Strong edge-coloring of planar graphs with girth at least seven

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Abstract. A *strong edge-coloring* of a graph G is that two edges e_1 and e_2 that are adjacent to each other or adjacent to the same edge must be colored with distinct colors. In this paper we prove that every planar graph G with girth $g \geq 7$ and maximum degree $\Delta \geq 5$ has a strong edge-coloring using at most $3\Delta - 1$ colors. In addition, we prove that every planar graph G without adjacent $7-cycles$, with girth $g \geq 7$ and the maximum degree $\Delta \geq 4$ has a strong edge-coloring using at most $3\Delta - 1$ colors.

Keywords: strong edge-coloring, planar graph, discharging method.
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1. Introduction

All graphs considered in this paper are finite, loopless and undirected. Let G be a simple undirected graph. A vertex of degree k, at least k or at most k is denoted by a $k-vertex$, a $k^+-vertex$ or a $k^--vertex$ respectively. A neighbor of v of degree k, at most k or at least k is denoted by a $k-neighbor$, a $k^--neighbor$ or a $k^+-neighbor$, respectively.

A *strong edge-coloring* of a graph G is that two edges e_1 and e_2 that are adjacent to each other or adjacent to the same edge must be colored with distinct colors. The *strong chromatic index* of G is denoted by $\chi'_s(G)$, which is the minimum number of colors for a strong edge-coloring of G.

We denote the minimum and maximum degree of vertices in G by $\delta(G)$ and $\Delta(G)$ (δ and Δ for short), respectively. The degree of vertex v in G is denoted by $d_G(v)$. The *girth* of a graph G, denoted by $g(G)$ (g for short),

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is the length of its shortest cycle. Define that the *maximum average degree* of a graph G is the largest average degree of its subgraphs and is denoted by $mad(G) = \max_{H \subseteq G}\{\frac{2|E(H)|}{|V(H)|}\}$. The *distance* of two edges e_1 and e_2 refers to the length of the shortest path from u to v, where u is an arbitrary endvertice of e_1 and v is an arbitrary endvertice of e_2. So, equivalently, a strong edge-coloring is an assignment of colors to all edges such that every two edges with distance at most 1 receive distinct colors.

Using greedy algorithm, we may easily see that $\chi'_s(G) \leq 2\Delta^2 - 2\Delta + 1$ for every graph G. In 1989, Erdős and Nešetřil [3] conjectured the following upper bounds.

Conjecture 1.1 ([2],[3])**.** For every graph G with maximum degree Δ,

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even,} \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}, & \text{if } \Delta \text{ is odd.} \end{cases}$$

These bounds would be tight, as Erdős and Nešetřil [3] gave examples of graphs that get these bounds. For the case when $\Delta = 2$, this conjecture is clearly true. For the case of $\Delta = 3$, Andersen [1] and Horák *et al.*[5] proved this conjecture to be correct, independently. Moreover, for subcubic graph G, that is, maximum degree is at most 3, Faudree *et al.*[4] proposed some conjectures. Steger and Yu [10]showed that $\chi'_s(G) \leq 9$ for every subcubic bipartite graph G. For subcubic planar graph G with girth at least 6, Hudák et al.[8] proved the same result above. This conjecture is still open for $\Delta \geq 4$. For $\Delta = 4$, the best bound is 21, which was recently established by Huang, Santana, and Yu [9]. The bound of 21 is still one larger than the conjectured bound of 20. For every planar graph G with $\Delta = 4$, Wang *et al.*[11] proved that $\chi'_s(G) \leq 19$ and Jian-Bo Lv *et al.*[13] proved that if $mad(G) < \frac{61}{18}$ (*resp.* $\frac{7}{2}, \frac{18}{5}, \frac{15}{4}, \frac{51}{13}$), then $\chi'_s(G) \leq 16$ (*resp.* $17, 18, 19, 20$).

Recently, a great deal of research has been done on planar graphs with different values of girth. In 2014, Hudák *et al.*[8] proved that $\chi'_s(G) \leq 3\Delta$ for every planar graph G with $g \geq 7$. For every planar graph G with $g \geq 10\Delta - 4$, Wang *et al.*[12] further reduced this bound and proved that $\chi'_s(G) \leq 2\Delta - 1$. By maximum degree restriction, Choi *et al.*[6] showed two results, namely, $\chi'_s(G) \leq 3\Delta$ for every planar graph G with $g \geq 6$ and $\Delta \geq 7$, and $\chi'_s(G) \leq 3\Delta - 3$ for every graph G with $g \geq 8$ and $\Delta \geq 9$. Guo *et al.*[7] also came to two conclusions that $\chi'_s(G) \leq 3\Delta - 2$ for every planar graph G with $g \geq 8$ and $\Delta \geq 4$, and $\chi'_s(G) \leq 3\Delta - 3$ for every planar graph G with $g \geq 10$ and $\Delta \geq 5$.

In this paper, we take into account the girth and the maximum degree of planar graphs and prove the following results.

Theorem 1.2. *If G is a planar graph with $g \geq 7$ and $\Delta \geq 5$, then $\chi'_s(G) \leq 3\Delta - 1$.*

Theorem 1.3. *If G is a planar graph without adjacent $7-cycles$, with $g \geq 7$ and $\Delta \geq 4$, then $\chi'_s(G) \leq 3\Delta - 1$.*

Note that $mad(G) < \frac{2g}{g-2}$ for every planar graph. Thus, when $g \geq 7$, we have that $mad(G) < \frac{14}{5}$. Therefore, there is the following corollary.

Corollary 1.4. *If G is a planar graph with $mad(G) < \frac{14}{5}$ and $\Delta \geq 5$, then $\chi'_s(G) \leq 3\Delta - 1$.*

By adding the condition that $\Delta \geq 5$, our results improve the bound of Hudák et al.[8] and are reduce by one color.

Let G^* be obtained by removing all vertices of degree one in graph G. The paper is organized as follows. In Section 2, we assume that G is a minimal counterexample with the fewest edges to Theorem 1.2. We first prove some structural properties of the minimal counterexample G and its subgraph G^*. Next, we use the discharging method to show that G^* cannot exist. In Section 3, with a weaker maximum degree restriction and a stronger cycle constraint, we still obtain the same bound.

2. Proof of Theorem 1.2

In this section, G is a counterexample to Theorem 1.2 with the $|V(G)|$ minimized, subject to that, assume that $|E(G)|$ is as small as possible. It is obvious that G and G^* are connected. A *strong partial edge-coloring* of a graph G is a proper edge-coloring of a proper subgraph G' such that every two edges of G' with distance at most 1 in G receive different colors. Suppose that G has a strong partial edge-coloring. For every uncolored edge e of G, we use $A(e)$ to denote the set of colors that are available at the edge e. The $2-$*neighborhood* of an edge e refers to the set of edges whose distance at most 2 from e.

We first state some structural properties regarding G and G^* as follows.

Lemma 2.1. $\delta(G^*) \geq 2$. *Moreover, $d_{G^*}(v) = 2$ if and only if $d_G(v) = 2$.*

Proof. Suppose to the contrary that $\delta(G^*) \leq 1$. If $\delta(G^*) = 0$, then G is a star since G and G^* are connected. Clearly, G has a strong edge-coloring with Δ colors, a contradiction. If $\delta(G^*) = 1$, then there must be $d_{G^*}(v) = 1$. Then, there must be $d_G(v) > d_{G^*}(v) = 1$, otherwise v will not appear in G^*. Therefore, v must have at least one $1-$neighbor in G, denoted by v_1, as shown in Fig.1(1). By the minimality of G, $G - v_1$ has a strong edge-coloring ϕ with $(3\Delta - 1)$ colors. Note that there are at most $2\Delta - 2$ colored edges in the $2-$neighborhood of the edge vv_1. Therefore, $|A(vv_1)| \geq \Delta + 1 \geq 6$. Thus, we can extend ϕ to G, a contradiction. So, $\delta(G^*) \geq 2$.

If $d_G(v) = 2$, then $d_{G^*}(v) = 2$ since $d_G(v) \geq d_{G^*}(v)$. Suppose that $d_{G^*}(v) = 2$. We assume that $d_G(v) > 2$. Then, v has at least one $1-$neighbor v_1 in G, as shown in Fig.1(2). By the minimality of G, $G - v_1$ has a strong edge-coloring ϕ using $(3\Delta - 1)$ colors. Clearly, there are at most $3\Delta - 3$ colored edges in the $2-$neighborhood of the edge vv_1. Hence, $|A(vv_1)| \geq 2$, which means that we can extend ϕ to G, a contradiction. So $d_G(v) \leq 2$. Since $d_G(v) \geq d_{G^*}(v)$, $d_G(v) = 2$. □

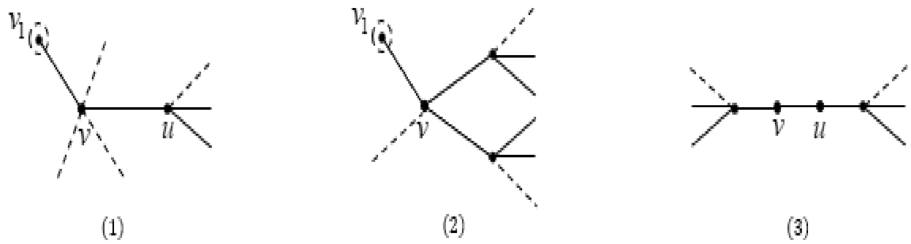

Fig.1
(The solid lines represent the edges that exist in G.
The dashed lines represent the edges that might exist in G.)

Lemma 2.2. *Let v be a $2-vertex$ in G^*. Then, both of neighbors of v in G^* are $3^+ - neighbor$.*

Proof. Suppose otherwise that v has a $2-$neighbor, say u, in G^*. Since $d_{G^*}(v) = d_{G^*}(u) = 2$, by Lemma 2.1, $d_G(v) = d_G(u) = 2$, as shown in Fig.1(3). By the minimality of G, $G - uv$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. Since there are at most 2Δ colored edges in the $2-$neighborhood of the edge uv, $|A(uv)| \geq \Delta - 1 \geq 4$. Then, we can color uv with one of the available colors, a contradiction. □

Lemma 2.3. *Let v be a $3-vertex$ in G^*. Then, v has at least two $3^+-neighbors$ in G^*.*

Proof. Suppose otherwise that v has at most a 3^+-neighbor in G^*. Let u_1, u_2 be two $2-$neighbors of v in G^*. By Lemma 2.1, $d_G(u_1) = d_G(u_2) = 2$. Assume that $d_G(v) > d_{G^*}(v)$. Then, v has at least one $1-$neighbor v_1 in G, as shown in Fig.2(1). By the minimality of G, $G - v_1$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. It is easy to see that vv_1 has at most 2Δ colored edges within distance one. Thus, $|A(vv_1)| \geq \Delta - 1 \geq 4$. Then, we can color vv_1 with one of the available colors, a contradiction. Therefore, $d_G(v) = d_{G^*}(v) = 3$, as shown in Fig.2(2). Let ϕ be a strong $(3\Delta - 1)$-edge-coloring of $G - vu_1$. Note that vu_1 in G has at most $2\Delta + 2$ colored edges in its $2-$neighborhood. Hence, $|A(vu_1)| \geq \Delta - 3 \geq 2$, which implies that vu_1 has at least one available color, a contradiction. □

By Lemma 2.3, a $3 - vertex$ v in G^* is adjacent to at most one $2 - neighbor$ in G^*. We call a $3 - vertex$ v *weak* if it is adjacent to a $2 - vertex$, otherwise we call it *strong*.

Lemma 2.4. *Let v be a weak $3 - vertex$ in G^*. Then, $d_G(v) = d_{G^*}(v) = 3$.*

Proof. Suppose to the contrary that $d_G(v) > d_{G^*}(v)$. Then, v has at least one $1-$neighbor in G, denoted by v_1. Let u be a $2-$neighbor of v in G^*. By Lemma 2.1, $d_G(u) = 2$, as shown in Fig.2(3). By the minimality of G, $G - v_1$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. Note that vv_1 in G has at most $3\Delta - 2$ colored edges in its $2-$neighborhood. So $|A(vv_1)| \geq 3\Delta - 1 - (3\Delta - 2) = 1$, which implies that vv_1 has at least one available color, a contradiction. □

Lemma 2.5. *Assume that v is a weak $3-vertex$ in G^*. Then, v is not adjacent to a weak $3-vertex$.*

Proof. Suppose otherwise that v has a weak $3-$neighbor, say v_1, in G^*. By Lemma 2.4, $d_G(v) = d_{G^*}(v) = 3$ and $d_G(v_1) = d_{G^*}(v_1) = 3$. Let u be the $2-$neighbor of v in G^*, as shown in Fig.2(4). By the minimality of G, $G - uv$ has a strong $(3\Delta-1)$-edge-coloring ϕ. Then, uv in G has at most $2\Delta+3$ colored edges in its $2-$neighborhood. So, $|A(uv)| \geq 3\Delta-1-(2\Delta+3) = \Delta-4 \geq 1$. Thus, we can extend ϕ to a strong $(3\Delta - 1)$-edge-coloring of G, a contradiction. □

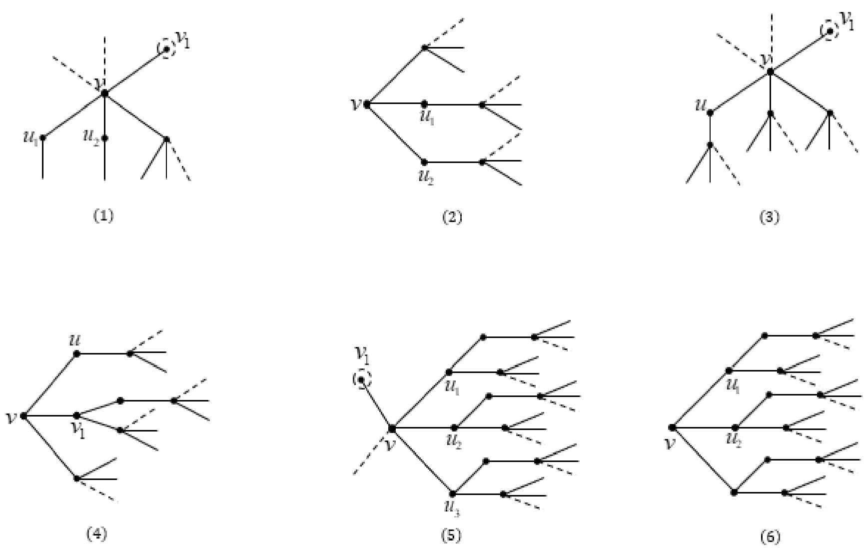

Fig.2

Lemma 2.6. *Let v be a strong $3-vertex$ in G^*. Then, v has at most two weak $3-neighbors$ in G^*.*

Proof. Suppose otherwise that the three neighbors of v are all weak $3-$neighbors in G^*. Let u_1, u_2, u_3 be three weak $3-$neighbors of v in G^*. By Lemma 2.4, $d_G(u_1) = d_G(u_2) = d_G(u_3) = 3$. Assume that $d_G(v) > d_{G^*}(v)$. Then, v has at least one $1-$neighbor v_1 in G, as shown in Fig.2(5). By the minimality of G,

$G - v_1$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. It is easy to see that vv_1 has at most $\Delta + 5$ colored edges in its $2-$neighborhood. Thus, $|A(vv_1)| \geq 2\Delta - 6 \geq 4$. Then, we can color vv_1 with one of the available colors, a contradiction. Therefore, $d_G(v) = d_{G^*}(v) = 3$, as shown in Fig.2(6). Let ϕ be a strong $(3\Delta - 1)$-edge-coloring of $G - vu_1$. Note that vu_1 in G has at most $\Delta + 8$ colored edges in its $2-$neighborhood. Hence, $|A(vu_1)| \geq 2\Delta - 9 \geq 1$, which implies that vu_1 has at least one available color, a contradiction. $\square$

Lemma 2.7. *Every $4 - vertex$ v in G^* has at most three $2 - neighbors$.*

Proof. Suppose otherwise that the four neighbors of v are all $2-$vertices. Let u be one of neighbors of v in G^*, as shown in Fig.3(1). By the minimality of G, $G - uv$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. Note that uv in G has at most $2\Delta + 2$ colored edges in its $2-$neighborhood. Thus, $|A(uv)| \geq 3\Delta - 1 - (2\Delta + 2) = \Delta - 3 \geq 2$. So, ϕ can be extended to a strong $(3\Delta - 1)$-edge-coloring of G, a contradiction. $\square$

If a $4-vertex$ has just three $2-neighbors$, we call it a $4_3-vertex$. Otherwise, if a $4 - vertex$ has at most two $2 - neighbors$, we call it a $4_2 - vertex$.

Lemma 2.8. *If v is a $4_3 - vertex$ in G^*, then $d_G(v) = d_{G^*}(v) = 4$.*

Proof. Suppose otherwise that that $d_G(v) > 4$. Then, v has at least one $1-$neighbor in G, denoted by v_1. Let u be a $2-$neighbor of v in G^*. By Lemma 2.1, all three $2-$neighbors of v have degree 2 in G, as shown in Fig.3(2). By the minimality of G, $G - v_1$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. Note that vv_1 in G has at most $2\Delta + 1$ colored edges in its $2-$neighborhood. So $|A(vv_1)| \geq \Delta - 2 \geq 3$, which implies that vv_1 has at least one available color, a contradiction. $\square$

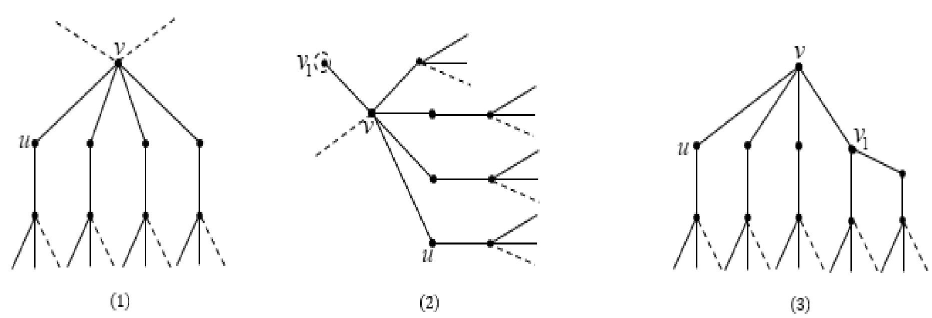

Fig.3

Lemma 2.9. *Assume that v is a $4_3-vertex$ in G^*. Then, v is not adjacent to a weak $3-vertex$.*

Proof. Suppose otherwise that v has a weak $3-$neighbor, say v_1, in G^*. By Lemma 2.7, $d_G(v) = d_{G^*}(v) = 4$. Let u be one of $2-$neighbors of v in G^*, as shown in Fig.3(3). By the minimality of G, $G - uv$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. Then, uv in G has at most $\Delta+7$ colored edges in its $2-$neighborhood. So, $|A(uv)| \geq 3\Delta - 1 - (\Delta + 7) = 2\Delta - 8 \geq 2$. Thus, we can extend ϕ to a strong $(3\Delta - 1)$-edge-coloring of G, a contradiction. □

The total charge remains unchanged when we transfer the charge between vertices and faces. Now we will use discharging method and Euler's formula to get a contradiction and complete the proof of Theorem 1.2. We assign the initial charge $\rho(v) = \frac{1}{2}d_{G^*}(v) - 3$ for each vertex $v \in V(G^*)$ and $\rho(f) = d_{G^*}(f) - 3$ for each face $f \in F(G^*)$.

By Euler's formula, we have the following equality.

$$\sum_{v\in V(G^*)} \rho(v) + \sum_{f\in F(G^*)} \rho(f) = \sum_{v\in V(G^*)} (\frac{1}{2}d(v) - 3) + \sum_{f\in F(G^*)} (d(f) - 3) = -6$$

We will design appropriate discharging rules and redistribute charges among vertices and faces so that the final charges of every vertex and every face are non-negative. The discharging rules are shown as follows.

(R1) Every vertex receives $\frac{4}{7}$ from the incident face.

(R2) Every weak $3 - vertex$ sends $\frac{3}{7}$ to the adjacent $2 - vertex$.

(R3) Every $4^+ - vertex$ sends $\frac{3}{7}$ to the adjacent $2 - vertex$.

(R4) Every $4^+ - vertex$ sends $\frac{3}{28}$ to the adjacent weak $3 - vertex$.

(R5) Every strong $3 - vertex$ sends $\frac{3}{28}$ to the adjacent weak $3 - vertex$.

Let $\rho'(x)$ denote the finial charge of each element x in $V(G^*) \cup F(G^*)$ after the discharging process. We first consider the final charge of each face. By (R1), $\rho'(f) = d_{G^*}(f) - 3 - \frac{4}{7} \times d_{G^*}(f) = \frac{3}{7} \times d_{G^*}(f) - 3 \geq 0$. So, the final charge of each face is at least 0.

Next, we consider the final charge of each vertex v. Let $d_{G^*}(v) = k$. By (R1), it can get $\frac{4}{7} \times d_{G^*}(v) = \frac{4}{7} \times k$ from faces incident to v.

Assume that $d_{G^*}(v) = 2$. By Lemma 2.2, (R2) and (R3), we have that $\rho'(v) = \frac{1}{2} \times 2 - 3 + \frac{4}{7} \times 2 + \frac{3}{7} \times 2 = 0$.

Assume that $d_{G^*}(v) = 3$. If v is a weak $3 - vertex$, by Lemma 2.5, (R4) and (R5), we have that $\rho'(v) = \frac{1}{2} \times 3 - 3 + \frac{4}{7} \times 3 + \frac{3}{28} \times 2 - \frac{3}{7} = 0$. If v is a strong $3 - vertex$, then $\rho'(v) \geq \frac{1}{2} \times 3 - 3 + \frac{4}{7} \times 3 - \frac{3}{28} \times 2 = 0$ by Lemma 2.6 and (R5).

Assume that $d_{G^*}(v) = 4$. If v is a $4_2 - vertex$, then by (R3) and (R4), we have that $\rho'(v) \geq \frac{1}{2} \times 4 - 3 + \frac{4}{7} \times 4 - \frac{3}{7} \times 2 - \frac{3}{28} \times 2 = \frac{3}{14} > 0$. If v is a $4_3 - vertex$, then $\rho'(v) = \frac{1}{2} \times 4 - 3 + \frac{4}{7} \times 4 - \frac{3}{7} \times 3 = 0$ by Lemma 2.8 and (R3).

Assume that $d_{G^*}(v) = k \geq 5$. By (R3) and (R4), we have $\rho'(v) \geq \frac{1}{2} \times k - 3 + \frac{4}{7} \times k - \frac{3}{7} \times k = \frac{9}{14} \times k - 3 \geq \frac{9}{14} \times 5 - 3 = \frac{3}{14} > 0$.

Hence, the final charge of each vertex is at least 0.

By Euler's formula, we can obtain the following contradiction:

$$0 \leq \sum_{v \in V(G^*)} (\frac{1}{2} d(v) - 3) + \sum_{f \in F(G^*)} (d(f) - 3) = -6$$

Therefore, such a minimal counterexample to Theorem 1.2 does not exist.

3. Proof of Theorem 1.3

In this section, we still assume that G is a counterexample to Theorem 1.3 with the $|V(G)|$ minimized, subject to that, assume that $|E(G)|$ is as small as possible. We use the same method as Theorem 1.2 to prove Theorem 1.3. It is obvious that G satisfies the following structures.

Lemma 3.1. (1) $\delta(G^*) \geq 2$. *Moreover,* $d_{G^*}(v) = 2$ *if and only if* $d_G(v) = 2$.

(2) *Let* v *be a* $2 - vertex$ *in* G^*. *Then, both of neighbors of* v *in* G^* *are* $3^+ - neighbor$.

(3) *Let* v *be a* $3 - vertex$ *in* G^*. *Then,* v *has at least two* $3^+ - neighbors$ *in* G^*.

(4) *Let* v *be a weak* $3 - vertex$ *in* G^*. *Then,* $d_G(v) = d_{G^*}(v) = 3$.

(5) *Every* $4 - vertex$ v *in* G^* *has at most three* $2 - neighbors$.

Lemma 3.2. *Assume that* v *is a weak* $3 - vertex$ *in* G^*. *Then,* v *has at least one* $4^+ - neighbor$ *or one strong* $3 - neighbor$ *in* G^*.

Proof. Suppose otherwise that the other two neighbors of v in G^* are weak $3-$vertices, denoted by v_1, v_2. By Lemma 3.1(4), $d_G(v) = d_G(v_1) = d_G(v_2) = 3$. Let u be the $2-$neighbor of v in G^*. By the minimality of G, $G - uv$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. Then, uv in G has at most $\Delta + 6$ colored edges in its $2-$neighborhood. So, $|A(uv)| \geq 3\Delta - 1 - (\Delta + 6) = 2\Delta - 7 \geq 1$. Thus, we can extend ϕ to a strong $(3\Delta - 1)$-edge-coloring of G, a contradiction. □

The total charge remains unchanged when we transfer the charge between vertices and faces. Now we assign the initial charge $\rho(v) = \frac{1}{2} d_{G^*}(v) - 3$ for each vertex $v \in V(G^*)$ and $\rho(f) = d_{G^*}(f) - 3$ for each face $f \in F(G^*)$. The discharging rules are shown as follows.

(R1) Every vertex receives $\frac{4}{7}$ from the incident $7 - face$.

(R2) Every vertex receives $\frac{5}{8}$ from the incident $8^+ - face$.

(R3) Every $4^+ - vertex$ sends $\frac{3}{7}$ to the adjacent $2 - vertex$.

(R4) Every $4^+ - vertex$ sends $\frac{3}{28}$ to the adjacent weak $3 - vertex$.

(R5) Every strong $3 - vertex$ sends $\frac{3}{28}$ to the adjacent weak $3 - vertex$.

(R6) Every weak $3 - vertex$ sends $\frac{3}{7}$ to the adjacent $2 - vertex$.

Let $\rho'(x)$ denote the finial charge of each element x in $V(G^*) \cup F(G^*)$ after the discharging process. We first consider the final charge of each face.

If $d_{G^*}(f) = 7$, then by (R1), we have that $\rho'(f) \geq d_{G^*}(f) - 3 - \frac{4}{7} \times 7 = 0$. If $d_{G^*}(f) \geq 8$, then we have that $\rho'(f) \geq d_{G^*}(f) - 3 - \frac{5}{8} \times d_{G^*}(f) = \frac{3}{8} \times d_{G^*}(f) - 3 \geq 0$ by (R2). Obviously, the final charge of each face is at least 0.

Next, we consider the final charge of each vertex. Let $d_{G^*}(v) = k$. Since there is no adjacent $7 - faces$ in G, by (R1) and (R2), it can at least get $\frac{4}{7} \times \lfloor \frac{k}{2} \rfloor + \frac{5}{8} \times \lceil \frac{k}{2} \rceil$ from 7^+−faces incident to v.

Assume that $d_{G^*}(v) = 2$. By Lemma 3.1, (R3) and (R6), $\rho'(v) \geq \frac{1}{2} \times 2 - 3 + \frac{5}{8} + \frac{4}{7} + \frac{3}{7} \times 2 = \frac{3}{56} > 0$.

Assume $d_{G^*}(v) = 3$. If v is a weak $3 - vertex$, then by Lemma 3.2, (R4) and (R6), we have that $\rho'(v) \geq \frac{1}{2} \times 3 - 3 + \frac{5}{8} \times 2 + \frac{4}{7} - \frac{3}{7} + \frac{3}{28} = 0$. If v is a strong $3 - vertex$, then $\rho'(v) \geq \frac{1}{2} \times 3 - 3 + \frac{5}{8} \times 2 + \frac{4}{7} - \frac{3}{28} \times 3 = 0$ by (R5).

Assume that $d_{G^*}(v) = 4$. By Lemma 3.1, (R3) and (R4), $\rho'(v) \geq \frac{1}{2} \times 4 - 3 + \frac{5}{8} \times 2 + \frac{4}{7} \times 2 - \frac{3}{7} \times 3 - \frac{3}{28} = 0$.

Assume that $d_{G^*}(v) = k \geq 5$. By (R3) and (R4), $\rho'(v) \geq \frac{1}{2} \times k - 3 + \frac{5}{8} \times \lceil \frac{k}{2} \rceil + \frac{4}{7} \times \lfloor \frac{k}{2} \rfloor - \frac{3}{7} \times k \geq \frac{75}{112} \times k - 3 \geq \frac{39}{112} > 0$.

Hence, the final charge of each vertex is at least 0.

By Euler's formula, we can obtain the following contradiction:

$$0 \leq \sum_{v \in V(G^*)} (\frac{1}{2} d(v) - 3) + \sum_{f \in F(G^*)} (d(f) - 3) = -6$$

Therefore, such a minimal counterexample to Theorem 1.3 does not exist.

Acknowledgements

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