On the localization of a type B semigroup

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Abstract. This paper mainly investigates the localization of a type B semigroup. Firstly, the unique localization of a type B semigroup on its idempotent semilattice is given, and some properties of the localization of a type B semigroup are studied. It is proved that the localization of a type B semigroup on its idempotent semilattice is the maximum cancellative monoid homomorphic image. Finally, the relationships between localizations and the minimum cancellative congruence of a type B semigroup are discussed.

Keywords: type B semigroup, idempotent semilattice, cancellative monoid homomorphic image, localization.
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1. Introduction

In recent years, abundant semigroups have attracted more and more attention from semigroup scholars (see, [4-5, 7-8, 16]). As an important subclass of abundant semigroups, type B semigroups (see, [12-15, 17-19]) are called generalized inverse semigroups together with ample semigroups (see, [2-3, 6]) because of their similar properties to inverse semigroups (see, [1, 11, 23]). The localization (see, [9, 20-22]) is a good method to construct a new algebraic structure, and it plays an important role in commutative algebra. Localizations of inverse semigroups and ample semigroups have been studied by many authors (see, [9,

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21-22]). As an application of the localization, this paper will give some new characterizations of localizations of a type B semigroup.

2. Preliminaries

Firstly, some definitions, notations and known results used in this paper are provided.

In 1951, the concept of Green's relations were introduced by Green in [10]. Let $a, b \in S$, we have

$$a\mathcal{L}b \Longleftrightarrow S^1a = S^1b; \quad a\mathcal{R}b \Longleftrightarrow aS^1 = bS^1.$$

In the 1970s, Fountain extended Green's relations to Green's * relations. Let S be a semigroup. Recall, from [5] that two elements a and b in S are $\mathcal{L}^*$-[$\mathcal{R}^*$-] related if and only if they are $\mathcal{L}$-[$\mathcal{R}$-]related in some oversemigroup of S. The equivalent definitions of $\mathcal{L}^*$-relation and $\mathcal{R}^*$-relation are given as follows:

Lemma 2.1 ([5]). *Let S be a semigroup and $a, b \in S$. Then, the following statements hold:*

(1) *$a\mathcal{L}^*b$ if and only if, for all $x, y \in S^1$, $ax = ay \Leftrightarrow bx = by$;*

(2) *$a\mathcal{R}^*b$ if and only if, for all $x, y \in S^1$, $xa = ya \Leftrightarrow xb = yb$.*

Corollary 2.2 ([5]). *Let S be a semigroup and $a, e = e^2 \in S$. Then, the following statements are equivalent:*

(1) *$a\mathcal{L}^*e$ [$a\mathcal{R}^*\ e$];*

(2) *$ae = a$ [$a = ea$] and for all $x, y \in S^1$, $ax = ay$ [$xa = ya$]implies $ex = ey$ [$xe = ye$].*

Obviously, let S be a semigroup. The relation $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence on S. Usually, $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$ on S. But, if a and b are regular elements of a semigroup S, then we obtain that $a\mathcal{L}^*b$ if and only if $a\mathcal{L}b$, and that $a\mathcal{R}^*b$ if and only if $a\mathcal{R}b$. That is, $\mathcal{L}^* \cap (RegS \times RegS) = \mathcal{L}$, $\mathcal{R}^* \cap (RegS \times RegS) = \mathcal{R}$, where $RegS$ denotes the set of all regular elements of S. For convenience, ${\mathcal{L}^*}_a$ and ${\mathcal{R}^*}_a$ denote the $\mathcal{L}^*-$class and $\mathcal{R}^*-$class containing a, respectively; $E(S)$ denotes the set of idempotents of S; a^+ and a^* denote the idempotent of the $\mathcal{L}^*-$class and $\mathcal{R}^*-$class containing a, respectively.

As in [4], a semigroup S is said to be *right (left) abundant* if each $\mathcal{L}^*-$ $(\mathcal{R}^*)-$class of S contains an idempotent. A semigroup S is *abundant* if it is both right and left abundant. A right (left) abundant semigroup S is *right (left) adequate* if $E(S)$ is a semilattice ([5]). A semigroup S is said to be *adequate* if it is both left and right adequate.

Definition 2.1 ([4]). *Let S be a right adequate semigroup. Then, S is said to be right type B, if it satisfies the following conditions:*

$(B1)$ *for all* $e, f \in E(S^1), a \in S$, $(efa)^* = (ea)^*(fa)^*$;

$(B2)$ *for all* $a \in S, e \in E(S)$, *if* $e \leq a^*$, *then there is* $f \in E(S^1)$ *such that* $e = (fa)^*$, *where* $\leq$ *is a natural partial order on* $E(S)$.

Definition 2.2 ([4])**.** *Let* S *be a left adequate semigroup. Then,* S *is left type* B, *if it satisfies the following conditions:*

$(B1')$ *for all* $e, f \in E(S^1), a \in S$, $(aef)^+ = (ae)^+(af)^+$;

$(B2')$ *for all* $a \in S, e \in E(S)$, *if* $e \leq a^+$, *then there is* $f \in E(S^1)$ *such that* $e = (af)^+$, *where* $\leq$ *is a natural partial order on* $E(S)$.

A semigroup is said to be type B *if it is both left and right type* B.

Lemma 2.3 ([12])**.** *Let* S *be a type* B *semigroup. The relation* σ *is defined as follows:*

$$(a, b) \in \sigma \Longleftrightarrow (\exists e \in E(S)) \; eae = ebe.$$

Then, σ *is the least cancellative congruence on* S.

Definition 2.3 ([21])**.** *Let* T *be a monoid,* S *be a semigroup and* H *be a subsemigroup of* S. *Then,* T *is said to be a localization of* S *on* H, *if it satisfies the following conditions:*

(1) *There is a surjective homomorphism* $\phi : S \to T$ *such that* $\phi(a)$ *is inverse on* T, *for all* $a \in H$.

(2) *If there are a monoid* S' *and a homomorphism* $\alpha : S \to S'$ *such that* $\alpha(a)$ *is inverse on* S', *for all* $a \in H$, *then there is a unique homomorphism* $\psi : T \to S'$ *such that* $\psi\phi = \alpha$.

Lemma 2.4 ([9])**.** *Let* S *be a semigroup and* H *be a subsemigroup of* S. *If there exists a localization of* S *on* H, *then the localization is unique in the sense of isomorphism. For convenience, we denote the unique localization by* $S[H^{-1}]$.

3. The localization of a type B semigroup on its idempotent semilattice

In this section, we shall characterize the localization of a type B semigroup on its idempotents. For convenience, we denote the idempotent set $E(S)$ of a semigroup S by E.

Proposition 3.1. *Let* S *be a type* B *semigroup and* E *be its idempotent semilattice. Define a relation on set* $S \times E$ *as follows:*

$$(\forall (x, e) \in S \times E)(x, e) \sim (y, f) \Longleftrightarrow (\exists h \in E) hfxfh = heyeh,$$

then the following statements hold:

(1) *The relation* $\sim$ *is an equivalence relation on* S.

(2) *For all* $x \in S$, $e, f \in E$, $(x, e) \sim (x, f)$.

(3) *For all* $(x, e) \in S \times E$, *we denote the equivalence class containing* (x, e) *by* x/e. *Then, for all* $e_1, e_2, e_3, e_4 \in E$, $e_1/e_2 \sim e_3/e_4$. *In particular, for* $e \in E$, *we denote* $\sim$ *−class containing all* (e_1, e_2) *by* e/e, *where* $e_1, e_2 \in E$.

(4) *Put* $T = (S \times E)/\sim = \{x/e \mid x \in S\}$. *Define a multiplication* "$\cdot$" *on* T *as follows:*

$$(\forall x/e, y/e \in T)\ x/e \cdot y/e = (xy)/e.$$

Then, T *is a monoid whose identity element is* e/e *under the multiplication* "$\cdot$".

Proof. (1) Obviously, "$\sim$" is reflexive and symmetric. Now, we prove that "$\sim$" is transitive. To see it, let $(x, e), (y, f), (z, g) \in S \times E$ such that $(x, e) \sim (y, f)$, $(y, f) \sim (z, g)$. Then, there exist $e_1, e_2 \in E$ such that $e_1fxfe_1 = e_1eyee_1$ and $e_2gyge_2 = e_2fzfe_2$. Hence,

$$\begin{aligned} e_1e_2fgxge_1e_2f &= e_2ge_1fxfe_1e_2g = e_2ge_1eyee_1e_2g = e_1ee_2gyge_2e_1e \\ &= e_1ee_2fzfe_2e_1e = e_1e_2fezee_1e_2f. \end{aligned}$$

Let $h = e_1e_2f \in E$. Then, $hgxgh = hezeh$. This shows that $(x, e) \sim (z, g)$. Therefore, "$\sim$" is an equivalence relation on S.

(2) For all $x \in S$, $e, f \in E$, we have that $effxfef = efxef = eefxeef = efexeef$. Let $h = ef \in E$. Then, $hfxfh = hexeh$. Therefore, $(x, e) \sim (x, f)$.

(3) Since E is the idempotent semilattice of S, we have that $h = e_1e_2e_3e_4 \in E$, for all $e_1, e_2, e_3, e_4 \in E$. Again, since $he_4e_1e_4h = he_2e_3e_2h$, we have $(e_1, e_2) \sim (e_3, e_4)$. That is, $e_1/e_2 \sim e_3/e_4$. In particular, we choose one element $e \in E$, it is easy to see that $(e_1, e_2) \in e/e$, for all $e_1, e_2 \in E$.

(4) Firstly, we prove that the multiplication operation "$\cdot$" on T is well-defined. Let $x_1/e, x_2/e, y_1/e, y_2/e \in T$ with $x_1/e = x_2/e$, $y_1/e = y_2/e$. Then, there exist $f, g \in E$ such that $fex_1ef = fex_2ef$ and $gey_1eg = gey_2eg$. Notice that $x_1^*ef \leq x_1^*$, $x_2^*ef \leq x_2^*$. We have that there exist $e_1, e_2 \in E(S^1)$ such that $x_1^*ef = (e_1x_1)^*$ and $x_2^*ef = (e_2x_2)^*$ from Condition (B2). Hence,

$$\begin{aligned} e_1e_2fex_1ef &= e_1e_2fex_1x_1^*ef = e_1e_2fex_1(e_1x_1)^* \\ &= e_1e_2fee_1x_1(e_1x_1)^* = e_1e_2fee_1x_1. \end{aligned}$$

Similarly, $e_1e_2fex_2ef = e_1e_2fee_2x_2$. Again, $fex_1ef = fex_2ef$. Multiplying it on the left by e_1e_2, we obtain that $e_1e_2fex_1ef = e_1e_2fex_2ef$. Thus, $e_1e_2fee_1x_1 = e_1e_2fee_2x_2$. On the other hand, it is clear that $gey_1^+ \leq y_1^+$ and $gey_2^+ \leq y_2^+$. Therefore, there exist $e_3, e_4 \in E(S^1)$ such that $gey_1^+ = (y_1e_3)^+$ and $gey_2^+ = (y_2e_4)^+$ from Condition (B2'), and so

$$\begin{aligned} gey_1ege_3e_4 &= gey_1^+y_1ege_3e_4 = (y_1e_3)^+y_1ege_3e_4 \\ &= (y_1e_3)^+y_1e_3ege_3e_4 = y_1e_3ege_3e_4. \end{aligned}$$

Similarly, $gey_2ege_3e_4 = y_2e_4ege_3e_4$. Again, $gey_1eg = gey_2eg$. Multiplying it on the right by e_3e_4, we obtain that $gey_1ege_3e_4 = gey_2ege_3e_4$. Thus, $y_1e_3ege_3e_4 = y_2e_4ege_3e_4$. For some $h = e_1e_2e_3e_4fg \in E$, we have

$$\begin{aligned} hex_1y_1eh &= e_1e_2e_3e_4fgex_1y_1ee_1e_2e_3e_4fg = e_3e_4ge_1e_2fee_1x_1y_1e_3ege_3e_4e_1e_2f \\ &= e_3e_4ge_1e_2fee_2x_2y_2e_4ege_3e_4e_1e_2f = e_1e_2e_3e_4fgex_2y_2ee_1e_2e_3e_4fg \\ &= hex_2y_2eh. \end{aligned}$$

Hence, $(x_1y_1)/e = (x_2y_2)/e$. This means that the multiplication operation " $\cdot$ " on T is good.

Next, we show that T is a monoid whose identity element is e/e under the multiplication " $\cdot$ ". Let $x/e, y/e, z/e \in T$. We have

$$\begin{aligned} (x/e \cdot y/e) \cdot z/e &= (xy)/e \cdot z/e = (xyz)/e \\ &= x/e \cdot (yz)/e = x/e \cdot (y/e \cdot z/e). \end{aligned}$$

This shows that T is associative under the multiplication operation " $\cdot$ ". It is clear that T is closed. Thus, T is a semigroup with respect to the multiplication " $\cdot$ ". Obviously, we have $ee(xe)ee = eexee$, for all $e \in E, x/e \in T$. Hence, $(xe, e) \sim (x, e)$. That is, $(xe)/e = x/e \cdot e/e = x/e$. On the other hand, for all $e \in E, x/e \in T$, we have $ee(ex)ee = eexee$. Thus, $(ex, e) \sim (x, e)$. That is, $(ex)/e = e/e \cdot x/e = x/e$. Therefore, T is a monoid whose identity element is e/e under the multiplication " $\cdot$ ". □

The following theorem shows that the existence of localization of a type B semigroup on its idempotent semilattice.

Theorem 3.2. *Let S be a type B semigroup and E be its idempotent semilattice. Then, there is a localization of S on E.*

Proof. Define a mapping as follows:

$$\phi : S \longrightarrow T = (S \times E)/\sim,\ x \mapsto x/e,$$

where T is a monoid which is constructed in Proposition 3.1(4). It is clear that ϕ is a surjection from S into T. For all $x, y \in S$, we have

$$\phi(xy) = (xy)/e = x/e \cdot y/e = \phi(x) \cdot \phi(y).$$

Hence, ϕ is a surjective homomorphism from S into T. By Proposition 3.1, we have $\phi(f) = f/e = e/e$, for all $f \in E$. Thus, $\phi(f)$ is an identity element of T. This means that $\phi(f)$ is inverse on T.

Suppose that there are a monoid S' and a homomorphism $\alpha : S \to S'$ such that $\alpha(f)$ is inverse on S', for all $f \in E$. Define a mapping as follows:

$$\psi : T = (S \times E)/\sim \longrightarrow S',\ x/e \mapsto \alpha(x).$$

Let $x/e, y/e \in T$ with $x/e = y/e$. Then, there exists $h \in E$ such that $hexeh = heyeh$. Let $f = eh = he \in E$. It follows that $fxf = fyf$. Hence,

$$\alpha(f)\alpha(x)\alpha(f) = \alpha(f)\alpha(y)\alpha(f).$$

Multiplying it on the left and right by $\alpha(f)^{-1}$, we have $\alpha(x) = \alpha(y)$ since $\alpha(f)$ is inverse on S'. Thus, ψ is a well defined. Let $x/e, y/e \in T$. Then,

$$\psi(x/e \cdot y/e) = \psi((xy)/e) = \alpha(xy) = \alpha(x)\alpha(y) = \psi(x/e)\psi(y/e).$$

Hence, ψ is a homomorphism. It is easy to see that $\psi\phi(x) = \psi(x/e) = \alpha(x)$, for all $x \in S$. That is, $\psi\phi = \alpha$. Finally, we prove that ψ is unique. Suppose that there exists a homomorphism $\psi' : T \to S'$ such that $\psi'\phi = \alpha$. Then, for all $x/e \in T$, we have $\psi'(x/e) = \psi'(\phi(x)) = (\psi'\phi)(x) = \alpha(x) = \psi(x/e)$. Thus, $\psi' = \psi$. To sum up, T is a localization of S on E. This completes the proof. □

4. The cancellative monoid homomorphic image of a type B semigroup

In this section, we shall characterize the relations between localizations and the minimum cancellative congruence of a type B semigroup.

By Lemma 2.4, we have the localization T of S on E is unique. we denote the localization T by $S[E^{-1}]$.

Proposition 4.1. *Let S be a type B semigroup and E be its idempotent semilattice. Then, the localization $S[E^{-1}]$ of S on E is cancellative.*

Proof. Let $x/e, y/e, z/e \in S[E^{-1}]$ with $x/e \cdot y/e = x/e \cdot z/e$. Then, $(xy)/e = (xz)/e$. Hence, there exists $h \in E$ such that $hexyeh = hexzeh$, and so

$$\begin{aligned} hexyeh = hexzeh &\Rightarrow (hex)yeh = (hex)zeh \\ &\Rightarrow (hex)^*yeh = (hex)^*zeh \\ &\Rightarrow (hex)^*heye(hex)^*h = (hex)^*heze(hex)^*h. \end{aligned}$$

Thus, $y/e = z/e$ since $(hex)^*h \in E$. This shows that $S[E^{-1}]$ is left cancellative. Dually, $S[E^{-1}]$ is right cancellative. That is, $S[E^{-1}]$ is cancellative. □

Proposition 4.2. *Let S be a type B semigroup and E be its idempotent semilattice. Then, the localization $S[E^{-1}]$ of S on E is the maximum cancellative monoid homomorphic image of S.*

Proof. Let ϕ be a surjective homomorphism from S onto $S[E^{-1}]$ such that $\phi(f)$ is inverse on $S[E^{-1}]$, for all $f \in E$. If S' is the cancellative monoid homomorphic image of S, then there exists a homomorphism $\alpha : S \to S'$. By the definition of localization, there is a unique homomorphism $\psi : S[E^{-1}] \to S$ such that $\psi\phi = \alpha$. Thus, $S[E^{-1}]$ is the maximum cancellative monoid homomorphic image of S. □

Proposition 4.3. *Let S be a type B semigroup and E be its idempotent semilattice, H be a subsemigroup of S. If $E \subseteq H \subseteq RegS$, then there is the localization $S[H^{-1}]$ of S on H with $S[H^{-1}] = S[E^{-1}]$. In particular, $S[(RegS)^{-1}] = S[E^{-1}]$.*

Proof. Since S is a type B semigroup, H is a subsemigroup of S and $E \subseteq H \subseteq RegS$, we have that $x^*\mathcal{L}x\mathcal{R}x^+$, for all $x \in H$. Again, since $S[E^{-1}]$ is the localization of S on E, there exists a surjective homomorphism $\phi : S \to S[E^{-1}]$. Hence,

$$\phi(x)\mathcal{H}(S[E^{-1}])\phi(x^*) = \phi(x^+) = e/e.$$

This means that $\phi(x)$ is inverse on $S[E^{-1}]$. On the other hand, if there are a monoid S' and a homomorphism $\alpha : S \to S'$ such that $\alpha(x)$ is inverse on S', for all $x \in H$, then $\alpha(f)$ is inverse on S', for all $f \in E \subseteq H$. By the definition of localization, there is a unique homomorphism $\psi : S[E^{-1}] \to S'$ such that $\psi\phi = \alpha$. Therefore, $S[E^{-1}]$ is the localization of S on H. That is, $S[H^{-1}] = S[E^{-1}]$.

Note that, E is an idempotent semilattice of S. we have that $RegS$ is a subsemigroup of S. Again, $E \subseteq RegS$. Therefore, $S[(RegS)^{-1}] = S[E^{-1}]$. □

Theorem 4.4. *Let S be a type B semigroup and E be its idempotent semilattice. Then, $S[E^{-1}] = S/\sigma$, where σ is the least cancellative congruence on S.*

Proof. Define a mapping as follows:

$$\varphi : S[E^{-1}] \longrightarrow S/\sigma,\ x/e \mapsto x\sigma.$$

Now, we prove that φ is an isomorphism. Let $x/e, y/e \in S[E^{-1}]$ with $x/e = y/e$. Then, there exists $h \in E$ such that $hexeh = heyeh$. Hence, $fxf = fyf$ for some $f = eh = he \in E$, and so $(x, y) \in \sigma$. That is, $x\sigma = y\sigma$. This means that φ is well defined. Let $x\sigma, y\sigma \in S/\sigma$ with $x\sigma = y\sigma$. Then, there is $g \in E$ such that $gxg = gyg$, and $gexeg = geyeg$. Thus, $x/e = y/e$. Obviously, φ is a surjection. Hence, φ is a bijection from $S[E^{-1}]$ onto S/σ. Finally, we show that φ is a homomorphism. Obviously, for all $x/e, y/e \in S[E^{-1}]$, we have

$$\varphi(x/e \cdot y/e) = \varphi((xy)/e) = (xy)\sigma = x\sigma \cdot y\sigma = \varphi(x/e) \cdot \varphi(y/e).$$

This completes the proof.

□

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