Topological factor groups relative to normal soft int-groups

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Abstract. Given a group $\mathcal{G}$, let $\alpha_{\mathcal{G}}$ be a normal soft int-group in $\mathcal{G}$. We construct the factor group $\mathcal{G}/\alpha$ relative to $\alpha_{\mathcal{G}}$ by defining a congruence relation on $\mathcal{G}$. Using this construction, we establish soft Isomorphism Theorems which generalize the classical group Isomorphism Theorems. Finally, we give some topological structures on $\mathcal{G}$ and $\mathcal{G}/\alpha$.

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1. Introduction

In 1965, the concept of fuzzy set theory has been introduced by Zadeh [18]. The application of fuzzy sets can be found in many branches of mathematics and engineering sciences. Molodtsov in [11] introduced the soft set as a generalization of the fuzzy set to deal with uncertainty. A soft set (fuzzy soft sets, see[4]) is a set-valued function from a set of parameters to the power set(all fuzzy sets) of a universe set. The concept of soft groups (semigroups) is defined in [1, 2] as a collection of subgroups (subsemigroups) of a group (semigroup). In this direction, new types of soft ideals over semigroups are presented in recent works [6, 12, 13]. Cagman et al. [3], based on intersection and inclusion relation of sets, defined the soft int-group which are unlike that in [1, 14]. Some properties of soft int-groups and normal soft int-groups are introduced in [8, 9, 15]. Ideal theory in semigroups and ordered semigroups based on soft int- (uni-)semigroup is investigated in [5, 7, 17]. In this paper, we introduce a method to construct

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factor groups related to normal soft int-groups. We apply this construction to establish soft Isomorphism Theorems which generalize the classical group Isomorphism Theorems. Topological structures on $\mathcal{G}$ and the factor group $\mathcal{G}/\alpha$ are introduced.

2. Preliminaries

In this Section, we recall some definitions and results of soft set. Throughout our discussion, $\mathcal{U}$ refers to a universal set, $\mathcal{P}(\mathcal{U})$ the power set of $\mathcal{U}$ and $\mathcal{E}$ the set of parameters where $A, B, C, ... \subseteq \mathcal{E}$.

Definition 2.1 ([3]). *A soft set (α, A) over $\mathcal{U}$ is a set of ordered pairs*

$$(\alpha, A) := \{(x, \alpha(x)) : x \in \mathcal{E}, \alpha(x) \in \mathcal{P}(\mathcal{U})\},$$

where $\alpha : \mathcal{E} \longrightarrow \mathcal{P}(\mathcal{U})$ such that $\alpha(x) = \phi$ if $x \notin A$.

From now on, we write α_A instead of (α, A).

Definition 2.2 ([3]). *Let α_A and α_B be soft sets over $\mathcal{U}$. Then, union $\alpha_A \sqcup \alpha_B$ and intersection $\alpha_A \sqcap \alpha_B$ of α_A and α_B are defined by*

$$(\alpha_A \sqcup \alpha_B)(x) = \alpha_A(x) \cup \alpha_B(x), \qquad (\alpha_A \sqcap \alpha_B)(x) = \alpha_A(x) \cap \alpha_B(x)$$

respectively, for all $x \in \mathcal{E}$.

Definition 2.3 ([3]). *Let $\mathcal{G}$ be a group and $\alpha_{\mathcal{G}}$ be a soft set over $\mathcal{U}$. Then, $\alpha_{\mathcal{G}}$ is called a soft intersection group (**soft int-group**) over $\mathcal{U}$ if*

1. *$\alpha_{\mathcal{G}}(xy) \supseteq \alpha_{\mathcal{G}}(x) \cap \alpha_{\mathcal{G}}(y)$ for all $x, y \in \mathcal{G}$, and*
2. *$\alpha_{\mathcal{G}}(x^{-1}) = \alpha_{\mathcal{G}}(x)$ for all $x \in \mathcal{G}$.*

Or, equivalenty, if $\alpha_{\mathcal{G}}(xy^{-1}) \supseteq \alpha_{\mathcal{G}}(x) \cap \alpha_{\mathcal{G}}(y)$ for all $x, y \in \mathcal{G}$.

Theorem 2.1 ([8]). *Let $\alpha_{\mathcal{G}}$ be a soft int-group and $x, y \in \mathcal{G}$. Then*

1. *$\alpha_{\mathcal{G}}(e) \supseteq \alpha_{\mathcal{G}}(x)$,*
2. *$\alpha_{\mathcal{G}}(xy^{-1}) = \alpha_{\mathcal{G}}(e) \Rightarrow \alpha_{\mathcal{G}}(x) = \alpha_{\mathcal{G}}(y)$.*

Definition 2.4 ([3]). *A soft int-group $\alpha_{\mathcal{G}}$ over $\mathcal{U}$ is called **normal**, if for all $x, y \in \mathcal{G}$, it satisfies one of the following equivalent conditions:*

1. *$\alpha_{\mathcal{G}}(xyx^{-1}) \supseteq \alpha_{\mathcal{G}}(y)$,*
2. *$\alpha_{\mathcal{G}}(xyx^{-1}) \subseteq \alpha_{\mathcal{G}}(y)$,*
3. *$\alpha_{\mathcal{G}}(xy) = \alpha_{\mathcal{G}}(yx)$.*

Definition 2.5 ([9]). *Let α_A be a soft set over $\mathcal{U}$ and $V \in \mathcal{P}(\mathcal{U})$). Then, V-inclusion of the soft set α_A, denoted by α^V is defined as*

$$\alpha^V = \{x \in A : \alpha(x) \supseteq V\}.$$

It is proved in [9] that "A soft set $\alpha_{\mathcal{G}}$ is a (normal) soft int-group over $\mathcal{U}$ iff for all $V \in \mathcal{P}(\mathcal{U})$, α^V is either empty or a (normal) subgroup of $\mathcal{G}$".

Definition 2.6 ([15]). *Let $f : \mathcal{G} \to \mathcal{H}$ be a function between groups. Then, the soft image $f(\alpha_{\mathcal{G}})$ of a soft set $\alpha_{\mathcal{G}}$ and the soft preimage $f^{-1}(\beta_{\mathcal{H}})$ of a soft set $\beta_{\mathcal{H}}$ under f are defined as*

$$f(\alpha_{\mathcal{G}})(y) = \begin{cases} \bigcup\{\alpha_{\mathcal{G}}(x) : x \in \mathcal{G}, f(x) = y\}, & \text{for } y \in f(\mathcal{G}), \\ \phi, & \text{otherwise,} \end{cases}$$

and

$$f^{-1}(\beta_{\mathcal{H}})(x) = \beta_{\mathcal{H}}(f(x)), \quad \forall x \in \mathcal{G}.$$

Theorem 2.2 ([15]). *If $f : \mathcal{G} \to \mathcal{H}$ is an epimorphism of groups, and $\alpha_{\mathcal{G}}$ is a normal soft int-group, then $f(\alpha_{\mathcal{G}})$ is a normal soft int-group.*

3. Construction of the factor group

In this Section, we represent our main findings. Given a group $\mathcal{G}$ we denote the identity element of $\mathcal{G}$ by $e_{\mathcal{G}}$, and the set of all soft int-groups over $\mathcal{U}$ with $\mathcal{G}$ as a set of parameters by $\mathcal{S}(\mathcal{G},\mathcal{U})$.

Recall that an equivalence relation δ on $\mathcal{G}$ is called a **congruence relation** if

$$x\delta y \Rightarrow xz\delta yz,\ zx\delta zy$$

for all $x, y, z \in \mathcal{G}$.

Let $\alpha_{\mathcal{G}} \in \mathcal{S}(\mathcal{G},\mathcal{U})$ be a normal soft int- group. For any $x, y \in \mathcal{G}$, we define the relation R on $\mathcal{G}$ by

$$xRy \Leftrightarrow \alpha_G(xy^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}}).$$

Lemma 3.1. *R is a congruence relation on $\mathcal{G}$.*

Proof. Clearly, R is reflexive and symmetric. Also, R is transitive. Indeed, let xRy and yRz, then $\alpha_{\mathcal{G}}(xy^{-1}) = \alpha_{\mathcal{G}}(yz^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})$. Then, $\alpha_{\mathcal{G}}(xz^{-1}) = \alpha_{\mathcal{G}}(xy^{-1}yz^{-1}) \supseteq \alpha_{\mathcal{G}}(xy^{-1}) \cap \alpha_{\mathcal{G}}(yz^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})$. Hence, $\alpha_{\mathcal{G}}(xz^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})$, which proves that xRz and so R is an equivalence relation. If xRy, then $\alpha_{\mathcal{G}}(xy^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})$. Thus, for all $z \in \mathcal{G}$ we have

$$\alpha_{\mathcal{G}}((xz)(yz)^{-1}) = \alpha_{\mathcal{G}}(xzz^{-1}y^{-1}) = \alpha_{\mathcal{G}}(xy^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}}).$$

Hence, $xzRyz$. Since $\alpha_{\mathcal{G}}$ is normal, we get $\alpha_{\mathcal{G}}((zx)(zy)^{-1}) = \alpha_{\mathcal{G}}(zxy^{-1}z^{-1}) = \alpha_{\mathcal{G}}(xy^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})$. This gives $zxRzy$, and we conclude that R is a congruence relation on $\mathcal{G}$. □

By $[x]_\alpha$, we denote the equivalence class containing $x \in \mathcal{G}$ and by $\mathcal{G}/\alpha$ the corresponding factor set relative to $\alpha_\mathcal{G}$.

Theorem 3.1. *$\mathcal{G}/\alpha$ is a group with the operation $[x]_\alpha[y]_\alpha = [xy]_\alpha$.*

Proof. Straightforward. □

Example 3.1. Assume that $\mathcal{U} = S_3$ is the set of permutations on $\{1, 2, 3\}$. Let $\mathcal{G} = Z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ be the set of parameters. We define a soft set $\alpha_\mathcal{G}$ over U by

$$\begin{aligned}
\alpha_\mathcal{G}(\bar{0}) &= U,\\
\alpha_\mathcal{G}(\bar{1}) &= \{(12), (13), (132)\},\\
\alpha_\mathcal{G}(\bar{2}) &= \{(12), (13), (23), (123), (132)\},\\
\alpha_\mathcal{G}(\bar{3}) &= \{(1), (12), (13), (132)\},\\
\alpha_\mathcal{G}(\bar{4}) &= \{(12), (13), (23), (123), (132)\},\\
\alpha_\mathcal{G}(\bar{5}) &= \{(12), (13), (132)\}.
\end{aligned}$$

Clearly, $\alpha_\mathcal{G}$ is a normal soft int-group over $\mathcal{U}$ and

$$\mathcal{G}/\alpha = \{[\bar{0}]_\alpha, [\bar{1}]_\alpha, [\bar{2}]_\alpha, [\bar{3}]_\alpha, [\bar{4}]_\alpha, [\bar{5}]_\alpha\}.$$

By Definition 2.5, the set $K_{\alpha_\mathcal{G}} = \{x \in \mathcal{G} : \alpha_\mathcal{G}(x) = \alpha_\mathcal{G}(e_\mathcal{G})\}$ is a (normal) subgroup of $\mathcal{G}$ iff $\alpha_\mathcal{G}$ is a (normal)soft int-group over $\mathcal{U}$.

Proposition 3.1. *Let $f : \mathcal{G} \longrightarrow \mathcal{H}$ be a homomorphism of groups and $\alpha_\mathcal{G} \in \mathcal{S}(\mathcal{G},\mathcal{U})$, then*

(*i*) $f(K_{\alpha_\mathcal{G}}) \subseteq K_{f(\alpha_\mathcal{G})}$,

(*ii*) *If $\alpha_\mathcal{G}$ is constant on $Kerf$, then $f(\alpha_\mathcal{G})(f(x)) = \alpha_\mathcal{G}(x)$ for all $x \in \mathcal{G}$.*

Proof. (i) Let $y \in f(K_{\alpha_\mathcal{G}})$, then $y = f(x)$ for some $x \in K_{\alpha_\mathcal{G}}$. Since $\alpha_\mathcal{G}(x) = \alpha_\mathcal{G}(e_\mathcal{G})$, then

$$f(\alpha_\mathcal{G})(y) = \bigcup_{x \in f^{-1}(y)} \{\alpha_\mathcal{G}(x)\} = \alpha_\mathcal{G}(e_\mathcal{G}) = f(\alpha_\mathcal{G})(e_\mathcal{H}).$$

Therefore, $y \in K_{f(\alpha_\mathcal{G})}$.

(ii) Let $y = f(x)$, then $f(\alpha_\mathcal{G})(y) = \bigcup_{z \in f^{-1}(y)}\{\alpha_\mathcal{G}(z)\}$. But $f(zx^{-1}) = e_\mathcal{H}$ for all $z \in f^{-1}(y)$. Hence, $\alpha_\mathcal{G}(zx^{-1}) = \alpha_\mathcal{G}(e_\mathcal{G})$ because $\alpha_\mathcal{G}$ is constant on $Kerf$. By Theorem 2.1, we have $\alpha_\mathcal{G}(z) = \alpha_\mathcal{G}(x)$ for all $z \in f^{-1}(y)$. Therefore, $f(\alpha_\mathcal{G})(f(x)) = \alpha_\mathcal{G}(x)$. □

Theorem 3.2. *Let $f : \mathcal{G} \longrightarrow \mathcal{H}$ be an epimorphism of groups and $\alpha_\mathcal{G} \in \mathcal{S}(\mathcal{G},\mathcal{U})$ be normal with $kerf \subseteq K_{\alpha_\mathcal{G}}$, then $\mathcal{G}/\alpha \cong \mathcal{H}/f(\alpha_\mathcal{G})$.*

Proof. From Theorem 2.2, $f(\alpha_{\mathcal{G}})$ is a normal soft int-group and hence $\mathcal{H}/f(\alpha_{\mathcal{G}})$ is a group. We define $\theta : \mathcal{G}/\alpha \longrightarrow \mathcal{H}/f(\alpha_{\mathcal{G}})$, such that $\theta([x]_\alpha) = [f(x)]_{f(\alpha_{\mathcal{G}})}$. Firstly, θ is well defined since $[x]_\alpha = [y]_\alpha$ implies $\alpha_{\mathcal{G}}(xy^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})$. Since $kerf \subseteq K_{\alpha_{\mathcal{G}}}$, then $\alpha_{\mathcal{G}}$ is constant on $kerf$, and by Proposition 3.1, we have

$$f(\alpha_{\mathcal{G}})(f(xy^{-1})) = f(\alpha_{\mathcal{G}})(f(e_{\mathcal{G}})).$$

Then, $f(\alpha_{\mathcal{G}})(f(x)f(y)^{-1}) = f(\alpha_{\mathcal{G}})(e_{\mathcal{H}})$, and so $[f(x)]_{f(\alpha_{\mathcal{G}})} = [f(y)]_{f(\alpha_{\mathcal{G}})}$. Therefore, θ is well defined.

Secondly, θ is a homomorphism because:

$$\begin{aligned}\theta([x]_\alpha[y]_\alpha) = \theta([xy]_\alpha) &= [f(xy)]_{f(\alpha_{\mathcal{G}})} = [f(x)f(y)]_{f(\alpha_{\mathcal{G}})}\\ &= [f(x)]_{f(\alpha_{\mathcal{G}})}[f(y)]_{f(\alpha_{\mathcal{G}})} = \theta([x]_\alpha)\theta([y]_\alpha).\end{aligned}$$

Now, we show that θ is an epimorphism. For any $[y]_{f(\alpha_{\mathcal{G}})} \in \mathcal{H}/f(\alpha_{\mathcal{G}})$, there exists $x \in \mathcal{G}$ such that $f(x) = y$ (since f is onto). So $\theta([x]_\alpha) = [f(x)]_{f(\alpha_{\mathcal{G}})} = [y]_{f(\alpha_{\mathcal{G}})}$, which means that θ is an epimorphism. Finally, θ is a 1-1 homomorphism since

$$\begin{aligned}[f(x)]_{f(\alpha_{\mathcal{G}})} = [f(y)]_{f(\alpha_{\mathcal{G}})}&\\ &\Longrightarrow f(\alpha_{\mathcal{G}})(f(x)f(y)^{-1}) = f(\alpha_{\mathcal{G}})(e_{\mathcal{H}})\\ &\Longrightarrow f(\alpha_{\mathcal{G}})(f(xy^{-1})) = f(\alpha_{\mathcal{G}})(f(e_{\mathcal{G}}))\\ &\Longrightarrow \alpha_{\mathcal{G}}(xy^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})\\ &\Longrightarrow [x]_\alpha = [y]_\alpha,\end{aligned}$$

which proves that θ is injective. We conclude that θ is an isomorphism. □

Corollary 3.1. *Let $f : \mathcal{G} \longrightarrow \mathcal{H}$ be an onto homomorphism of groups and $\beta_{\mathcal{H}} \in \mathcal{S}(\mathcal{H},\mathcal{U})$ be normal , then $\mathcal{G}/f^{-1}(\beta_{\mathcal{H}}) \cong \mathcal{H}/\beta$.*

Proof. It is known that $f^{-1}(\beta_{\mathcal{H}})$ is a normal soft int-group over $\mathcal{U}$ (see, [15]). Consequently, $\mathcal{G}/f^{-1}(\beta_{\mathcal{H}})$ and $\mathcal{H}/\beta$ are groups. Since f is onto, then $\beta_{\mathcal{H}} = f(f^{-1}(\beta_{\mathcal{H}}))$ [9]. Let x be an element in $kerf$, then $f(x) = f(e_{\mathcal{G}})$, and so $\beta_{\mathcal{H}}(f(x)) = \beta_{\mathcal{H}}(f(e_{\mathcal{G}}))$, that is $f^{-1}(\beta_{\mathcal{H}})(x) = f^{-1}(\beta_{\mathcal{H}})(e_{\mathcal{G}})$. Hence, $x \in K_{f^{-1}(\beta_{\mathcal{H}})}$, which means that $kerf \subseteq K_{f^{-1}(\beta_{\mathcal{H}})}$. By applying Theorem 3.2, we get the desired result. □

For a nonempty subset $\mathcal{A}$ of $\mathcal{G}$, define a map $\chi_{\mathcal{A}} : \mathcal{G} \longrightarrow \mathcal{P}(\mathcal{U})$ as follows:

$$\chi_{\mathcal{A}}(x) = \begin{cases}\mathcal{U}, & \text{if } x \in \mathcal{A},\\ \phi, & \text{otherwise.}\end{cases}$$

Then, $\chi_{\mathcal{A}}$ is a soft set over $\mathcal{U}$, which is called the characteristic soft set (see, [17]).

Theorem 3.3. *$\mathcal{A}$ is a (normal) subgroup of $\mathcal{G}$ if and only if $\chi_{\mathcal{A}}$ is a (normal) soft int-group over $\mathcal{U}$.*

Proof. Assume that $\chi_{\mathcal{A}}$ is a normal soft int-group over $\mathcal{U}$. For any $x, y \in \mathcal{A}$ we have $\chi_{\mathcal{A}}(xy^{-1}) \supseteq \chi_{\mathcal{A}}(x) \cap \chi_{\mathcal{A}}(y) = \mathcal{U}$. Thus, $\chi_A(xy^{-1}) = U$ and $xy^{-1} \in A$. Therefore $\mathcal{A}$ is a subgroup of $\mathcal{G}$. Similarly, for any $y \in \mathcal{A}, x \in \mathcal{G}$ we have $\chi_{\mathcal{A}}(xyx^{-1}) \supseteq \chi_{\mathcal{A}}(y) = \mathcal{U}$. Hence, $\chi_{\mathcal{A}}(xyx^{-1}) = \mathcal{U}$ and $xyx^{-1} \in \mathcal{A}$. This proves that $\mathcal{A}$ is a normal subgroup of $\mathcal{G}$. Conversely, suppose that $\mathcal{A}$ is a normal subgroup of $\mathcal{G}$. If $x, y \in \mathcal{A}$, then $\chi_{\mathcal{A}}(xy^{-1}) = \chi_{\mathcal{A}}(x) = \chi_{\mathcal{A}}(y) = \mathcal{U}$. Hence, $\chi_{\mathcal{A}}(xy^{-1}) = \chi_{\mathcal{A}}(x) \cap \chi_{\mathcal{A}}(y)$. If at least one of x and y is not in $\mathcal{A}$, then at least one of $\chi_{\mathcal{A}}(x)$ and $\chi_{\mathcal{A}}(y)$ is ϕ. Therefore $\chi_{\mathcal{A}}(xy^{-1}) \supseteq \chi_{\mathcal{A}}(x) \cap \chi_{\mathcal{A}}(y)$. Hence, $\chi_{\mathcal{A}}$ is a soft int-group over $\mathcal{U}$. Moreover, for any $x, y \in \mathcal{G}$, if $y \in \mathcal{A}$, then $xyx^{-1} \in \mathcal{A}$ and $\chi_{\mathcal{A}}(xyx^{-1}) = \mathcal{U} = \chi_{\mathcal{A}}(y)$. If $y \notin \mathcal{A}$, then $\chi_{\mathcal{A}}(xyx^{-1}) \supseteq \chi_{\mathcal{A}}(y) = \phi$. Hence, $\chi_{\mathcal{A}}$ is normal. □

Corollary 3.2. *Let $f : \mathcal{G} \longrightarrow \mathcal{H}$ be an onto homomorphism. Then, $\mathcal{G}/\chi_{kerf} \cong \mathcal{H}$.*

Proof. By Theorem 3.3, the characteristic soft set $\chi_{\{e_{\mathcal{H}}\}} \in \mathcal{S}(\mathcal{H},\mathcal{U})$ is normal. It is easy to see that the soft preimage $f^{-1}(\chi_{\{e_{\mathcal{H}}\}})$ is the soft set χ_{kerf}. Hence, the factor group $\mathcal{H}/\chi_{\{e_{\mathcal{H}}\}}$ is isomorphic to $\mathcal{H}$. By applying Corollary 3.1, we get $\mathcal{G}/\chi_{kerf} \cong \mathcal{H}/\chi_{\{e_{\mathcal{H}}\}} \cong \mathcal{H}$. □

In group theory, on the factor group $\mathcal{G}/kerf$ we can define an equivalence relation by $x \sim y \Leftrightarrow xy^{-1} \in kerf$. Easily, one shows that $x \sim y$ iff xRy relative to the normal soft int-group χ_{kerf}. Therefore, we have $\mathcal{G}/\chi_{kerf} \cong \mathcal{G}/kerf$ and Corollary 3.2 becomes the First Group Isomorphism Theorem.

Lemma 3.2. *Let $\mathcal{A}$ be a normal subgroup of $\mathcal{G}$ and $\alpha_{\mathcal{G}}$ a normal soft int-group over $\mathcal{U}$. Then, the restriction $\alpha_{\mathcal{G}} \mid \mathcal{A}$ is a normal soft int-group over $\mathcal{U}$ and $\mathcal{A}/\alpha$ is a normal subgroup of $\mathcal{G}/\alpha$.*

Proof. It is obvious from [9, Theorem 2.13] that $\alpha_{\mathcal{G}} \mid \mathcal{A}$ is a soft int-group. Since $\mathcal{A}$ is normal, $(\alpha_{\mathcal{G}} \mid \mathcal{A})(xy) = (\alpha_{\mathcal{G}} \mid \mathcal{A})(yx)$ for any $x, y \in \mathcal{A}$. Hence, $\alpha_{\mathcal{G}} \mid A$ is a normal soft int-group. If $[a]_\alpha, [b]_\alpha \in A/\alpha$, where $a, b \in \mathcal{A}$, then $([a]_\alpha)([b]_\alpha)^{-1} = ([a]_\alpha)([b^{-1}]_\alpha) = [ab^{-1}]_\alpha \in \mathcal{A}/\alpha$. Hence, $\mathcal{A}/\alpha$ is a subgroup of $\mathcal{G}/\alpha$. If $[a]_\alpha \in \mathcal{A}/\alpha, [x]_\alpha \in \mathcal{G}/\alpha$, where $a \in \mathcal{A}$ and $x \in \mathcal{G}$, then $xax^{-1} \in \mathcal{A}$ and

$$([x]_\alpha)([a]_\alpha)([x]_\alpha)^{-1} = ([x]_\alpha)([a]_\alpha)([x^{-1}]_\alpha) = [xax^{-1}]_\alpha \in \mathcal{A}/\alpha.$$

Thus, $\mathcal{A}/\alpha$ is a normal subgroup of $\mathcal{G}/\alpha$. □

Notation. For $\mathcal{A}, \mathcal{B} \subseteq \mathcal{G}$, we set $\mathcal{A} \cdot \mathcal{B} = \{ab : a \in \mathcal{A}, b \in \mathcal{B}\}$.

Theorem 3.4. *If $\alpha_{\mathcal{G}}$ and $\beta_{\mathcal{G}}$ are normal soft int-groups over $\mathcal{U}$ such that $\alpha_{\mathcal{G}}(e_{\mathcal{G}}) = \beta_{\mathcal{G}}(e_{\mathcal{G}})$, then $(K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}})/\beta_{\mathcal{G}} \cong K_{\alpha_{\mathcal{G}}}/(\alpha_{\mathcal{G}} \sqcap \beta_{\mathcal{G}})$.*

Proof. Before we proceed and for simplicity, put $\gamma_{\mathcal{G}} = \alpha_{\mathcal{G}} \sqcap \beta_{\mathcal{G}}$. Since $\gamma_{\mathcal{G}}$ is a normal soft int-group over $\mathcal{U}$ (see, [9]) and by Lemma 3.2, the restrictions $\beta_{\mathcal{G}} \mid (K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}})$ and $\gamma_{\mathcal{G}} \mid K_{\alpha_{\mathcal{G}}}$ are a normal soft int-groups over $\mathcal{U}$. Then, the factor sets $(K_{\alpha_{\mathcal{G}}} \cdot K_{\beta})/\beta$ and $K_{\alpha_{\mathcal{G}}}/\gamma$ are groups by Lemma 3.1. For any $x \in K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}}, x = ab$, where $a \in K_{\alpha_{\mathcal{G}}}$ and $b \in K_{\beta_{\mathcal{G}}}$, we define $\Omega : (K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}})/\beta \longrightarrow K_{\alpha_{\mathcal{G}}}/\gamma$ such that $f([x]_\beta) = [a]_\gamma$. The mapping Ω is well-defined. Indeed, if $[x]_\beta = [y]_\beta$, where $y = wz \in K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}}$, then

$$\begin{aligned}\beta_{\mathcal{G}}(xy^{-1}) = \beta_{\mathcal{G}}(ab(wz)^{-1}) = \beta_{\mathcal{G}}(abz^{-1}w^{-1}) &= \beta_{\mathcal{G}}(w^{-1}abz^{-1}) \\ &= \beta_{\mathcal{G}}(w^{-1}a(zb^{-1})^{-1}) = \beta_{\mathcal{G}}(e_{\mathcal{G}}).\end{aligned}$$

Hence, $\beta_{\mathcal{G}}(w^{-1}a) = \beta_{\mathcal{G}}(zb^{-1}) = \beta_{\mathcal{G}}(e_{\mathcal{G}})$. Thus,

$$\begin{aligned}\gamma_{\mathcal{G}}(aw^{-1}) = \alpha_{\mathcal{G}}(aw^{-1}) \cap \beta_{\mathcal{G}}(aw^{-1}) &= \alpha_{\mathcal{G}}(e_{\mathcal{G}}) \cap \beta_{\mathcal{G}}(w^{-1}a) \\ &= \alpha_{\mathcal{G}}(e_{\mathcal{G}}) \cap \beta_{\mathcal{G}}(e_{\mathcal{G}}) = \gamma_{\mathcal{G}}(e_{\mathcal{G}}),\end{aligned}$$

that is $[a]_\gamma = [w]_\gamma$.

Now, we prove that Ω is a homomorphism. Let $[x]_\beta, [y]_\beta \in (K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}})/\beta$, where $x = ab, y = wz, a, w \in K_{\alpha_{\mathcal{G}}}$ and $b, z \in K_{\beta_{\mathcal{G}}}$, then $xy = abwz$. Since $K_{\alpha_{\mathcal{G}}}$ is normal, $bwz \in K_{\alpha_{\mathcal{G}}}$. Hence,

$$\Omega([x]_\beta[y]_\beta) = \Omega([xy]_\beta) = [a(bwz)]_\gamma = [a]_\gamma[bwz]_\gamma$$

and

$$\begin{aligned}\gamma_{\mathcal{G}}((bwz)w^{-1}) &= \alpha_{\mathcal{G}}((bwz)w^{-1}) \cap \beta_{\mathcal{G}}((b(wzw^{-1})) \\ &= \alpha_{\mathcal{G}}(e_{\mathcal{G}}) \cap \beta_{\mathcal{G}}(e_{\mathcal{G}}) = \gamma_{\mathcal{G}}(e_{\mathcal{G}}).\end{aligned}$$

Hence, $[w]_\gamma = [bwz]_\gamma$, i.e.

$$\Omega([x]_\beta[y]_\beta) = [a]_{\gamma)}[w]_\gamma = \Omega([x]_\beta)\Omega([y]_\beta),$$

which implies that Ω is a homomorphism. It is also onto, since for any $[a]_\gamma \in K_{\alpha_{\mathcal{G}}}/\gamma$ and $b \in K_{\beta_{\mathcal{G}}}$, we have $x = ab \in K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}}$ and $\Omega([x]_\beta) = [a]_{\gamma)}$. Finally, we show that Ω is injective. Let $x, y \in K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}}$, where $x = ab, y = wz$. Assume that $[a]_\gamma = [w]_\gamma$, then $\gamma_{\mathcal{G}}(aw^{-1}) = \gamma_{\mathcal{G}}(e_{\mathcal{G}})$, that is

$$\alpha_{\mathcal{G}}(aw^{-1}) \cap \beta_{\mathcal{G}}(aw^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}}) \cap \beta_{\mathcal{G}}(e_{\mathcal{G}}).$$

But $\alpha_{\mathcal{G}}(e_{\mathcal{G}}) = \beta_{\mathcal{G}}(e_{\mathcal{G}})$ and $\alpha_{\mathcal{G}}(aw^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})$ imply that $\beta_{\mathcal{G}}(aw^{-1}) = \beta_{\mathcal{G}}(e_{\mathcal{G}})$. Therefore,

$$\begin{aligned}\beta_{\mathcal{G}}(xy^{-1}) &= \beta_{\mathcal{G}}(ab(wz)^{-1}) = \beta_{\mathcal{G}}(abz^{-1}w^{-1}) = \beta_{\mathcal{G}}(w^{-1}abz^{-1}) \\ &\supseteq \beta_{\mathcal{G}}(w^{-1}a) \cap \beta_{\mathcal{G}}(bz^{-1}) = \beta_{\mathcal{G}}(aw^{-1}) \cap \beta_{G}(bz^{-1}) = \beta_{\mathcal{G}}(e_{\mathcal{G}}).\end{aligned}$$

Hence, $[x]_\beta = [y]_\beta$. Therefore, Ω is an isomorphism. □

In case $\mathcal{A}, \mathcal{B} \subseteq \mathcal{G}$ are normal subgroups, the result $(\mathcal{A} \cdot \mathcal{B})/\chi_{\mathcal{B}} \cong \mathcal{B}/\chi_{\mathcal{A}\cap\mathcal{B}}$ comes as a corollary of Theorem 3.4. and then we get the Second Group Isomorphism Theorem. Finally, the Third Group Isomorphism Theorem is outcome of the following result.

Theorem 3.5. *Let $\alpha_{\mathcal{G}}, \beta_{\mathcal{G}} \in \mathcal{S}(\mathcal{G},\mathcal{U})$ be normal such that $\beta_{\mathcal{G}} \sqsubseteq \alpha_{\mathcal{G}}$ and $\alpha_{\mathcal{G}}(e_{\mathcal{G}}) = \beta_{\mathcal{G}}(e_{\mathcal{G}})$. Then, $(\mathcal{G}/\beta)/(K_{\alpha_{\mathcal{G}}}/\beta) \cong \mathcal{G}/\alpha$*

Proof. For all $x \in \mathcal{G}$, we define $\theta : \mathcal{G}/\beta \longrightarrow \mathcal{G}/\alpha$ by $\theta([x]_\beta) = [x]_\alpha$. The mapping is well defined since $[x]_\beta = [y]_\beta$ implies $\beta_{\mathcal{G}}(xy^{-1}) = \beta_{\mathcal{G}}(e_{\mathcal{G}}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})$. By assumption, we get $\alpha_{\mathcal{G}}(xy^{-1}) \supseteq \beta_{\mathcal{G}}(xy^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})$ and hence $\alpha_{\mathcal{G}}(xy^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})$, that is $[x]_\alpha) = [y]_\alpha$. By definition, θ is an onto homomorphism. We have $k_{\alpha_{\mathcal{G}}}/\beta = \{[z]_\beta : z \in k_{\alpha_{\mathcal{G}}}\} = \{[z]_\beta : \alpha_{\mathcal{G}}(z) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})\} = \{[z]_\beta : [z]_\alpha = [e_{\mathcal{G}}]_\alpha\} = \{[z]_\beta \in \mathcal{G}/\beta : \theta([z]_\alpha) = [e_{\mathcal{G}}]_\alpha\} = ker\theta$. Therefore, it follows that $(\mathcal{G}/\beta)/(K_{\alpha_{\mathcal{G}}}/\beta) \cong \mathcal{G}/\alpha$. □

4. Topological structures on $\mathcal{G}/\alpha$

Group $\mathcal{G}$ with the congruence relation R construct an approximation space ([16]). The lower and upper approximations of $\mathcal{H} \subseteq \mathcal{G}$ are defined respectively as

$$R_\star(\mathcal{H}) = \{x \in \mathcal{G} : [x]_\alpha \subseteq \mathcal{H}\},$$
$$R^\star(\mathcal{H}) = \{x \in \mathcal{G} : [x]_\alpha \cap \mathcal{H} \neq \phi\}.$$

The lower approximation induces a topology on $\mathcal{G}$.

Proposition 4.1 ([10])**.** $T_R = \{\mathcal{H} \subseteq \mathcal{G} : R_\star(\mathcal{H}) = \mathcal{H}\}$ *is a topology on* $\mathcal{G}$.

Furthermore, we have the following result.

Theorem 4.1. $(\mathcal{G}, T_R)$ *is a topological group.*

Proof. Let x and y be elements in $\mathcal{G}$. Every open set $U \in T_R$ containing the element xy satisfies the condition $R_\star(U) = U$. This implies $[xy]_\alpha \subseteq U$. Since R is a congruence relation on $\mathcal{G}$, we have $[x]_\alpha[y]_\alpha \subseteq [xy]_\alpha \subseteq U$. Notice that, $[x]_\alpha$ and $[y]_\alpha$ are open sets containing x, y respectively such that $[x][y] \subseteq U$. Hence, the group operation : $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a continuous mapping. To complete the proof, we have to verify continuity of the inversion mapping $x \to x^{-1}$. Let x be an element in $\mathcal{G}$ and $V \in T_R$ an open set containing the element x^{-1}, then $[x^{-1}]_\alpha \subseteq V$. Let $y^{-1} \in [x]^{-1} = \{y^{-1} : y \in [x]\}$ then

$$\alpha_{\mathcal{G}}(x^{-1}(y^{-1})^{-1}) = \alpha_{\mathcal{G}}(x^{-1}y) = \alpha_{\mathcal{G}}(yx^{-1}) = \alpha_{\mathcal{G}}(e).$$

That is, $y^{-1} \in [x^{-1}]$. Since $[x]$ is an open set containing x such that $[x]^{-1} \subseteq [x^{-1}] \subseteq V$, then the inverse operation on $\mathcal{G}$ is continuous. Therefore, $(\mathcal{G}, T_R)$ is a topological group. □

Example 4.1. Assume that $\mathcal{G} = S_3$ is the set of permutations on $\{1, 2, 3\}$ and $\mathcal{U} = \mathbb{Z}$ is the set of parameters. We define a soft set $\alpha_{\mathcal{G}}$ over U by

$$\begin{aligned} \alpha_{\mathcal{G}}(e) &= \mathbb{Z}, \\ \alpha_{\mathcal{G}}((12)) = \alpha_{\mathcal{G}}((13)) = \alpha_{\mathcal{G}}((23)) &= \{-2, -1, 0, 1, 2\}, \\ \alpha_{\mathcal{G}}((123)) = \alpha_{\mathcal{G}}((132)) &= \{-3, -2, -1, 0, 1, 2, 3\}. \end{aligned}$$

$\alpha_{\mathcal{G}}$ is a soft int-group ([3]). Easily, one can verify that $\alpha_{\mathcal{G}}$ is a normal soft int-group over $\mathcal{U}$.

Obviously, the equivalence class $[p]_\alpha$ contains only the element p, for every $p \in \mathcal{G}$. This implies that the topology T_R is the discrete topology, that is $T_R = \mathcal{P}(\mathcal{G})$. Then, group $\mathcal{G}$ endowed with the topology T_R is a topological group.

Consider the quotient map $\pi : \mathcal{G} \longrightarrow \mathcal{G}/\alpha$ defined by $x \to [x]_\alpha$, for all $x \in \mathcal{G}$. We equip the factor group $\mathcal{G}/\alpha$ with the quotient topology $\tau = \{K \subseteq \mathcal{G}/\alpha : \pi^{-1}(K) \in T_R\}$. In general topology, not every quotient map is open.

Proposition 4.2. *The quotient map $\pi : (\mathcal{G}, T_R) \longrightarrow (\mathcal{G}/\alpha, \tau)$ is open.*

Proof. For any open set $V \in T_R$, we show that $\pi(V) \in \tau$,

$$\pi^{-1}(\pi(V)) = \pi^{-1}(\bigcup_{x \in V} [x]_\alpha) = \bigcup_{x \in V} \pi^{-1}([x]_\alpha) = V.$$

So $\pi^{-1}(\pi(V))$ is open set and hence, by definition of quotient topology, $\pi(V)$ is open □

Theorem 4.2. *$(\mathcal{G}/\alpha, \tau)$ is a topological group.*

Proof. For $x, y \in \mathcal{G}$, let $[x]_\alpha, [y]_\alpha$ be elements in $\mathcal{G}/\alpha$ such that $[x]_\alpha[y]_\alpha = [xy]_\alpha \in W \in \tau$. Since $\pi(xy) = \pi(x)\pi(y) = [xy]_\alpha$ then $xy \in \pi^{-1}(W)$. Being $(\mathcal{G}, T_R)$ a topological group and $xy \in \pi^{-1}(W)$, there exists $V_x, V_y \in T_R$ containing x, y respectively and $V_xV_y \subseteq \pi^{-1}(W)$. Notice that $\pi(V_x)\pi(V_y) = \pi(V_xV_y) \in \pi(\pi^{-1}(W)) = W$. Since $\pi(x) = [x]_\alpha \in \pi(V_x), \pi(y) = [y]_\alpha \in \pi(V_y)$ and by Proposition 4.2, we verified that the product operation on $\mathcal{G}/\alpha$ is continuous. Now, we have to show that the inverse operation is also continuous. Let $[x]_\alpha$ be an element in $\mathcal{G}/\alpha$ and $V \in \tau$ an open set containing the element $[x]_\alpha^{-1} = [x^{-1}]_\alpha$, then $\pi(x^{-1}) = [x^{-1}]_\alpha \in V$ which implies $x^{-1} \in \pi^{-1}(V)$. Since $(\mathcal{G}, T_R)$ is a topological group, there exists $U \in T_R$ containing $x^{-1} \in \mathcal{G}$ such that $U^{-1} = \{z^{-1} \in \mathcal{G} : z \in U\} \subseteq \pi^{-1}(V)$. Since $\pi(x) = [x]_\alpha \in \pi(U) \in \tau$ and $\pi(U^{-1}) = \pi(U)^{-1}$ then we have $\pi(U)^{-1} \subseteq \pi(\pi^{-1}(V)) = V$. Therefore the mapping $[x]_\alpha \to [x^{-1}]_\alpha$ is continuous and hence $(\mathcal{G}/\alpha, \tau)$ is a topological group. □

5. Conclusion

In this paper, we constructed factor groups caused by normal soft int-groups. With the help of this construction, we established the group Isomorphism theorems. Further research can examine the factor groups caused by normal soft uni-groups.

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