$\mathcal{J}$-ω^*-open sets and $\mathcal{J}$-ω^*-topology in ideal topological spaces

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Abstract. The aim of this study is to introduce $\mathcal{J}$-ω^*-open sets as a new set in ideal space which form topology on $\mathcal{X}$ known as $\mathcal{T}_{\mathcal{J}\omega^*}$ (or $\mathcal{J}$-ω^*-topology) which is strictly placed between $\mathcal{T}_{\omega^*}$ and $\mathcal{T}_{\omega}$. Additionally, we investigate the relationships of $\mathcal{J}$-ω^*-open sets with some other classes of sets.

Keywords: Ideal topological spaces, ω-topology, ω^*-topology, $\mathcal{J}$-ω^*-topology, ω-open sets, ω^*-open sets and $\mathcal{J}$-ω^*-open sets.
MSC 2020: 54A05, 54A10, 54A15, 54A20

1. Introduction

Kuratowski [17] and Vaidyanathaswamy [22] introduced the ideal topological space. Jankovic and Hamlett [4] presented the $\mathcal{J}$-open set in 1990. Dontchev [15] introduced *pre*-$\mathcal{J}$-open set in 1996. The concept of α-$\mathcal{J}$-open (resp., semi-$\mathcal{J}$-open, β-$\mathcal{J}$-open) set introduced by Hatir and Noiri [5]. $(\mathcal{X}, \mathcal{T})$ will be used to denote topological space in this paper, without losing any of separation qualities. The set of all real (resp., rational, irrational, and natural) numbers is denoted by R (resp., Q, Irr, N). Also, $P(\mathcal{X})$ mean the collection of all subsets of $\mathcal{X}$. We will denote the closure of any $\mathcal{H} \subseteq \mathcal{X}$ (resp., ω-closure, interior, ω-interior, θ-interior and δ-interior) of $\mathcal{H}$ by $cl\mathcal{H}$ (resp., $cl_\omega\mathcal{H}$, $int\mathcal{H}$,

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$int_\omega\mathcal{H}$, $int_\theta\mathcal{H}$ and $int_\delta\mathcal{H}$). An ideal $\mathcal{J}$ on $(\mathcal{X},\mathcal{T})$ is a nonempty collection of subsets of $\mathcal{X}$ which satisfies the following conditions:

1. If $\mathcal{H}\in\mathcal{J}$, $\mathcal{L}\subseteq\mathcal{H}$ implies $\mathcal{L}\in\mathcal{J}$.
2. If $\mathcal{H}\in\mathcal{J}$, $\mathcal{L}\in\mathcal{J}$ implies $\mathcal{H}\cup\mathcal{L}\in\mathcal{J}$.

If $\mathcal{J}$ is an ideal on $\mathcal{X}$, then $(\mathcal{X},\mathcal{T},\mathcal{J})$ is called an ideal topological space or ideal space. A set operator $(.)^*: P(\mathcal{X})\to P(\mathcal{X})$, called a local function of $\mathcal{H}$ with respect to $\mathcal{T}$ and $\mathcal{J}$ is defined as follows for: $\mathcal{H}\subseteq\mathcal{X}$, $\mathcal{H}^*=\{x\in\mathcal{X}: \mathcal{L}\cap\mathcal{H}\notin\mathcal{J}$ for each $x\in\mathcal{L}\subseteq\mathcal{T}\}$ [22]. Furthermore, in [17], [4] Kuratowski introduced $cl^*(.)$ defined by $cl^*(\mathcal{H})=\mathcal{H}\cup\mathcal{H}^*$ which construct a new topology on $\mathcal{X}$ finer than $\mathcal{T}$, it denoted by $\mathcal{T}^*$ called $*$-topology on $\mathcal{X}$, the members of $\mathcal{T}^*$ are called $\mathcal{T}^*$-open ($*$-open) sets. We will write the interior of $\mathcal{H}$ by $int^*(\mathcal{H})$ in $(\mathcal{X},\mathcal{T}^*)$ for every subset $\mathcal{H}$ of an ideal space$(\mathcal{X},\mathcal{T},\mathcal{J})$. The notion of ω-open set defined by Hdeib [13] several types of ω-open sets are introduced such as (ω^o-open, ω_θ-*open*, ω_δ-open, ω_p-open and ω^*-open) by (Al-Hamary et. al. [24], Ekici et. al. [8], Darwesh [11], Darwesh [10] and Darwesh and Shareef [12]). $(\mathcal{T}_\omega, \mathcal{T}_\theta, \mathcal{T}_{\omega^*})$ denote the families of (ω-open, θ-open, ω^*-open) which they are forms a topology on $\mathcal{X}$. Besides, O. Ravi, P. Sekar and K. Vidhyalakshmi [21] defined the notion of α-$\mathcal{J}_\omega$-open (resp., *pre*-$\mathcal{J}_\omega$-open, b-$\mathcal{J}_\omega$-open, β-$\mathcal{J}_\omega$-open) set in ideal space, which is weaker than the ω-*open* set.

In this study, by using a new notion $\mathcal{J}$-ω^*-open sets we construct a new topology $\mathcal{T}_{\mathcal{J}\omega^*}$ on $(\mathcal{X},\mathcal{T})$. Then, we show that $\mathcal{T}_{\mathcal{J}\omega^*}$ is strictly stronger than $\mathcal{T}_{\omega^*}$ (ω^*-topology) and weaker than $\mathcal{T}_\omega$ (ω-topology). Finally, we discussed several basic properties

2. Preliminaries

Definition 2.1. *A subset $\mathcal{H}$ of a space $(\mathcal{X},\mathcal{T})$ is said to be θ-open [20] (resp., θ_ω-open [23]), if for any $x\in\mathcal{H}$, there is an open set $\mathcal{F}$ containing x such that $x\in\mathcal{F}\subseteq cl\mathcal{F}\subseteq\mathcal{H}$ (resp., $x\in\mathcal{F}\subseteq cl_\omega\mathcal{F}\subseteq\mathcal{H}$).*

Definition 2.2. *A subset $\mathcal{H}$ of a space $(\mathcal{X},\mathcal{T})$ is said to be ω-open [13] (resp., ω^*-open [12], ω^o-open [24], ω_θ-open [8], ω_δ-open [11]), if for each $x\in\mathcal{H}$, there is an open $\mathcal{F}$ set containing x such that $\mathcal{F}\backslash\mathcal{H}$ (resp., $cl\mathcal{F}\backslash\mathcal{H}$, $\mathcal{F}\backslash int\mathcal{H}$, $\mathcal{F}\backslash int_\theta\mathcal{H}$, $\mathcal{F}\backslash int_\delta\mathcal{H}$) is a countable subset of $\mathcal{X}$.*

Definition 2.3. *A subset $\mathcal{H}$ of a space $(\mathcal{X},\mathcal{T})$ is said to be ω_p-open [10], if $\mathcal{H}\subseteq intcl_\omega(\mathcal{H})$.*

Definition 2.4. *A subset $\mathcal{H}$ of an ideal space $(\mathcal{X},\mathcal{T},\mathcal{J})$ is said to be α-$\mathcal{J}$-open [5] (resp., semi-$\mathcal{J}$-open [5], pre-$\mathcal{J}$-open [15], b-$\mathcal{J}$-open [3], strongly-β-$\mathcal{J}$-open[6])set, if $\mathcal{H}\subseteq int(cl^*(int\mathcal{H}))$ (resp., $\mathcal{H}\subseteq cl^*(int\mathcal{H}), \mathcal{H}\subseteq int(cl^*\mathcal{H})$, $\mathcal{H}\subseteq int(cl^*\mathcal{H})\cup cl^*(int\mathcal{H})$, $\mathcal{H}\subseteq cl^*(int(cl^*\mathcal{H}))$).*

The next definitions and result from [21]:

Definition 2.5. *A subset $\mathcal{H}$ of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is said to be α-$\mathcal{J}_\omega$-open (resp., pre-$\mathcal{J}_\omega$-open, b-$\mathcal{J}_\omega$-open, β-$\mathcal{J}_\omega$-open), if $\mathcal{H} \subseteq int_\omega(cl^*(int_\omega \mathcal{H}))$ (resp., $\mathcal{H} \subseteq int_\omega(cl^* \mathcal{H})$, $\mathcal{H} \subseteq int_\omega(cl^* \mathcal{H}) \cup cl^*(int_\omega \mathcal{H})$, $\mathcal{H} \subseteq cl^*(int_\omega(cl^* \mathcal{H}))$.*

Theorem 2.1. *For a subset of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$, the following properties hold:*

1. *Every ω-open set is α-$\mathcal{J}_\omega$ -open.*
2. *Every α-$\mathcal{J}_\omega$- open set is pre-$\mathcal{J}_\omega$ -open.*
3. *Every pre-$\mathcal{J}_\omega$- open set is b-$\mathcal{J}_\omega$ -open.*
4. *Every b-$\mathcal{J}_\omega$- open set is β-$\mathcal{J}_\omega$ -open.*

Proposition 2.1 ([1])**.** *Let $\mathcal{H}$ be a subset of $(\mathcal{X}, \mathcal{T}, \mathcal{J})$. If $\mathcal{J} = \{\emptyset\}$ (resp., $\mathcal{J} = P(\mathcal{X})$), then $\mathcal{H}^* = cl\mathcal{H}$ (resp., $\mathcal{H}^* = \emptyset$) and $cl^*\mathcal{H} = cl\mathcal{H}$ (resp., $cl^*\mathcal{H} = \mathcal{H}$).*

The next definition and result from [25]:

Definition 2.6. *Let $\mathcal{H} \subseteq \mathcal{X}$, is said to be α-ω-open (resp., pre-ω-open, b-ω-open, β-ω-open), if $\mathcal{H} \subseteq int_\omega(cl(int_\omega \mathcal{H}))$ (resp., $\mathcal{H} \subseteq int_\omega(cl\mathcal{H})$, $\mathcal{H} \subseteq int_\omega(cl\mathcal{H}) \cup cl(int_\omega \mathcal{H})$, $\mathcal{H} \subseteq cl(int_\omega(cl\mathcal{H}))$.*

Lemma 2.1. *For a subset of a topological space $(\mathcal{X}, \mathcal{T})$, the following properties hold:*

1. *Every ω-open set is α-ω-open.*
2. *Every α-ω-open set is pre-ω-open.*
3. *Every pre-ω-open set is b-ω-open.*
4. *Every b-ω-open set is β-ω-open.*

Definition 2.7. *A space $(\mathcal{X}, \mathcal{T})$ is defined as:*

1. *Locally countable [18] if every point of $\mathcal{X}$ has a countable open neighbourhood.*
2. *Hyperconnected [14] if each nonempty open subsets of $\mathcal{X}$ is dense in $\mathcal{X}$.*

Definition 2.8. *A subset $\mathcal{H}$ of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is called:*

1. *$*$-dense in itself [16] if and only if $\mathcal{H} \subseteq \mathcal{H}^*$.*
2. *$*$-dense [7] if $cl^*(\mathcal{H}) = \mathcal{X}$.*
3. *$\mathcal{J}$-open set [19] if $\mathcal{H} \subseteq int\mathcal{H}^*$.*

Definition 2.9 ([7])**.** *An ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is said to be $*$-hyperconnected. If $\mathcal{H}$ is $*$-dense, for any nonempty open subset $\mathcal{H}$ of $\mathcal{X}$.*

Lemma 2.2 ([27]). *If $\mathcal{H}$ is $*$- dense in itself, then $\mathcal{H}^* = cl\,(\mathcal{H}) = cl^*(\mathcal{H})$.*

Definition 2.10 ([2]). *An ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is a $R\mathcal{J}$-space if for each $x \in \mathcal{X}$ and every open set $\mathcal{F}$ containing x, there exists an open set $\mathcal{L}$ such that $x \in \mathcal{L} \subseteq cl^*\mathcal{L} \subseteq \mathcal{F}$.*

Lemma 2.3 ([9]). *Let $\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{X}$. Then, $cl^*_{\mathcal{M}}\mathcal{N} = cl^*(\mathcal{N}) \cap \mathcal{M}$.*

3. $\mathcal{J}$-ω^*-open sets with their relations with some other types of sets

This section establishes a new topology in the ideal space and introduces a new set called $\mathcal{J}$-ω^*- open sets. We also investigated their connections to other types of sets.

Definition 3.1. *A subset $\mathcal{H}$ of an ideal topological space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is said to be an $\mathcal{J}$-ω^*- open set, if for each $x \in \mathcal{H}$,* there is an *open* set *$\mathcal{F}$ containing x such that $cl^*\mathcal{F}\backslash\mathcal{H}$ is a countable set. Also, $\mathcal{H}$ is said to be $\mathcal{J}$-ω^*-closed, if $\mathcal{X}\backslash\mathcal{H}$ is $\mathcal{J}$-ω^*-open.*

Remark 3.1. *In any ideal space $(X, \tau, \mathcal{J})$, it is clear that X and $\emptyset$ are always $\mathcal{J}$-ω^*-open sets.*

Theorem 3.1. *A subset $\mathcal{M}$ of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is $\mathcal{J}$-ω^*-open if and only if for every $x \in \mathcal{M}$, there is an open set $\mathcal{F}_x$ containing x and a countable set $\mathcal{C}_x$ which does not containing x such that $cl^*\mathcal{F}_x\backslash\mathcal{C}_x \subseteq \mathcal{M}$.*

Proof. Let $\mathcal{M}$ be an $\mathcal{J}$-ω^*-open subset of $\mathcal{X}$ and $x \in \mathcal{M}$, there exists $\mathcal{F}_x \in \mathcal{T}$ such that $x \in \mathcal{F}_x$ and $cl^*\mathcal{F}_x\backslash\mathcal{M}$ is a countable set. Then, $\mathcal{C}_x = cl^*\mathcal{F}_x\backslash\mathcal{M}$ is a countable set and $x \notin \mathcal{C}_x$. It remains to show that $cl^*\mathcal{F}_x\backslash\mathcal{C}_x \subseteq \mathcal{M}$. Then, $cl^*\mathcal{F}_x\backslash\mathcal{C}_x = cl^*\mathcal{F}_x\backslash(cl^*\mathcal{F}_x\backslash\mathcal{M}) = cl^*\mathcal{F}_x\backslash(cl^*\mathcal{F}_x \cap \mathcal{X}\backslash\mathcal{M}) = (cl^*\mathcal{F}_x\backslash cl^*\mathcal{F}_x) \cup (cl^*\mathcal{F}_x \cap \mathcal{M})$. Hence, $cl^*\mathcal{F}_x\backslash\mathcal{C}_x \subseteq \mathcal{M}$.

Conversely, let $x \in \mathcal{M}$, consequently by our hypothesis, there are open set $\mathcal{F}_x$ containing x and countable set $\mathcal{C}_x$ such that $x \notin \mathcal{C}_x$ and $cl^*\mathcal{F}_x\backslash C_x \subseteq \mathcal{M}$. This implies, $cl^*\mathcal{F}_x\backslash\mathcal{M} \subseteq \mathcal{C}_x$. Therefore, $\mathcal{M}$ is an $\mathcal{J}$-ω^*-open subset of $\mathcal{X}$. □

Theorem 3.2. *If $\mathcal{M}$ is an $\mathcal{J}$-ω^*-closed subset of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$, then $\mathcal{M} \subseteq int^*\mathcal{H} \cup \mathcal{C}$ for a countable set $\mathcal{C}$ and a closed set $\mathcal{H}$.*

Proof. If $\mathcal{M}$ is equal to $\mathcal{X}$. Putting $\mathcal{H} = \mathcal{M}$ and $\mathcal{C} = \emptyset$, we get $\mathcal{M} \subseteq int^*\mathcal{H} \cup \mathcal{C}$. Otherwise, let x be an arbitrary point in $\mathcal{X}$ such that $x \notin \mathcal{M}$. Since $\mathcal{X}\backslash\mathcal{M}$ is $\mathcal{J}$ -ω^*-open, consequently by Theorem 3.1, there exists $\mathcal{F} \in \mathcal{T}$ containing x and a countable set $\mathcal{C}_x$ which does not contains x such that $cl^*\mathcal{F}_x\backslash\mathcal{C}_x \subseteq \mathcal{X}\backslash\mathcal{M}$. Then, $\mathcal{H} = \mathcal{X}\backslash\mathcal{F}$ and $\mathcal{C}$ are the requisite closed set and a countable set. □

Theorem 3.3. *A subset $\mathcal{M}$ of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is $\mathcal{J}$ -ω^*-closed if and only if $\mathcal{M} = \mathcal{X}$ or for any x not belong to $\mathcal{M}$, there is a closed set $\mathcal{H}$ and a countable set $\mathcal{C}$ such that $\mathcal{M} \subseteq int^*\mathcal{H} \cup \mathcal{C}$.*

Proof. Let $\mathcal{M}$ be an $\mathcal{J}$-ω^*-*closed* subset of $\mathcal{X}$. Then, either $\mathcal{M} = \mathcal{X}$ or $\mathcal{M} \subset \mathcal{X}$. If $\mathcal{M} = \mathcal{X}$, then there is nothing to prove, otherwise $\mathcal{M}$ is a proper $\mathcal{J}$ -ω^*-*closed* subset of $\mathcal{X}$, then by Theorem 3.2, a closed set $\mathcal{H}$ and a countable set $\mathcal{C}$ exist such that $\mathcal{M} \subseteq int^*\mathcal{H} \cup \mathcal{C}$.

Conversely, if $\mathcal{M} = \mathcal{X}$, then it is $\mathcal{J}$-ω^*-*closed.* Otherwise, let for each $x \in \mathcal{X}\backslash\mathcal{M}$, there is a closed set $\mathcal{H}$ and a countable set $\mathcal{C}$ such that $\mathcal{M} \subseteq int^*\mathcal{H} \cup \mathcal{C}$. Then, $\mathcal{F} = \mathcal{X}\backslash\mathcal{H}$ is an *open* subset of $\mathcal{X}$ which contains x and $cl^*\mathcal{F}\backslash\mathcal{C} = cl^*(\mathcal{X}\backslash\mathcal{H})\backslash\mathcal{C}$. But, $cl^*(\mathcal{X}\backslash\mathcal{H}) = (\mathcal{X}\backslash int^*\mathcal{H})$ [26] thus, $cl^*\mathcal{F}\backslash\mathcal{C} = cl^*(\mathcal{X}\backslash\mathcal{H})\backslash\mathcal{C} = (\mathcal{X}\backslash int^*\mathcal{H})\backslash\mathcal{C} = (\mathcal{X}\backslash int^*\mathcal{H}) \cap (\mathcal{X}\backslash\mathcal{C}) = \mathcal{X}\backslash(int^*\mathcal{H} \cup \mathcal{C}) \subseteq \mathcal{X}\backslash\mathcal{M}$. This by Theorem 3.1 implies, $\mathcal{X}\backslash\mathcal{M}$ is $\mathcal{J}$-ω^*-open. Thus, $\mathcal{M}$ is $\mathcal{J}$-ω^*-*closed.* □

Theorem 3.4. *The intersection of two $\mathcal{J}$-ω^*-open sets is $\mathcal{J}$-ω^*-open.*

Proof. Let $\mathcal{M}$ and $\mathcal{P}$ be two $\mathcal{J}$-ω^*- *open* sets. If $\mathcal{M} \cap \mathcal{P} = \emptyset$, then there is nothing to prove. Otherwise, for $x \in \mathcal{M} \cap \mathcal{P}$, there are two open sets $\mathcal{G}$ and $\mathcal{L}$ containing x such that $cl^*\mathcal{G}\backslash\mathcal{M}$ and $cl^*\mathcal{L}\backslash\mathcal{P}$ are countable sets. Since $cl^*(\mathcal{G} \cap \mathcal{L}) \backslash (\mathcal{M} \cap \mathcal{P}) \subseteq (cl^*\mathcal{G} \cap cl^*\mathcal{L}) \cap (\mathcal{X}\backslash(\mathcal{M} \cap \mathcal{P}) = (cl^*\mathcal{G} \cap cl^*\mathcal{L}) \cap ((\mathcal{X}\backslash\mathcal{M}) \cup (\mathcal{X}\backslash\mathcal{P})) \subseteq (cl^*\mathcal{G} \cap (\mathcal{X}\backslash\mathcal{M})) \cup (cl^*\mathcal{L} \cap (\mathcal{X}\backslash\mathcal{P})) = (cl^*\mathcal{G}\backslash\mathcal{M}) \cup (cl^*\mathcal{L}\backslash\mathcal{P})$. Which means that, $cl^*(\mathcal{G} \cap \mathcal{L}) \backslash (\mathcal{M} \cap \mathcal{P})$ is countable. Hence, $\mathcal{M} \cap \mathcal{P}$ is $\mathcal{J}$-ω^*-open. □

Corollary 3.1. *The union of two $\mathcal{J}$-ω^*-closed sets is $\mathcal{J}$-ω^*-closed.*

Proof. It follows Theorem 3.4. □

Theorem 3.5. *The union (resp., intersection) of each family of $\mathcal{J}$-ω^*-open (resp., $\mathcal{J}$ -ω^*- closed) sets in any ideal topological space is $\mathcal{J}$ -ω^*-open (resp., $\mathcal{J}$ -ω^*-closed).*

Proof. Let $\{\mathcal{M}_\gamma\colon \gamma \in \Delta\}$ be any each family of $\mathcal{J}$-ω^*-open sets and $x \in \bigcup_{\gamma\in\Delta} \mathcal{M}_\gamma$. Then, there is $\gamma_\circ \in \Delta$ and an open set $\mathcal{F}$ such that $x \in \mathcal{F} \cap \mathcal{M}_{\gamma_\circ}$ and $cl^*\mathcal{F}\backslash\mathcal{M}_{\gamma_\circ}$ is a countable set. Since, $cl^*\mathcal{F}\backslash(\bigcup_{\gamma\in\Delta} \mathcal{M}_\gamma) \subseteq cl^*\mathcal{F}\backslash\mathcal{M}_{\gamma_\circ}$. Thus, $\bigcup_{\gamma\in\Delta} \mathcal{M}_\gamma$ $\mathcal{J}$ -ω^*-open. □

We denote $\mathcal{T}_{\mathcal{J}\omega^*}$ to the family of all $\mathcal{J}$-ω^*-open.

Corollary 3.2. *Let $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ be an ideal space. Then, $\mathcal{T}_{\mathcal{J}\omega^*}$ form topology on $\mathcal{X}$. Hence, $(X, \mathcal{T}_{\mathcal{J}\omega^*}, \mathcal{J})$ is an ideal topological space.*

Proof. It follows from Remark 3.1, Theorem 3.4 and Theorem 3.5. □

The new topology of the Corollary 3.2, known as $\mathcal{J}$-ω^*-topology.

Proposition 3.1. *Every ω^*-open set in any ideal space $(\mathcal{X}, \mathcal{T}, J)$ is $\mathcal{J}$-ω^*-open.*

Proof. Let $\mathcal{M}$ be an ω^*-open subset of $\mathcal{X}$ and x belong to $\mathcal{M}$. Consequently, there is an open set $\mathcal{F}$ containing x such that $cl\mathcal{F}\backslash\mathcal{M}$ is a countable set. Since $cl^*\mathcal{F} \subseteq cl\mathcal{F}$, then $cl^*\mathcal{F}\backslash\mathcal{M} \subseteq cl\mathcal{F}\backslash\mathcal{M}$ and hence $cl^*\mathcal{F}\backslash\mathcal{M}$ is a countable set. Therefore, $\mathcal{M}$ is an $\mathcal{J}$-ω^*-open set. □

The converse of Proposition 3.1, on the other hand does not have to be correct as demonstrated by the following example:

Example 3.1. In $(R, \mathcal{T})$ with $\mathcal{T} = \{\emptyset, Q, R\}$ and $\mathcal{J} = P(R)$. Then, the set $\mathcal{M} = Q$ is not ω^*-open but it is $\mathcal{J}$-ω^*-open. Since, for each $x \in Q$ there is $Q \in \mathcal{T}$ with $cl^*Q = Q$ but $clQ = R$.

Proposition 3.2. *In any ideal space* $(\mathcal{X}, \mathcal{T}, \mathcal{J})$, $\mathcal{T}_\theta \subseteq \mathcal{T}_{\omega^*} \subseteq \mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_\omega$.

Proof. From [[12], Theorem 3.2] we have $\mathcal{T}_\theta \subseteq \mathcal{T}_{\omega^*}$ and by Proposition 3.1, $\mathcal{T}_{\omega^*} \subseteq \mathcal{T}_{\mathcal{J}\omega^*}$ it remains to show that $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_\omega$. Let $\mathcal{M}$ be an $\mathcal{J}$-ω^*-open set. If $\mathcal{M}$ is empty, then $\mathcal{M} \in \mathcal{T}_\omega$. Otherwise, for any arbitrary point x in $\mathcal{M}$; there exists $\mathcal{F} \in \mathcal{T}$ containing x such that $cl^*\mathcal{F} \backslash \mathcal{M}$ is a countable set. Since $\mathcal{F} \backslash \mathcal{M} \subseteq cl^*\mathcal{F} \backslash \mathcal{M}$. Therefore, $\mathcal{F} \backslash \mathcal{M}$ is countable, implying that $\mathcal{M} \in \mathcal{T}_\omega$. Hence, $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_\omega$. □

In general, the converse of Proposition 3.2, is not true. As illustrated in the following examples:

Example 3.2. Let $\mathcal{X} = \{a, b, c, d\}$ with $\mathcal{T} = \{\emptyset, \mathcal{X}, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{J} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. The set $\mathcal{M} = \{a, d\}$ is an $\mathcal{J}$ -ω^*-open set, but it is not θ-open.

Example 3.3. In the space R with topology $\mathcal{T} = \{\emptyset, R, Q\}$ and $\mathcal{J} = \{\emptyset\}$, the set $\mathcal{M} = Q \in \mathcal{T}_\omega$. But, $clQ = cl^*Q = R$ then $cl^*Q \backslash \mathcal{M} = Irr$ is uncountable. Hence, M is not $\mathcal{J}$ -ω^*-open.

Proposition 3.3. *If* $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ *is any ideal space such that* $\mathcal{X}$ *is a locally countable space, then* $\mathcal{T}_{\mathcal{J}\omega^*} = P(\mathcal{X})$.

Proof. Let $\mathcal{M}$ be any subset of $\mathcal{X}$. If $\mathcal{M} = \emptyset$, then $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. Otherwise, for any $x \in \mathcal{M}$, the set $\mathcal{X}$ is open which contain x, and $cl^*\mathcal{X} = \mathcal{X}$ is countable, so $cl^*\mathcal{X} \backslash \mathcal{M}$ is also countable, therefore, $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. Hence, $\mathcal{T}_{\mathcal{J}\omega^*} = P(\mathcal{X})$. □

Corollary 3.3. *Every* $\mathcal{J}$-ω^**-open set is* α-$\mathcal{J}_\omega$*-open (resp. pre-*$\mathcal{J}_\omega$*-open, b-*$\mathcal{J}_\omega$*-open and* β-$\mathcal{J}_\omega$*-open).*

Proof. Proposition 3.2 and Theorem 2.1 provide the proof. □

The following example shows that the converse of Corollary 3.3, is not true:

Example 3.4. From [[21], Example 3.1], consider R be a space with $\mathcal{T} = \{\emptyset, R, Q\}$ and $\mathcal{J} = \{\emptyset\}$. Then, $\mathcal{N} = Q \cup \{\sqrt{2}\}$ is an α-$\mathcal{J}_\omega$ *–open* set, since $int_\omega \mathcal{N} = Q$, $cl^*(int_\omega \mathcal{N}) = cl(Q) = R$. Therefore, $\mathcal{N} \subseteq int_\omega(cl^*(int_\omega \mathcal{N}))$. Thus, $\mathcal{N}$ is *pre*-$\mathcal{J}_\omega$-open (*resp. b*-$\mathcal{J}_\omega$ -open and β-$\mathcal{J}_\omega$ -open). But, $\mathcal{N} \notin \mathcal{T}_{\mathcal{J}\omega^*}$.

Corollary 3.4. *Every* $\mathcal{J}$-ω^**-open set is* α-ω*-open (resp., pre-*ω*-open, b-*ω*-open and* β-ω*–open).*

Proof. It follows from Proposition 3.2 and Lemma 2.1. □

However, as shown in the following example, the converse of Corollary 3.4 is incorrect:

Example 3.5. In the space R with $\mathcal{T} = \{\emptyset, R, Q\}$ and ideal $\mathcal{J} = \{\emptyset\}$. If the set $\mathcal{P} = Q$, then $\mathcal{P}$ is α-ω-open (*resp.* *pre*-ω-open, b-ω-open and β-ω-open). Since $int_\omega \mathcal{P} = Q$, $clint_\omega \mathcal{P} = R$, $int_\omega (clint_\omega \mathcal{P}) = R$. Thus, $\mathcal{P} \subseteq int_\omega (clint_\omega \mathcal{P})$. This implies that $\mathcal{P}$ is α-ω–open and from Lemma 2.1, $\mathcal{P}$ is *pre*-ω-open, b-ω-open and β-ω-open. But, $\mathcal{P} \notin \mathcal{T}_{\mathcal{J}\omega^*}$.

The examples below show that the concept of $\mathcal{J}$ -ω^*-open is independent of the classes open (*preopen*, $\mathcal{J}$-*open*, α-$\mathcal{J}$-open, *pre*-$\mathcal{J}$-open, *semi*-$\mathcal{J}$-open, b-$\mathcal{J}$-open and *strongly* β-$\mathcal{J}$-open) sets.

Example 3.6. 1. In the space R with $\mathcal{T} = \{\emptyset, R, Q\}$ and ideal $\mathcal{J} = \{\emptyset\}$. If the set $\mathcal{P} = Q$, then $\mathcal{P}$ is open (*peropen*, α-$\mathcal{J}$-*open*, *pre*-$\mathcal{J}$-open, *semi*-$\mathcal{J}$-open, b-$\mathcal{J}$-open and *strongly* β-$\mathcal{J}$-open). But, $\mathcal{P} \notin \mathcal{T}_{\mathcal{J}\omega^*}$. Since for each $x \in Q$, there is $Q \in \mathcal{T}$ and $cl\,(Q) = cl^*\,(Q) = R$.

2. Let $\mathcal{X} = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, \{a\}, \mathcal{X}\}$ and $\mathcal{J} = \{\emptyset, \{a\}\}$. Then, the set $\mathcal{M} = \{a, c\}$ is $\mathcal{J}$ -ω^*-open but not *open*, $\mathcal{J}$-open, *semi*-$\mathcal{J}$-open and α-$\mathcal{J}$-open.

3. Let $\mathcal{X} = \{a, b\}$, $\mathcal{T} = \{\emptyset, \{a\}, \mathcal{X}\}$ and $\mathcal{J} = \{\emptyset\}$. Then, the set $\mathcal{M} = \{b\}$ is $\mathcal{J}$ -ω^*-open but not *pre*-$\mathcal{J}$-open, b-$\mathcal{J}$-open and *strongly*β-$\mathcal{J}$-open.

4. In R with usual topology $\mathcal{T}_u$ and $\mathcal{J} = \mathcal{F}$ (all finite subsets of R's ideal). Then, $\mathcal{P} = Q$ is $\mathcal{J}$-open since $\mathcal{P}^* = Q^* = R$. Implies that, $\mathcal{P} \subseteq int(\mathcal{P}^*)$ but $\mathcal{P}$ is not $\mathcal{J}$-ω^*-open since $cl^*\,(Q) = R$ and $cl^*\,(Q) \backslash Q$ is not countable.

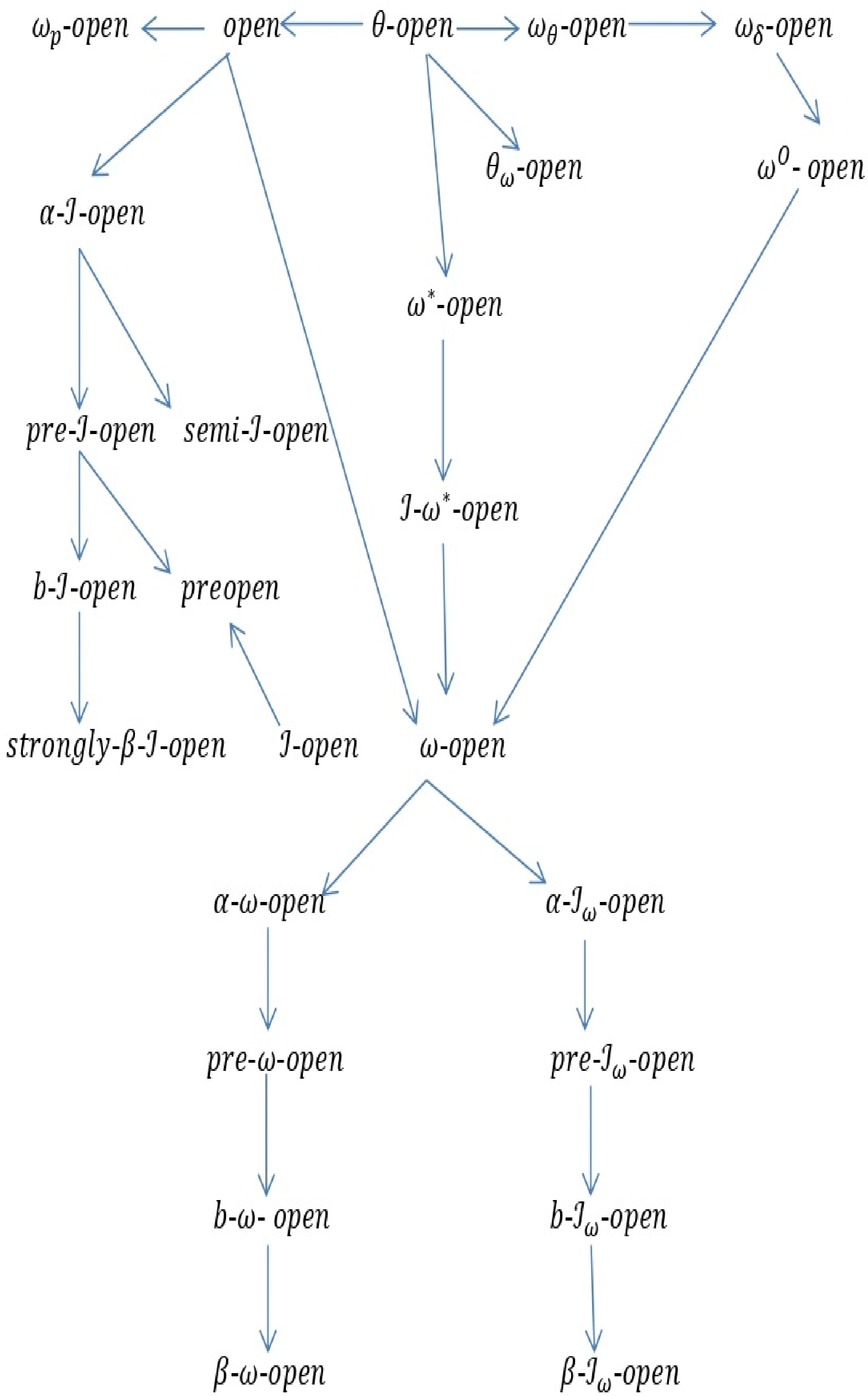
We have examples that demonstrate the independence of the notion of $\mathcal{J}$-ω^*-open set with each of the classes ω_p-open, ω_θ-open, ω_δ-open, ω^o -open and θ_ω-open is independent.

Example 3.7. 1. In the space$(R, \mathcal{T})$ with $\mathcal{T} = \{\emptyset, Q, R\}$ and $\mathcal{J} = \{\emptyset\}$. Then, the set $\mathcal{P} = Q$ is ω_p-open (*resp.* ω_θ-open, ω_δ-open, ω^o -open and θ_ω-open). But, $\mathcal{P} \notin \mathcal{T}_{\mathcal{J}\omega^*}$.

2. In the indiscrete space $(R, \mathcal{T}_{ind})$ and $\mathcal{J} = \{\emptyset\}$. Let $\mathcal{P} = R\backslash\{0\}$ is $\mathcal{J}$ -ω^*-open but it is not ω_θ-open, ω_δ-open, ω^o -open and θ_ω-open.

3. In Example 3.6.(3), assume $\mathcal{M} = \{b\}$ is $\mathcal{J}$ -ω^*-open but not ω_p-open since $\{b\} \notin \mathcal{T}$. Since $cl_\omega\,(\mathcal{M}) = \{b\}$, then $intcl_\omega\,(\mathcal{M}) = \emptyset$. This implies that, $\mathcal{M} \nsubseteq intcl_\omega\,(\mathcal{M})$.

Thus, from Proposition 3.1, Proposition 3.2, Corollary 3.3, Corollary 3.4, Example 3.6 and Example 3.7 we have the following diagram:



ωp-open
open
θ-open
ωθ-open
ωδ-open
θω-open
ω0- open
α-J-open
ω*-open
pre-J-open
semi-J-open
J-ω*-open
b-J-open
preopen
strongly-β-J-open
J-open
ω-open
α-ω-open
α-Jω-open
pre-ω-open
pre-Jω-open
b-ω- open
b-Jω-open
β-ω-open
β-Jω-open

4. Some other properties of $\mathcal{J}$-ω^*-open sets

This section investigates further aspects of $\mathcal{J}-\omega^*$-open sets and the topology $\mathcal{T}_{\mathcal{J}\omega^*}$, beginning with the following definition.

Definition 4.1. *A point x of a subset $\mathcal{M}$ of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is said to be an $\mathcal{J}$ -$*$-condensation point, if $cl^*\mathcal{F} \cap \mathcal{M}$ is an uncountable set for each open set $\mathcal{F}$ containing x. The set of all $\mathcal{J}$-$*$-condensation points of a set $\mathcal{M}$ is denoted by $\mathcal{J}$-cond$^*(\mathcal{M})$.*

Theorem 4.1. *A subset $\mathcal{M}$ of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is $\mathcal{J}$-ω^*-closed if and only if $\mathcal{J}$-cond$^*(\mathcal{M}) \subseteq \mathcal{M}$.*

Proof. Let $\mathcal{M}$ be an $\mathcal{J}$-ω^*-*closed* subset of $\mathcal{X}$ and $x \in \mathcal{J}$-*cond*$^*(\mathcal{M})$. On contrary we suppose that $x \notin \mathcal{M}$, there exists an open set $\mathcal{F}$ containing x such that $cl^*\mathcal{F}\backslash(\mathcal{X}\backslash\mathcal{M})$ is countable. This means that, $cl^*\mathcal{F}\cap\mathcal{M}$ is countable. Hence, $x \notin \mathcal{J}$-*cond*$^*(\mathcal{M})$ which is a contradiction. Then, $\mathcal{J}$-*cond*$^*(\mathcal{M}) \subseteq \mathcal{M}$.

Conversely, suppose that $\mathcal{J}$-*cond*$^*(\mathcal{M}) \subseteq \mathcal{M}$ and $x \notin \mathcal{M}$, then there is an *open* set $\mathcal{F}$ containing x such that $cl^*\mathcal{F} \cap \mathcal{M}$ is countable. This indicates that, $cl^*\mathcal{F}\backslash(\mathcal{X}\backslash\mathcal{M})$ is countable. So, $\mathcal{X}\backslash\mathcal{M}$ is $\mathcal{J}$-ω^*-open. Therefore, $\mathcal{M}$ is $\mathcal{J}$-ω^*-*closed*. □

Corollary 4.1. *Each countable subset of any ideal space is $\mathcal{J}$-ω^*-closed.*

Proof. If $\mathcal{M}$ is countable, then $\mathcal{J}$-*cond** $(\mathcal{M}) = \emptyset$. So, by Theorem 4.1, $\mathcal{M}$ is $\mathcal{J}$-ω^*-*closed*. □

Proposition 4.1. *If $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is any ideal space, then $\mathcal{T}_{coc} \subseteq \mathcal{T}_{\mathcal{J}\omega^*}$.*

Proof. If $\mathcal{M} \in \mathcal{T}_{coc}$, then $\mathcal{X}\backslash\mathcal{M}$ is countable subset of $\mathcal{X}$, subsequently by Corollary 4.1, $\mathcal{X}\backslash\mathcal{M}$ is $\mathcal{J}$-ω^*-*closed*. Therefore, $\mathcal{M}$ is $\mathcal{J}$ -ω^*-open subset of $\mathcal{X}$. □

Theorem 4.2. *If $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is a $*$- hyperconnected space, then $\mathcal{T}_{\mathcal{J}\omega^*}$ is the co-countable topology on $\mathcal{X}$.*

Proof. Let $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. If $\mathcal{M}$ is an empty set, then $\mathcal{M} \in \mathcal{T}_{coc}$. Otherwise, we choose any arbitrary point x in $\mathcal{M}$, and an open set $\mathcal{F}$ containing x such that $cl^*\mathcal{F}\backslash\mathcal{M} = \mathcal{C}$ where $\mathcal{C}$ is a countable set. Since $\mathcal{X}$ is $*$-hyperconnected, so $cl^*\mathcal{F} = R$ and $\mathcal{M} = R\backslash\mathcal{C}$. Thus, $\mathcal{M} \in \mathcal{T}_{coc}$. Hence, $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_{coc}$. By Proposition 4.1, we have $\mathcal{T}_{coc} \subseteq \mathcal{T}_{\mathcal{J}\omega^*}$. Therefore, $\mathcal{T}_{\mathcal{J}\omega^*} = \mathcal{T}_{coc}$. □

The opposite of Theorem 4.2 is generally incorrect, as illustrated in the next example:

Example 4.1. Let $\mathcal{X} = \{a, b, c, d\}$, $\mathcal{T} = \{\emptyset, \mathcal{X}, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $\mathcal{J} = \{\emptyset, \{b\}\}$. Then, by Proposition 3.3, $\mathcal{T}_{\mathcal{J}\omega^*} = P(\mathcal{X}) = \mathcal{T}_{coc}$. Clearly $(\mathcal{X}, \mathcal{T})$ is not a hyperconnected space and from [[7], Remark 3], $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is not $*$-hyperconnected space.

The following example shows that the requirement $*$- *hyperconnected* in Theorem 4.2, cannot be replaced by *hyperconnected*:

Example 4.2. Let $\mathcal{X} = R$, $\mathcal{T} = \{\emptyset, R, Q\}$ and $\mathcal{J} = P(R)$. Then, the space $(\mathcal{X}, \mathcal{T})$ is *hyperconnected* since $Q \in \mathcal{T}$ and $clQ = R$. As a result, $Q \in \mathcal{T}_{\mathcal{J}\omega^*}$ but $\mathcal{X} \backslash Q = Irr$ which is not countable. Thus, $Q \notin \mathcal{T}_{coc}$. Hence, $\mathcal{T}_{\mathcal{J}\omega^*} \neq \mathcal{T}_{coc}$.

In the Example 4.2, we can see that even if the space $\mathcal{X}$ is *hyperconneceted* $\mathcal{T}_{\mathcal{J}\omega^*} \neq \mathcal{T}_{coc}$, if $\mathcal{J} = \{\emptyset\}$. Consequently, $\mathcal{T}_{\mathcal{J}\omega^*} = \mathcal{T}_{coc}$ is yields the following result:

Corollary 4.2. *If $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is a hyperconnected space and $\mathcal{J} = \{\emptyset\}$, then $\mathcal{T}_{\mathcal{J}\omega^*}$ is the co-countable topology on $\mathcal{X}$.*

Proof. Let $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$ .If $\mathcal{M}$ is empty, then $\mathcal{M} \in \mathcal{T}_{coc}$. Otherwise, if $\mathcal{M} \neq \emptyset$, let $x \in \mathcal{M}$ there is an open set $\mathcal{G}$ containing x such that $cl^*\mathcal{F} \backslash \mathcal{M} = \mathcal{C}$ where $\mathcal{C}$ is a countable set. Since $\mathcal{X}$ is *hyperconnected*, so $cl\mathcal{F} = R$. Since $cl^*\mathcal{F} = cl\mathcal{F}$ then $cl^*\mathcal{F} = R$ and $\mathcal{M} = R \backslash \mathcal{C}$. Thus, $\mathcal{M} \in \tau_{coc}$. Hence, $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_{coc}$. However, according to Proposition 4.1, we have $\mathcal{T}_{coc} \subseteq \mathcal{T}_{\mathcal{J}\omega^*}$. As a result, $\mathcal{T}_{\mathcal{J}\omega^*} = \mathcal{T}_{coc}$. □

Theorem 4.3. *If $\mathcal{T}$ and $\mathcal{P}$ are two topologies on $\mathcal{X}$ and $\mathcal{J}$ is any ideal on $\mathcal{X}$ such that $\mathcal{T} \subseteq \mathcal{P}$, then $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{P}_{\mathcal{J}\omega^*}$.*

Proof. Let $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. If $\mathcal{M} = \emptyset$, then $\mathcal{M} \in \mathcal{P}_{\mathcal{J}\omega^*}$. Otherwise, if $\mathcal{M} \neq \emptyset$. Then, for each $x \in \mathcal{M}$, there is $\mathcal{F} \in \mathcal{T}$ containing x such that $cl^*_{\mathcal{T}}\mathcal{F} \backslash \mathcal{M}$ is a countable subset of $\mathcal{X}$. Since $\mathcal{T} \subseteq \mathcal{P}$ so $\mathcal{F} \in \mathcal{P}$ then $cl^*_{\mathcal{P}}\mathcal{F} \backslash \mathcal{M} \subseteq cl^*_{\mathcal{T}}\mathcal{F} \backslash \mathcal{M}$. Hence, $cl^*_{\mathcal{P}}\mathcal{F} \backslash \mathcal{M}$ is also a countable subset of $\mathcal{X}$. Thus, $\mathcal{M} \in \mathcal{P}_{\mathcal{J}\omega^*}$. Therefore, $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{P}_{\mathcal{J}\omega^*}$. □

Corollary 4.3. *If $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ be an ideal space, then $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq (\mathcal{T}^*)_{\mathcal{J}\omega^*}$.*

Proof. Since $\mathcal{T} \subseteq \mathcal{T}^*$, so by Theorem 4.3, $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq (\mathcal{T}^*)_{\mathcal{J}\omega^*}$. □

However, as the examples below show, the converse of Theorem 4.3 and Corollary 4.3, are not true:

Example 4.3.

1. Let $\mathcal{X} = N$, $\mathcal{J} = \{\emptyset\}$, $\mathcal{T} = \{\emptyset, \{0\}, N\}$ and $\sigma = \{\emptyset, \{1\}, N\}$. Then, by Proposition 3.3, $\mathcal{T}_{\mathcal{I}\omega^*} = P(\mathcal{X}) = \sigma_{\mathcal{I}\omega^*}$, but neither $\mathcal{T} \subseteq \sigma$ nor $\sigma \subseteq \mathcal{T}$.

2. In the space R with topology $\mathcal{T} = \{\emptyset, R, Irr\}$ and $\mathcal{J} = P(R)$. Then, the set $\mathcal{P} = Q \in (\mathcal{T}^*)_{\mathcal{J}\omega^*}$. But, $\mathcal{P} \notin \mathcal{T}_{\mathcal{J}\omega^*}$. Since $R \in \mathcal{T}$, so $cl^*(R) = R$. Implies that, $cl^*(R) \backslash \mathcal{P} = Irr$ which is uncountable.

Proposition 4.2. *Let $(\mathcal{X}, \mathcal{T})$ be a topological space and $\mathcal{J}, \mathcal{K}$ be two ideals on $\mathcal{X}$ in which $\mathcal{J} \subseteq \mathcal{K}$ Then, $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_{\mathcal{K}\omega^*}$.*

Proof. Let $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. If $\mathcal{M} = \emptyset$, then there is nothing to prove. Otherwise, for each $x \in \mathcal{M}$ there exists an open set $\mathcal{F}$ containing x such that $cl^*_{\mathcal{J}}\mathcal{F}\backslash\mathcal{M}$ is countable. Since $cl^*_{\mathcal{K}}\mathcal{F} \subseteq cl^*_{\mathcal{J}}\mathcal{F}$. As a result, $cl^*_{\mathcal{K}}\mathcal{F}\backslash\mathcal{M}$ is also countable. Hence, $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_{\mathcal{K}\omega^*}$. □

The following example demonstrates that the converse of Proposition 4.2 is incorrect:

Example 4.4. Consider $(\mathcal{X}, \mathcal{T})$ where $\mathcal{X} = N$ and $\mathcal{T}$ is the indiscrete topology on $\mathcal{X}$. Let $\mathcal{J} = \{\emptyset, \{1\}\}$ and $\mathcal{K} = \{\emptyset, \{2\}\}$. Then, every $\mathcal{K}$- ω^*-open set is $\mathcal{J}$ -ω^*-*open* when either $\mathcal{K}$ is not subfamily of $\mathcal{J}$.

Corollary 4.4. *Let $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ be an ideal space in which each open subset of it is $*$-dense in itself . Then, $\mathcal{T}_{\omega^*} = \mathcal{T}_{\mathcal{J}\omega^*}$.*

Proof. From Proposition 3.1, we have $\mathcal{T}_{\omega^*} \subseteq \mathcal{T}_{\mathcal{J}\omega^*}$. It remains only to show that $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_{\omega^*}$. Let $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. Then, for each $x \in \mathcal{M}$, there exists an open set $\mathcal{F}$ containing x such that $cl^*\mathcal{F}\backslash\mathcal{M}$ is a countable set. Since $\mathcal{F} \subseteq \mathcal{F}^*$, then according to Lemma 2.2, $cl^*\mathcal{F} = cl\mathcal{F}$. As a result, $cl\mathcal{F}\backslash\mathcal{M}$ is countable. Consequently, $\mathcal{M} \in \mathcal{T}_{\omega^*}$. So, we get $\mathcal{T}_{\omega^*} = \mathcal{T}_{\mathcal{J}\omega^*}$. □

Proposition 4.3. *Let $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ be an ideal space if $\mathcal{J} = \{\emptyset\}$. Then, $\mathcal{T}_{\omega^*} = \mathcal{T}_{\mathcal{J}\omega^*}$.*

Proof. Since $\mathcal{J} = \{\emptyset\}$, then $\mathcal{T} = \mathcal{T}^*$ and $cl^*\mathcal{G} = cl\mathcal{G}$. So, $\mathcal{T}_{\omega^*} = \mathcal{T}_{\mathcal{J}\omega^*}$. □

Theorem 4.4. *Let $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ be a $R\mathcal{J}$-space. Then, $\tau_\omega = \tau_{I\omega^*}$.*

Proof. From Proposition 3.2, it follows we have $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_\omega$. So, it remains only to show that $\mathcal{T}_\omega \subseteq \mathcal{T}_{\mathcal{J}\omega^*}$. Let $\mathcal{M} \in \mathcal{T}_\omega$. Then, for each point x belonging to $\mathcal{M}$, there exists an open set $\mathcal{F}$ containing x such that $\mathcal{F}\backslash\mathcal{M}$ is a countable set. Since $\mathcal{X}$ is $R\mathcal{J}$-space and $x \in \mathcal{F}$, there is an open set $\mathcal{L}$ such that $x \in \mathcal{L} \subseteq cl^*\mathcal{L} \subseteq \mathcal{F}$. Implying that, $cl^*\mathcal{L}\backslash\mathcal{M} \subseteq \mathcal{F}\backslash\mathcal{M}$. So, $cl^*\mathcal{L}\backslash\mathcal{M}$ is a countable set. Hence, $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. Therefore, $\mathcal{T}_\omega = \mathcal{T}_{\mathcal{J}\omega^*}$. □

Theorem 4.5. *Let $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ be an ideal space. Then, $(\mathcal{T}_{\mathcal{J}\omega^*})_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_{\mathcal{J}\omega^*}$.*

Proof. Let $x \in \mathcal{M} \in (\mathcal{T}_{\mathcal{J}\omega^*})_{\mathcal{J}\omega^*}$. Then, by Theorem 3.1, there is $\mathcal{U}_x \in \mathcal{T}_{\mathcal{J}\omega^*}$ and a countable set $\mathcal{H}_x$ such that $x \in \mathcal{U}_x$, $x \notin \mathcal{H}_x$ and $cl^*_{\mathcal{T}_{\mathcal{J}\omega^*}} \mathcal{U}_x\backslash\mathcal{H}_x \subseteq \mathcal{M}$. According to Theorem 3.1, there exists $\mathcal{G}_x \in \mathcal{T}$ and a countable set $\mathcal{K}_x$ such that $cl^*_{\mathcal{T}}\mathcal{G}_x\backslash\mathcal{K}_x \subseteq \mathcal{U}_x$. Since, $\mathcal{H}_x \cup \mathcal{K}_x$ is countable. Also, $cl^*_{\mathcal{T}}\mathcal{G}_x\backslash\mathcal{H}_x \cup \mathcal{K}_x \subseteq \mathcal{U}_x\backslash\mathcal{H}_x \subseteq cl^*_{\mathcal{T}_{\mathcal{J}\omega^*}} \mathcal{U}_x\backslash\mathcal{H}_x \subseteq \mathcal{M}$. Therefore, by Theorem 3.1, $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. □

Theorem 4.6. *Let $\mathcal{Y}$ be a subset of a space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$. Then, $(\mathcal{T}_{\mathcal{J}\omega^*})_{\mathcal{Y}} \subseteq (\mathcal{T}_{\mathcal{Y}})_{\mathcal{J}\omega^*}$.*

Proof. If $\mathcal{M} \in (\mathcal{T}_{\mathcal{J}\omega^*})_{\mathcal{Y}}$. Then, there is an $\mathcal{J}$ -ω^*-open set $\mathcal{F}$ in $\mathcal{X}$ such that $\mathcal{M} = \mathcal{F} \cap \mathcal{Y}$. For each point x in $\mathcal{M}$, there exists an open set $\mathcal{V}$ containing x such that $cl^*\mathcal{V} \backslash \mathcal{F}$ is countable. Since $\mathcal{U} = \mathcal{V} \cap \mathcal{Y} \in \mathcal{T}_{\mathcal{Y}}$, so $x \in \mathcal{U}$ and according to Lemma 2.3, $cl^*_{\mathcal{Y}}\, \mathcal{U} \subseteq cl^*\mathcal{V}$. Thus, $cl^*_{\mathcal{Y}}\, \mathcal{U} \backslash \mathcal{M} = cl^*_{\mathcal{Y}}\, \mathcal{U} \backslash (\mathcal{F} \cap \mathcal{Y}) = cl^*_{\mathcal{Y}}\, \mathcal{U} \backslash \mathcal{F} \subseteq cl^*\mathcal{V} \backslash \mathcal{F}$. This implies that, $cl^*_{\mathcal{Y}}\, \mathcal{U} \backslash \mathcal{M}$ is countable. Therefore, $\mathcal{M} \in (\mathcal{T}_{\mathcal{Y}})_{\mathcal{J}\omega^*}$. Hence, $(\mathcal{T}_{\mathcal{J}\omega^*})_{\mathcal{Y}} \subseteq (\mathcal{T}_{\mathcal{Y}})_{\mathcal{J}\omega^*}$. □

5. Conclusion

The ideal topological space was first introduced by Kuratowski and Vaidyanathaswamy. In ideal space, a variety of open sets were introduced, including the *α-$\mathcal{J}$-open* (*resp.*, *semi-$\mathcal{J}$-open*, β-$\mathcal{J}$-open) set. In this study, we introduce $\mathcal{J}$-ω^*-open sets as a new set in ideal space that constructs a new topology on $(\mathcal{X}, \mathcal{T})$ known as $\mathcal{T}_{\mathcal{J}\omega^*}$ that is stronger than $\mathcal{T}_{\omega^*}$ (ω^*-topology) and weaker than $\mathcal{T}_{\omega}$ (ω-topology). Additionally, we investigate the relationships of $\mathcal{J}$-ω^*-open sets with some other classes of sets. Finally, we discussed several basic properties. In the future, researchers will be able to define topological structures including separation axioms, compactness, and connectedness for the practical application via $\mathcal{J}$-ω^*-open sets.

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Accepted: June 30, 2023