$$\left(\lambda^2 - (b + (n-2)d + a(n-3))\,\lambda + a(n-3)\,(b + (n-2)d) - n(n-2)c^2\right).$$

4. Degree sum exponent distance energy of non-commuting graph for dihedral groups

This section will present the results of non-commuting graph energy for D_{2n}, using the corresponding $DSED$-matrix. Since for $n = 1$ and $n = 2$, D_{2n} is abelian, then strictly it is for $n \geq 3$. The following is an example of Γ_G for D_{2n}, where $n = 4$.

Example 4.1. Let $D_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ and $Z(D_8) = \{e, a^2\}$, where $C_{D_8}(a^i) = \{e, a, a^2, a^3\}$, $C_{D_8}(b) = \{e, a^2, b, a^2b\} = C_{D_8}(a^2b)$, $C_{D_8}(ab) = \{e, a^2, ab, a^3b\} = C_{D_8}(a^3b)$. For $G = D_8 \backslash Z(D_8)$, according to each element's centralizer in G, as a consequence, Γ_G is presented in Figure 1.

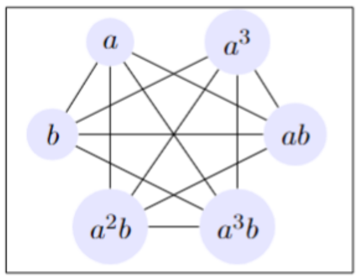

Figure 1: Non-commuting graph for D_8

The vertex degree of a and a^3 is four. Similarly, for $1 \leq i \leq 4$, and the degree of b, ab, a^2b, and a^3b is also four. The distance between a and b, between a^2b and a^3b, and between a^3 and ab are found to be equal, i.e. equal to one, otherwise it is two.

In the next theorem, we derive $DSED-$energy of Γ_G in terms of $G = G_1$ and $G = G_2$.

Theorem 4.1. *Let Γ_G be the non-commuting graph on G.*

1. *If $G = G_1$, then $E_{DSED}(\Gamma_G)$ is undefined, and*
2. *If $G = G_2$, then* $E_{DSED}(\Gamma_G) = \begin{cases} 4(n-1)^2, & \text{if } n \text{ is odd} \\ 4n(n-2)^2, & \text{if } n \text{ is even.} \end{cases}$.

Proof. 1. For $G = G_1$ case, by Theorem 2.3, $\Gamma_G \cong \bar{K}_m$, where $m = |G_1|$. Then, Γ_G consists of m isolated vertices which implies the distance of every pair vertices of G_1 is undefined.

2. For the second case when $G = G_2$, we first proceed for odd n. Again, by Theorem 2.3, $\Gamma_G \cong K_n$. Then, for every v_p of Γ_G, $d_{v_p} = (n-1)$ and every pair of vertices are at distance 1. Now, the $DSED-$matrix of Γ_G is $DSED(\Gamma_G) =$

$dsed_{pq}$, with $(p,q)-$entry if $v_p \neq v_q$ is $((n-1)+(n-1))^1 = 2(n-1)$, and zero if $v_p = v_q$. Hence,

$$\begin{aligned} DSED(\Gamma_G) &= \begin{bmatrix} 0 & 2(n-1) & 2(n-1) & \dots & 2(n-1) \\ 2(n-1) & 0 & 2(n-1) & \dots & 2(n-1) \\ 2(n-1) & 2(n-1) & 0 & \dots & 2(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(n-1) & 2(n-1) & 2(n-1) & \dots & 0 \end{bmatrix} \\ &= 2(n-1)A(K_n). \end{aligned}$$

In other words, $DSED(\Gamma_G)$ is the product of $2(n-1)$ and $A(K_n)$. Therefore, from Lemma 2.1, the $DSED-$energy of Γ_G is $2(n-1).2(n-1) = 4(n-1)^2$.

Meanwhile for the even n, by Theorem 2.3, $\Gamma_G \cong K_n - \frac{n}{2}K_2$, then every vertex has degree $(n-2)$ and the distance between every pair a^ib and $a^{\frac{n}{2}+i}$ for all $1 \leq i \leq n$ is 2, and 1, otherwise. Thus, $DSED(\Gamma_G) = dsed_{pq}$ and for $v_p \neq v_q$,

$$dsed_{ij} = \begin{cases} 4(n-2)^2, & \text{if } v_p = a^ib,\ v_q = a^{\frac{n}{2}+i}b,\ 1 \leq i \leq n \\ 2(n-2), & \text{if } v_p = a^ib,\ v_q \neq a^{\frac{n}{2}+i}b,\ 1 \leq i \leq n\ . \\ 0, & otherwise. \end{cases}$$

Now, we can construct $DSED(\Gamma_G)$ as follows:

$$DSED(\Gamma_G) = \begin{bmatrix} 0 & \dots & 2(n-2) & 4(n-2)^2 & \dots & 2(n-2) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(n-2) & \dots & 0 & 2(n-2) & \dots & 4(n-2)^2 \\ 4(n-2)^2 & \dots & 2(n-2) & 0 & \dots & 2(n-2) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(n-2) & \dots & 4(n-2)^2 & 2(n-2) & \dots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2(n-2)(J-I)_{\frac{n}{2}} & 2(n-2)(J-I)_{\frac{n}{2}} + 4(n-2)^2 I_{\frac{n}{2}} \\ 2(n-2)(J-I)_{\frac{n}{2}} + 4(n-2)^2 I_{\frac{n}{2}} & 2(n-2)(J-I)_{\frac{n}{2}} \end{bmatrix}.$$

In this case, we have four block matrices of $DSED(\Gamma_G)$:

$$DSED(\Gamma_G) = \begin{bmatrix} U & V \\ V & U \end{bmatrix}, \tag{12}$$

where U and V are $\frac{n}{2} \times \frac{n}{2}$ matrices. Matrix U consists of zero diagonal entries, otherwise, the entries are $2(n-2)$, while the diagonal entries of V are $4(n-2)^2$ and the non-diagonal entries are $2(n-2)$. By Theorem 3.2 with $a = 4(n-2)^2$ and $b = 2(n-2)$, Equation 12 is

(13)

$$P_{DSED(\Gamma_G)}(\lambda) = \left(\lambda + 4(n-2)^2\right)^{\frac{n}{2}} \left(\lambda - 4(n-2)(n-3)\right)^{\frac{n}{2}-1} \left(\lambda - 6(n-2)^2\right).$$

Therefore, using the roots of Equation 13, the $DSED-$energy of Γ_G is

$$\begin{aligned} E_{DSED}(\Gamma_G) &= \binom{n}{2}\left|-4(n-2)^2\right| + \left(\frac{n}{2}-1\right)\left|4(n-2)(n-3)\right| + \left|6(n-2)^2\right| \\ &= 4n(n-2)^2. \end{aligned}$$

□

Our next proposition will provide us with the characteristic polynomial of Γ_G for $G = G_1 \cup G_2$.

Theorem 4.2. *Let Γ_G be the non-commuting graph on G on $G = G_1 \cup G_2$, where $n \geq 3$. Then, the characteristic polynomial of Γ_G is*

1. *for n is odd:*

$$\begin{aligned} P_{DSED(\Gamma_G)}(\lambda) =& (\lambda + 4n^2)^{n-2}(\lambda + 4(n-1))^{n-1} \\ & ((\lambda - 4n^2(n-2))(\lambda - 4(n-1)^2) - (n-1)n(3n-2)^2), \end{aligned}$$

2. *for n is even:*

$$\begin{aligned} & P_{DSED(\Gamma_G)}(\lambda) = (\lambda+4n^2)^{n-3}(\lambda-8(n-2)(2n-5))^{\frac{n}{2}-1}(\lambda+16(n-2)^2)^{\frac{n}{2}} \\ & (\lambda^2 - (20(n-2)^2+4n^2(n-3))\lambda + 80n^2(n-3)(n-2)^2 - n(n-2)(3n-4)^2). \end{aligned}$$

Proof. 1. Let n is odd, from Theorem 2.2, we have $d_{a^i} = n$ and $d_{a^i b} = 2(n-1)$, for $1 \leq i \leq n$. Following Theorem 2.5, we then obtain the distance of every pair of vertices. Since $Z(D_{2n}) = \{e\}$, then there are $2n-1$ vertices for Γ_G, where $G = G_1 \cup G_2$. The vertex set consists of $n-1$ vertices of a^i, for $i = 1, 2, \ldots, n-1$, and n vertices of $a^i b$, $i = 1, 2, \ldots, n$. Then, from Definition 2.1, $DSED(\Gamma_G)$ is an $(2n-1) \times (2n-1)$ matrix as the following:

$$DSED(\Gamma_G) = \begin{bmatrix} 0 & \ldots & 4n^2 & 3n-2 & \ldots & 3n-2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 4n^2 & \ldots & 0 & 3n-2 & \ldots & 3n-2 \\ 3n-2 & \ldots & 3n-2 & 0 & \ldots & 4(n-1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 3n-2 & \ldots & 3n-2 & 4(n-1) & \ldots & 0 \end{bmatrix}.$$

It can be partitioned into four block matrices:

$$DSED(\Gamma_G) = \begin{bmatrix} 4n^2(J-I)_{n-1} & (3n-2)J_{(n-1)\times n} \\ (3n-2)J_{(n-1)\times n} & 4(n-1)(J-I)_n \end{bmatrix}. \quad (14)$$

Now, the characteristic polynomial of Equation 14 is

$$\begin{aligned} P_{DSED(\Gamma_G)}(\lambda) &= |\lambda I_{2n-1} - DSED(\Gamma_G)| \\ &= \begin{vmatrix} (\lambda + 4n^2)I_{n-1} - 4n^2 J_{n-1} & -(3n-2)J_{(n-1)\times n} \\ -(3n-2)J_{n\times(n-1)} & (\lambda + 4(n-1))I_n - 4(n-1)J_n) \end{vmatrix}. \end{aligned}$$

According to Lemma 3.1, with $a = 4n^2$, $b = 4(n-1)$, $c = d = 3n-2$, and $n_1 = n-1$, $n_2 = n$, then we obtain the formula of $P_{DSED(\Gamma_G)}(\lambda)$, and we obtain the desired outcome.

2. Let us prove the even n case. Based on Theorem 2.2, we know that $d_{(a^i)} = n$ and $d_{(a^ib)} = 2(n-2)$, for all $1 \leq i \leq n$. Since $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$, then there are $2n-2$ vertices in Γ_G. The vertex set contains $n-2$ vertices of a^i, for $1 \leq i < \frac{n}{2}, \frac{n}{2} < i < n$, and n vertices of a^ib, for $1 \leq i \leq n$. Following the result of Theorem 2.5 and by Definition 2.1, then matrix $DSED(\Gamma_G)$ of size $(2n-2) \times (2n-2)$ is as given below:

$$\begin{bmatrix}
0 & \dots & 4n^2 & 3n-4 & \dots & 3n-4 & 3n-4 & \dots & 3n-4 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
4n^2 & \dots & 0 & 3n-4 & \dots & 3n-4 & 3n-4 & \dots & 3n-4 \\
3n-4 & \dots & 3n-4 & 0 & \dots & 4(n-2) & 16(n-2)^2 & \dots & 4(n-2) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
3n-4 & \dots & 3n-4 & 4(n-2) & \dots & 0 & 4(n-2) & \dots & 16(n-2)^2 \\
3n-4 & \dots & 3n-4 & 16(n-2)^2 & \dots & 4(n-2) & 0 & \dots & 4(n-2) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
3n-4 & \dots & 3n-4 & 4(n-2) & \dots & 16(n-2)^2 & 4(n-2) & \dots & 0
\end{bmatrix}.$$

Now, we provide nine block matrices of $DSED(\Gamma_G)$ as follows:

$$\begin{bmatrix}
4n^2(J-I)_{n-2} & (3n-4)J_{(n-2)\times\frac{n}{2}} & (3n-4)J_{(n-2)\times\frac{n}{2}} \\
(3n-4)J_{\frac{n}{2}\times(n-2)} & 4(n-2)(J-I)_{\frac{n}{2}} & 4(n-2)(J-I)_{\frac{n}{2}} + 16(n-2)^2 I_{\frac{n}{2}} \\
(3n-4)J_{\frac{n}{2}\times(n-2)} & 4(n-2)(J-I)_{\frac{n}{2}} + 16(n-2)^2 I_{\frac{n}{2}} & 4(n-2)(J-I)_{\frac{n}{2}}
\end{bmatrix}.$$

By Theorem 3.3 with $r = 4n^2$, $s = 16(n-2)^2$, $t = 3n-4$, $u = 4(n-2)$, we then obtain the required result. □

As a result of Theorem 4.2, we proceed to the two following theorems.

Theorem 4.3. *Let Γ_G be a non-commuting graph on G, where $G = G_1 \cup G_2$, then $DSED-$spectral radius for Γ_G is*

1. *for n is odd:*

$$\rho_{DSED}(\Gamma_G) = 2n^2(n-2) + 2(n-1)^2 + \sqrt{\left(2n^2(n-2) - 2(n-1)^2\right)^2 + n(n-1)(3n-2)^2},$$

2. *for n is even:*

$$\rho_{DSED}(\Gamma_G) = 10(n-2)^2 + 2n^2(n-3) + \sqrt{(10(n-2)^2 - 2n^2(n-3))^2 + n(n-2)(3n-4)^2}.$$

Proof. 1. Consider the first case for odd n, $DSED(\Gamma_G)$ has four eigenvalues, where it follows the result of Theorem 4.2 (1). They are $\lambda_1 = -4n^2$ of multiplicity $(n-2)$ and $\lambda_2 = -4(n-1)$ of multiplicity $(n-1)$. The quadratic formula gives the other two eigenvalues, which are

$$\lambda_3, \lambda_4 = 2n^2\left(n-2\right) + 2\left(n-1\right)^2 \pm \sqrt{\left(2n^2\left(n-2\right) - 2\left(n-1\right)^2\right)^2 + \left(n-1\right)n\left(3n-2\right)^2}.$$

They are positive real numbers. Hence, the spectrum of Γ_G as the following:

$$Spec(\Gamma_G) = \left\{ \left(2n^2\left(n-2\right) + 2\left(n-1\right)^2 + \sqrt{\left(2n^2\left(n-2\right) - 2\left(n-1\right)^2\right)^2 + \left(n-1\right)n\left(3n-2\right)^2}\right)^1, \left(2n^2\left(n-2\right) + 2\left(n-1\right)^2 - \sqrt{\left(2n^2\left(n-2\right) - 2\left(n-1\right)^2\right)^2 + \left(n-1\right)n\left(3n-2\right)^2}\right)^1, (-4(n-1))^{n-1}, \left(-4n^2\right)^{n-2} \right\}.$$

By determining the maximum absolute eigenvalues, consequently, we derive the spectral radius of Γ_G as the desired result.

2. We may consider the even n case, it follows from Theorem 4.2 (2), $DSED(\Gamma_G)$ has five eigenvalues. Hence, we get $\lambda_1 = -4n^n$ of multiplicity $(n-3)$, the second is $\lambda_2 = 8(n-2)(2n-5)$ of multiplicity $\frac{n}{2} - 1$, and the third is $\lambda_3 = -16(n-2)^2$ of multiplicity $\frac{n}{2}$. From the quadratic formula we have $\lambda_4, \lambda_5 = 10\left(n-2\right)^2 + 2n^2\left(n-3\right) \pm \sqrt{(10(n-2)^2 - 2n^2(n-3))^2 + n(n-2)(3n-4)^2}$.

Hence, the spectrum of Γ_G as the following:

$$Spec(\Gamma_G) = \left\{ \left(10\left(n-2\right)^2 + 2n^2\left(n-3\right) + \sqrt{(10(n-2)^2 - 2n^2(n-3))^2 + n(n-2)(3n-4)^2}\right)^1, \left(10\left(n-2\right)^2 + 2n^2\left(n-3\right) - \sqrt{(10(n-2)^2 - 2n^2(n-3))^2 + n(n-2)(3n-4)^2}\right)^1, (8(n-2)(2n-5))^{\frac{n}{2}-1}, \left(-4n^2\right)^{n-3}, \left(-16(n-2)^2\right)^{\frac{n}{2}} \right\}.$$

Now, for $i = 1, 2, 3, 4$, the maximum of $|\lambda_i|$ is $DSED-$spectral radius of Γ_G. □

Theorem 4.4. *Let Γ_G be a non-commuting graph on G, where $G = G_1 \cup G_2$, then $DSED-$energy for Γ_G is*

1. *for n is odd: $E_{DSED}(\Gamma_G) = 8n^2\left(n-2\right) + 8\left(n-1\right)^2$*
2. *for n is even: $E_{DSED}(\Gamma_G) = 8n^2\left(n-3\right) + 8\left(n-2\right)^2 + 8n(n-2)^2$.*

Proof. 1. The proving part of Theorem 4.3 (1) was given the spectrum of Γ_G for odd n, then the $DSED-$energy of Γ_G can be calculated as follows:

$$\begin{aligned} E_{DSED}(\Gamma_G) =& (n-2)\left|-4n^2\right| + (n-1)\left|-4(n-1)\right| + \\ & \left|2n^2\left(n-2\right) + 2\left(n-1\right)^2 \pm \sqrt{\left(2n^2\left(n-2\right) - 2\left(n-1\right)^2\right)^2 + \left(n-1\right)n\left(3n-2\right)^2}\right| \\ =& 8n^2\left(n-2\right) + 8\left(n-1\right)^2 \end{aligned}$$

2. Let n is even, by Theorem 4.3 (2), the $DSED-$energy of Γ_G is derived as follows:

$$\begin{aligned} E_{DSED}(\Gamma_G) =&(n-3)\left|-4n^2\right| + \left(\frac{n}{2}-1\right)\left|-8(n-2)\right| + \left(\frac{n}{2}\right)\left|-16(n-2)^2\right| + \\ &\left|2n^2(n-3)+2(n-2)^2 \pm \sqrt{\left(2n^2(n-2)-2(n-1)^2\right)^2+(n-1)n(3n-2)^2}\right| \\ =&8n^2(n-3)+8(n-2)^2+8n(n-2)^2. \end{aligned}$$

□

Example 4.2. Following Example 4.1, we can construct 6×6 degree sum exponent distance matrix of Γ_G as follows:

$$DSED(\Gamma_G) = \begin{bmatrix} 0 & 64 & 8 & 8 & 8 & 8 \\ 64 & 0 & 8 & 8 & 8 & 8 \\ 8 & 8 & 0 & 8 & 64 & 8 \\ 8 & 8 & 8 & 0 & 8 & 64 \\ 8 & 8 & 64 & 8 & 0 & 8 \\ 8 & 8 & 8 & 64 & 8 & 0 \end{bmatrix}$$

Here $P_{DSED(\Gamma_G)}(\lambda)$ is derived as follows:

$$P_{DSED(\Gamma_G)}(\lambda) = (\lambda-48)^2(\lambda+64)^3(\lambda-96).$$

As a result of using Maple, we have determined that

$$Spec(\Gamma_G) = \left\{(96)^1, (48)^2, (-64)^3\right\}.$$

Therefore, the $DSED-$energy of Γ_G is as follows:

$$E_{DSED}(\Gamma_G) = (1)|96| + (2)|48| + (3)|-64| = 384.$$

5. Discussion

As in the previous result of Theorem 4.4 for $G = G_1 \cup G_2$, in the following, we get the classification of the $DSED-$Energy of Γ_G for D_{2n}.

Corollary 5.1. *Graph Γ_G associated with the degree sum exponent distance matrix is hyperenergetic.*

Moreover, based on the facts obtained in the previous section, the energies in Theorem 4.4 yield the following fact:

Corollary 5.2. *$DSED-$energy of Γ_G is always an even integer.*

The fact in Corollary 5.2 corresponds with the well-known statement from [4] and [18]. Furthermore, as a comparison of the energies from Theorems 2.4 and 4.4, as a consequence, we derive the following conclusion:

Corollary 5.3. $E_{DSED}(\Gamma_G) > E_A(\Gamma_G)$.

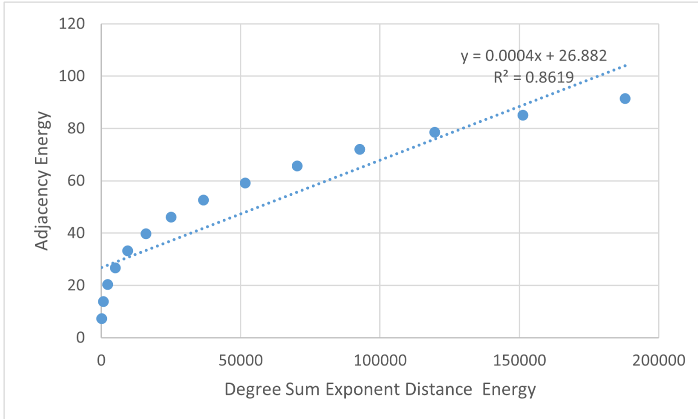

Figure 2: Correlation of $E_{DSED}(\Gamma_G)$ with $E_A(\Gamma_G)$ for odd n

In our graph, the $DSED-$energy of Γ_G for D_{2n}, where $n \geq 3$ is always greater than the adjacency energy. In addition, it can be seen from Figures 2 and 3 that $E_{DSED}(\Gamma_G)$ has a significant correlation with $E_A(\Gamma_G)$, with a correlation coefficient of 0.8619 for odd n, 0.865 for even n. Those results state that $E_{DSED}(\Gamma_G)$ and $E_A(\Gamma_G)$ have a strong correlation between them and comply with the result from [9]. However, it is slightly different from the claim from [20].

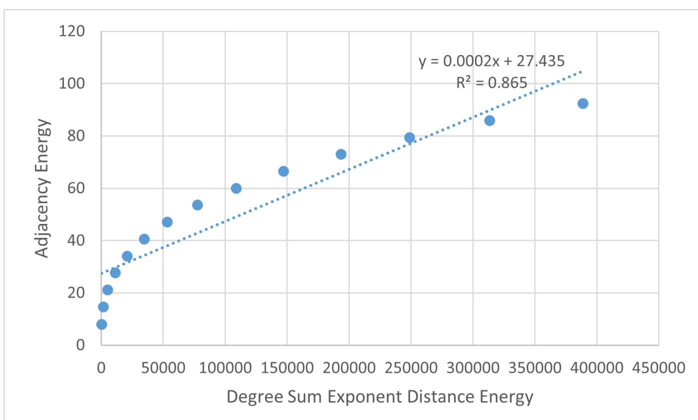

Figure 3: Correlation of $E_{DSED}(\Gamma_G)$ with $E_A(\Gamma_G)$ for even n

References

[1] A. Abdollahi, S. Akbari, H.R. Maimani, *Non-commuting graph of a group*, J. Algebra, 298 (2006), 468-492.

[2] A. Abdollahi, H. Shahverdi, *Characterization of the alternating group by*

its non-commuting graph, J. Algebra, 357 (2012), 203-207.

[3] M. Aschbacher, *Finite group theory*, 2nd ed., Cambridge University Press: Cambridge, UK, 2000.

[4] R.B. Bapat, S. Pati, *Energy of a graph is never an odd integer*, Bull. Kerala. Math. Assoc., 1 (2004), 129-132.

[5] A.E. Brouwer, W.H. Haemers, *Spectra of graphs*, vol. 1, Springer: New York, USA, 2011.

[6] M.R. Darafsheh, *Groups with the same non-commuting graph*, Discret. Appl. Math., 157 (2009), 833-837.

[7] S. Filipovski, R. Jajcay, *Bounds for the energy of graphs*, Mathematics, 9 (1687) (2021), 1-10.

[8] F.R. Gantmacher, *The theory of matrices*, vol. 1, Chelsea Publishing Company: New York, USA, 1959.

[9] K.J. Gowtham, N.N. Swamy, *On Sombor energy of graphs*, Nanosystems: Physics, Chemistry, Mathematics, 12 (2021), 411-417.

[10] I. Gutman, *The energy of graph*, Ber. Math.-Stat. Sekt. Forschungsz. Graz., 103 (1978), 1-2.

[11] I. Gutman, I. Redzepovic, B. Furtula, A.M. Sahal, *Energy of graphs with self-loops*, MATCH Commun. Math. Comput. Chem., 87 (2022), 645-652.

[12] G. Indulal, I. Gutman, A. Vijayakumar, *On distance energy of graphs*, MATCH Commun. Math. Comput. Chem., 60 (2008), 461-472.

[13] S.R. Jog, J.R. Gurjar, *Degree product distance energy of some graphs*, Asian Res. J. Math., 24 (2018), 42-49.

[14] S.R. Jog, J.R. Gurjar, *Degree sum exponent distance energy of some graphs*, J. Indones. Math. Soc., 27 (2021), 64-74.

[15] S.M.S. Khasraw, I.D. Ali, R.R. Haji, *On the non-commuting graph of dihedral group*, Electron. J. Graph Theory Appl., 8 (2020), 233-239.

[16] X. Li, Y. Shi, I. Gutman, *Graph energy*, Springer: New York, 2012.

[17] R. Mahmoud, N.H. Sarmin, A. Erfanian, *On the energy of non-commuting graph of dihedral groups*, AIP Conf. Proc., 1830 (2017), 1-5.

[18] S. Pirzada, I. Gutman, *Energy of a graph is never the square root of an odd integer*, Appl. Anal. Discrete Math., 2 (2008), 118-121.

[19] H.S. Ramane, S.S. Shinde, *Degree exponent polynomial of graphs obtained by some graph operations*, Electron. Notes Discret. Math., 63 (2017), 161-168.

[20] I. Redzepovic, I. Gutman, *Comparing energy and Sombor energy-an empirical study*, MATCH Commun. Math. Comput. Chem., 88 (2022), 133-140.

[21] M.U. Romdhini, A. Nawawi, *Degree sum energy of non-commuting graph for dihedral groups*, Malays. J. Sci., 41(sp1) (2022), 34-39.

[22] M.U. Romdhini, A. Nawawi, *Degree subtraction energy of commuting and non-commuting graphs for dihedral groups*, Int. J. Math. Comput. Sci., 18 (2023), 497-508.

[23] M.U. Romdhini, A. Nawawi, C.Y. Chen, *Neighbors degree sum energy of commuting and non-commuting graphs for dihedral groups*, Malaysian J. Math. Sci., 17 (2023), 53-65.

[24] M.U. Romdhini, A. Nawawi, C.Y. Chen, *Degree exponent sum energy of commuting graph for dihedral groups*, Malays. J. Sci., 41(sp1) (2022), 40-46.

[25] M.U. Romdhini, A. Nawawi, *Maximum and minimum degree energy of commuting graph for dihedral groups*, Sains Malays., 51 (2022), 4145-4151.

[26] M. U. Romdhini, A. Nawawi, F. Al-Sharqi, A. Al-Quran, *Closeness energy of non-commuting graph for dihedral groups*, Eur. J. Pure Appl. Math., 17 (2024).

[27] S. Sun, K.C. Das, Y. Shang, *On maximal energy distance*, Mathematics, 9 (2021), 1-7.

[28] B. Tolue, A.Erfanian, A. Jafarzadeh, *A kind of non-commuting graph of finite groups*, J. Sci. I. R. Iran, 25 (2014), 379-384.

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