Mycielskian of signed graphs

Albin Mathew
Department of Mathematics
Central University of Kerala
Kasaragod-671316, Kerala
India
albinmathewamp@gmail.com

Germina K.A.[*]
Department of Mathematics
Central University of Kerala
Kasaragod-671316, Kerala
India
srgerminaka@gmail.com

Abstract. In this paper, we define the Mycielskian of a signed graph and discuss the properties of balance and switching in the Mycielskian of a given signed graph. We provide a condition for ensuring the Mycielskian of a balanced signed graph remains balanced, leading to the construction of a balanced Mycielskian. We establish a relation between the chromatic numbers of a signed graph and its Mycielskian. We also study the structure of different matrices related to the Mycielskian of a signed graph.

Keywords: signed graph, signed graph coloring, Mycielskian of a signed graph.
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1. Introduction

A signed graph $\Sigma = (G, \sigma)$ consists of an underlying graph $G = (V, E)$, together with a function $\sigma : E \to \{-1, 1\}$, called the signature or sign function. The sign of a cycle C in Σ, denoted by $\sigma(C)$, is defined as the product of the signs of its edges, and the cycle C is said to be positive if $\sigma(C) = +1$. A signed graph Σ is said to be balanced if every cycle in it is positive, otherwise, Σ is unbalanced. A signed graph is called all-positive (all-negative) if all the edges are positive (negative).

A switching function for Σ is a function $\zeta : V(\Sigma) \to \{-1, 1\}$. For an edge $e = uv$ in Σ, the switched signature σ^ζ is defined as $\sigma^\zeta(e) = \zeta(u)\sigma(e)\zeta(v)$, and the switched signed graph is $\Sigma^\zeta = (G, \sigma^\zeta)$. The signs of cycles are unchanged by switching, and any balanced signed graph can be switched to an all-positive signed graph. If one signed graph can be switched from the other, they are said to be switching equivalent (see, [8, Section 3]).

∗. Corresponding author

The net-degree of a vertex v in a signed graph Σ, denoted by $d^{\pm}_{\Sigma}(v)$ is defined as $d^{\pm}_{\Sigma}(v) = d^{+}_{\Sigma}(v) - d^{-}_{\Sigma}(v)$, where $d^{+}_{\Sigma}(v)$ and $d^{-}_{\Sigma}(v)$ respectively denotes the number of positive and negative edges incident with v in Σ. The total number of edges incident with v in Σ is denoted by $d_{\Sigma}(v)$ and $d_{\Sigma}(v) = d^{+}_{\Sigma}(v) + d^{-}_{\Sigma}(v)$.

Throughout this paper, we consider only finite, simple, connected, and undirected graphs and signed graphs. For the standard notation and terminology in graphs and signed graphs not given here, the reader may refer to [3, 9, 12].

The Mycielski construction of a simple graph was introduced by J. Mycielski [7] in his search for triangle-free graphs with arbitrarily large chromatic number. The Mycielskian for a finite, simple, connected graph $G = (V, E)$ is defined as follows.

Definition 1.1 ([1]). *The Mycielskian $M(G)$ of G is a graph whose vertex set is the disjoint union $V \cup V' \cup \{w\}$, where $V' = \{v' : v \in V\}$, and whose edge set is $E \cup \{u'v : uv \in E\} \cup \{v'w : v' \in V'\}$. The vertex w is called the root of $M(G)$ and $v' \in V'$ is called the twin of v in $M(G)$.*

The Mycielski construction is useful in various applications, including the study of planar graphs and coloring problems, as triangle-free graphs have unique properties and often behave differently from graphs with triangles. When it comes to signed graphs, triangle-free signed graphs are even more important, as recent studies indicate that the negative triangles affects the balance of a signed graph more than other negative cycles.

1.1 Mycielskian of signed graphs

Motivated from the Definition 1.1, we define the Mycielskian $M(\Sigma)$ of the signed graph Σ as follows.

Definition 1.2 (Mycielskian). *The Mycielskian of Σ is the signed graph $M(\Sigma) = (M(G), \sigma_M)$, where $M(G)$ is the Mycielskian of the underlying graph G of Σ, and the signature function σ_M is defined as $\sigma_M(uv) = \sigma_M(u'v) = \sigma(uv)$ and $\sigma_M(v'w) = 1$*

The following are some immediate observations.

Observation 1.1. Let Σ be a signed graph with p vertices and q edges and let $M(\Sigma)$ be its Mycielskian. Then, we have the following.

(i) $M(\Sigma)$ has $2p + 1$ vertices and $3q + p$ edges.

(ii) If Σ contains r positive edges and $q-r$ negative edges, then $M(\Sigma)$ contains $3r + p$ positive edges and $3(q - r)$ negative edges.

(iii) If Σ is triangle-free, then $M(\Sigma)$ is also triangle-free.

(iv) For each vertex $v \in V$, $d^{\pm}_{M(\Sigma)}(v) = 2d^{\pm}_{\Sigma}(v)$ and $d_{M(\Sigma)}(v) = 2d_{\Sigma}(v)$.

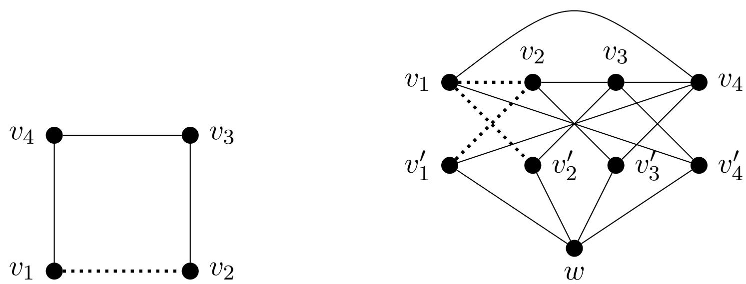
(v) For each vertex $v' \in V'$, $d^{\pm}_{M(\Sigma)}(v') = d^{\pm}_{\Sigma}(v) + 1$ and $d_{M(\Sigma)}(v') = d_{\Sigma}(v) + 1$.

(vi) $d^{\pm}_{M(\Sigma)}(w) = d_{M(\Sigma)}(w) = p$.

Note that, one can define the signature function for the Mycielskian of a signed graph in other ways. In this paper, we initiate a study on Mycielskian of a signed graph using this particular definition.

This particular construction of Mycielskian of a signed graph is illustrated in Example 1.1.

Example 1.1. Let Σ be the negative cycle C_4^-. The Mycielskian of C_4^- is constructed in Figure 1b.



(a) Σ (b) $M(\Sigma)$

Figure 1: A signed graph and its Mycielskian.

2. Balance and switching in Mycielskian of signed graphs

Balance and switching are two important concepts in signed graph theory.

In this section, we establish how the signed graph and its Mycielskian are related with respect to balance and switching. One may note that if Σ is unbalanced, then $M(\Sigma)$ is unbalanced. Also, in general, for a balanced signed graph Σ, the Mycielskian $M(\Sigma)$ need not be balanced.

The following is a characterization for $M(\Sigma)$ to be balanced.

Proposition 2.1. *The Mycielskian $M(\Sigma)$ is balanced if and only if Σ is all-positive.*

Proof. If Σ is all-positive, then so is $M(\Sigma)$, and hence is balanced. Conversely, If Σ has at least one negative edge, say v_iv_j, then $v_iv_jv_i'wv_j'v_i$ forms a negative 5 - cycle in $M(\Sigma)$, making it unbalanced. □

Consider any balanced signed graph Σ which is not all-positive. Then, Σ can be switched to an all-positive signed graph, say Σ'. By Proposition 2.1, $M(\Sigma)$ is not balanced, but $M(\Sigma')$ is balanced. Hence, the Mycielskians of two switching equivalent signed graphs need not to be switching equivalent.

The Mycielskian of an unbalanced signed graph is always unbalanced. However, for a balanced signed graph Σ, the Mycielskian $M(\Sigma) = (M(G), \sigma_M)$ can be made balanced by modifying the signature function σ_M. Though there are several ways to do so, to remain consistent with our original definition, we only look for changes that can be made in the signature of the edges incident to the root vertex w which makes the Mycielskian balanced, and leave the signatures of the other edges unchanged.

We need the following theorem [4].

Theorem 2.1 (Harary's bipartition theorem [4])**.** *A signed graph Σ is balanced if and only if there is a bipartition of its vertex set, $V = V_1 \cup V_2$, such that every positive edge is induced by V_1 or V_2 while every negative edge has one endpoint in V_1 and one in V_2. The bipartition $V = V_1 \cup V_2$ is called a Harary bipartition for Σ.*

Note that, if $V = V_1 \cup V_2$ is a Harary bipartition for Σ, then every path in Σ joining vertices in V_1 (similarly V_2) is positive, and every path between V_1 and V_2 is negative.

Theorem 2.2 provides a method to construct a balanced Mycielskian signed graph from a balanced signed graph.

Theorem 2.2. *Let Σ be a balanced signed graph and $M(\Sigma) = (M(G), \sigma_M)$ be its Mycielskian. If σ'_M is a signature function satisfying $\sigma'_M = \sigma_M$ on $M(G)\backslash\{w\}$ and satisfies the relation $\sigma'_M(v'_i w)\sigma'_M(v'_j w) = \sigma(v_i v_j)$ for every edge $v_i v_j$ in Σ, then the signed graph $M'(\Sigma) = (M(G), \sigma'_M)$ is balanced.*

Proof. Since Σ is balanced, by Harary bipartition theorem, there exists a bipartition $V = V_1 \cup V_2$ of V such that every negative edge in Σ has its one end vertex in V_1 and the other in V_2. We construct a Harary bipartition for $M'(\Sigma)$ as follows.

Let $V'_1 = \{v'_i : v_i \in V_1\}$ and $V'_2 = \{v'_i : v_i \in V_2\}$ be the subsets of V' corresponding to the subsets V_1 and V_2 of V. Since $V = V_1 \cup V_2$, we have $V' = V'_1 \cup V'_2$. Now, every edge with both its end vertices in V_1 is positive and no vertices in V'_1 are adjacent. Also, for edges of the form $v_i v'_j$, where $v_i \in V_1$ and $v'_j \in V'_1$, we have, $\sigma'_M(v_i v'_j) = \sigma_M(v_i v'_j) = \sigma(v_i v_j) = +1$. Thus, every edge with both its end vertices in $V_1 \cup V'_1$ is positive. Similarly, every edge with both its end vertices in $V_2 \cup V'_2$ is positive. Finally, consider any edge e having one end vertex in $V_1 \cup V'_1$ and the other in $V_2 \cup V'_2$. Without loss of generality, we can assume that $e = v_i v_j$, where $v_i \in V_1$ and $v_j \in V_2$. Then, $\sigma'_M(e) = \sigma_M(e) = \sigma_M(v_i v_j) = \sigma(v_i v_j) = -1$. Hence, every edge joining $V_1 \cup V'_1$ and $V_2 \cup V'_2$ is negative.

We now claim that *if $\sigma'_M(v'_k w)$ is positive for some $v_k \in V_1$, then $\sigma'_M(v'_i w)$ is positive for all $v_i \in V_1$ and $\sigma'_M(v'_j w)$ is negative for all $v_j \in V_2$.* To prove the claim, we first observe that if the relation $\sigma'_M(v'_iw)\sigma'_M(v'_jw) = \sigma(v_iv_j)$ holds for every edge v_iv_j in Σ, then for any path $P_{v_iv_j}$ joining the vertices v_i and v_j in Σ, the sign $\sigma(P_{v_iv_j})$ satisfies the relation $\sigma'_M(v'_iw)\sigma'_M(v'_jw) = \sigma(P_{v_iv_j})$. To prove this, consider a $v_i - v_j$ path, say $P_{v_iv_j} = v_iv_{i+1}v_{i+2}\cdots v_{j-1}v_j$, in Σ. Then, we have

$$\begin{aligned}
\sigma(P_{v_iv_j}) &= \sigma(v_iv_{i+1}v_{i+2}\cdots v_{j-1}v_j)\\
&= \sigma(v_iv_{i+1})\sigma(v_{i+1}v_{i+2})\cdots\sigma(v_{j-1}v_j)\\
&= (\sigma'_M(v'_iw)\sigma'_M(v'_{i+1}w))(\sigma'_M(v'_{i+1}w)\sigma'_M(v'_{i+2}w))\cdots(\sigma'_M(v'_{j-1}w)\sigma'_M(v'_jw))\\
&= \sigma'_M(v'_iw)(\sigma'_M(v'_{i+1}w)\sigma'_M(v'_{i+2}w)\cdots\sigma'_M(v'_{j-1}w))^2\sigma'_M(v'_jw))\\
&= \sigma'_M(v'_iw)\sigma'_M(v'_jw).
\end{aligned}$$

Now, consider $v_k \in V_1$ and let $v_i \in V_1$ and $v_j \in V_2$ be arbitrary. Then, every $v_i - v_k$ path is positive (i.e., $\sigma(P_{v_iv_k}) = +1$) and every $v_j - v_k$ path is negative (i.e., $\sigma(P_{v_jv_k}) = -1$). The connectedness of Σ guarantees the existence of such paths. Now, $\sigma'_M(v'_iw)\sigma'_M(v'_kw) = \sigma(P_{v_iv_k}) = +1$. Thus, $\sigma'_M(v'_iw)$ and $\sigma'_M(v'_kw)$ must have the same sign. Similarly, since $\sigma'_M(v'_jw)\sigma'_M(v'_kw) = \sigma(P_{v_jv_k}) = -1$, $\sigma'_M(v'_jw)$ and $\sigma'_M(v'_kw)$ are of the opposite sign. Thus, if $\sigma'_M(v'_kw)$ is positive for some $v_k \in V_1$, then $\sigma'_M(v'_iw)$ is positive for all $v_i \in V_1$ and $\sigma'_M(v'_jw)$ is negative for all $v_j \in V_2$. Hence, the claim is proved.

Now, consider the edges v'_iw, where $v'_i \in V'_1 \cup V'_2$. Because of the claim, if $\sigma'_M(v'_kw)$ is positive for some $v_k \in V_1$, then $\sigma'_M(v'_iw)$ is positive for all $v_i \in V_1$ and $\sigma'_M(v'_jw)$ is negative for all $v_j \in V_2$. In this case, let $(V_M)_1 = V_1 \cup V'_1 \cup \{w\}$ and $(V_M)_2 = V_2 \cup V'_2$. Similarly, if $\sigma'_M(v'_kw)$ is negative for some $v_k \in V_1$, then $\sigma'_M(v'_iw)$ is negative for all $v_i \in V_1$ and $\sigma'_M(v'_jw)$ is positive for all $v_j \in V_2$. In this case, let $(V_M)_1 = V_1 \cup V'_1$ and $(V_M)_2 = V_2 \cup V'_2 \cup \{w\}$.

Thus, in either case, $V_M = (V_M)_1 \cup (V_M)_2$ forms a Harary bipartition for $M'(\Sigma)$, and hence $M'(\Sigma)$ is balanced. □

Remark 2.1. One may note that σ'_M is a different signature on $M(G)$ that coincides with σ_M on $M(G)\backslash\{w\}$. The signature function σ'_M for the remaining edges v'_iw of $M(G)$ has to be defined using the relation stated in Theorem 2.2. One such construction is discussed in Section 2.1.

It is also worth noting that if $\sigma'_M = \sigma_M$ on $M(G)$, then Theorem 2.2 reduces to Proposition 2.1.

2.1 A balance-preserving construction

Given any balanced signed graph $\Sigma = (G, \sigma)$, there exist a switching function $\zeta : V(\Sigma) \to \{-1, +1\}$ that switches Σ to all-positive. Define $M_B(\Sigma)$ as the signed graph with underlying graph $M(G)$ and having the signature function

σ_B defined as

$$\sigma_B(v_iv_j) = \sigma(v_iv_j),$$
$$\sigma_B(v_i'v_j) = \sigma_B(v_iv_j') = \sigma(v_iv_j),$$
$$\sigma_B(v_i'w) = \zeta(v_i).$$

Define a switching function $\zeta_B : V(M_B(\Sigma)) \to \{-1, +1\}$ by

$$\zeta_B(v_i) = \zeta(v_i),$$
$$\zeta_B(v_i') = \zeta(v_i),$$
$$\zeta_B(w) = 1.$$

Since ζ switches Σ to all-positive, for edges v_iv_j,

$$\begin{aligned}\sigma_B^{\zeta_B}(v_iv_j) &= \zeta_B(v_i)\sigma_B(v_iv_j)\zeta_B(v_j)\\ &= \zeta(v_i)\sigma(v_iv_j)\zeta(v_j)\\ &= \sigma^{\zeta}(v_iv_j)\\ &= +1.\end{aligned}$$

Similarly, for edges $v_i'v_j$,

$$\begin{aligned}\sigma_B^{\zeta_B}(v_i'v_j) &= \zeta_B(v_i')\sigma_B(v_i'v_j)\zeta_B(v_j)\\ &= \zeta(v_i)\sigma(v_iv_j)\zeta(v_j)\\ &= \sigma^{\zeta}(v_iv_j)\\ &= +1.\end{aligned}$$

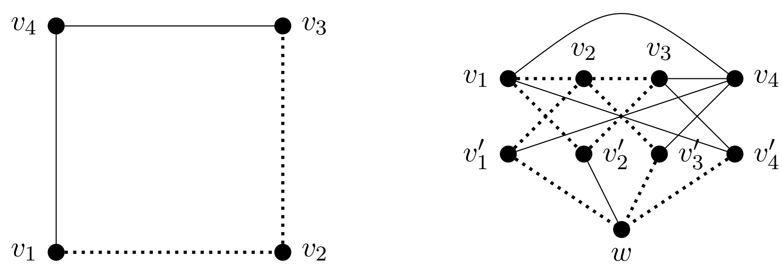
Also, for edges $v_i'w$,

$$\begin{aligned}\sigma_B^{\zeta_B}(v_i'w) &= \zeta_B(v_i')\sigma_B(v_i'w)\zeta_B(w)\\ &= \zeta(v_i)\zeta(v_i)(+1)\\ &= (\zeta(v_i))^2\\ &= +1.\end{aligned}$$

Hence, ζ_B switches $M_B(\Sigma)$ to all-positive. Thus, $M_B(\Sigma) = (M(G), \sigma_B)$ is balanced, and we call it as the balanced Mycielskian of Σ.

Definition 2.1 (Balanced Mycielskian). *Let $\Sigma = (G, \sigma)$ be a balanced signed graph, where the underlying graph $G = (V, E)$, is a finite simple connected graph. The signed graph $M_B(\Sigma) = (M(G), \sigma_B)$ is called the balanced Mycielskian of Σ.*

One can observe that under this construction, if two balanced signed graphs Σ_1 and Σ_2 are switching equivalent, then their corresponding balanced Mycielskians $M_B(\Sigma_1)$ and $M_B(\Sigma_2)$ are also switching equivalent.



(a) Σ (b) $M_B(\Sigma)$

Figure 2: A balanced signed graph Σ and its balanced Mycielskian $M_B(\Sigma)$.

Example 2.1. Let Σ be the balanced 4-cycle shown in Figure 2a. The switching function $\zeta : V(\Sigma) \to \{-1, 1\}$ defined by $\zeta(v_1) = \zeta(v_3) = \zeta(v_4) = -1$ and $\zeta(v_2) = 1$ switches Σ to all-positive. The corresponding balanced Mycielskian is constructed in Figure 2b.

Remark 2.2. Note that, since $\sigma^{\zeta}(v_iv_j) = +1$, for every edge v_iv_j in Σ, we have $\zeta(v_i)\zeta(v_j) = \sigma(v_iv_j)$. Thus,

$$\sigma_B(v_i'w)\sigma_B(v_i'w) = \zeta(v_i)\zeta(v_j) = \sigma(v_iv_j).$$

Hence, the signature function defined for the balanced Mycielskian satisfies the condition given in Theorem 2.2.

3. The chromatic number of Mycielskian of signed graphs

In 1981, Zaslavsky [10] introduced the concept of coloring a signed graph. For a signed graph Σ, he defined the signed coloring of Σ in μ colors, or in $2\mu+1$ signed colors as a mapping $c : V(\Sigma) \to \{-\mu, -\mu+1, \ldots, 0, \ldots, \mu-1, \mu\}$. Whenever a coloring never assumes the value 0, it is referred to as a zero-free coloring. A coloring c is said to be proper if $c(u) \neq \sigma(e)c(v)$ for every edge $e = uv$ of Σ (see, [10, Section 1]).

Máčajová *et al.* in [5] defined the chromatic number of a signed graph as follows.

Definition 3.1 ([5]). *An n - coloring of a signed graph Σ is a proper coloring that uses colors from the set M_n, which is defined for each $n \geq 1$ as*

$$M_n = \begin{cases} \{\pm 1, \pm 2, \ldots \pm k\}, & \text{if } n = 2k \\ \{0, \pm 1, \pm 2, \ldots \pm k\}, & \text{if } n = 2k+1. \end{cases}$$

The smallest n such that Σ admits an n - coloring is called the chromatic number of Σ and is denoted by $\chi(\Sigma)$.

The chromatic number of a balanced signed graph coincides with the chromatic number of its underlying unsigned graph.

Proposition 3.1. *Let $M(\Sigma)\backslash\{w\}$ be the signed graph obtained by removing the root vertex w (and the corresponding edges) from $M(\Sigma)$. Then, $\chi(M(\Sigma)\backslash\{w\}) = \chi(\Sigma)$.*

Proof. Let $\chi(\Sigma) = n$ and let $c : V(\Sigma) \to M_n$ be an n - coloring for Σ. Define $c' : V((M(\Sigma)\backslash\{w\}) \to M_n$ by $c'(v_i') = c'(v_i) = c(v_i)$ for all i. Since $c(v_i) \neq \sigma(v_iv_j)c(v_j)$, it follows that $c'(v_i) \neq \sigma_M(v_iv_j)c'(v_j)$ and $c'(v_i') \neq \sigma_M(v_i'v_j)c'(v_j)$. Hence, c' is an n - coloring for $M(\Sigma)\backslash\{w\}$. □

For any given signed graph Σ, there exist a signed graph $-\Sigma$ obtained by reversing the signs of all edges of Σ. We say Σ is antibalanced when $-\Sigma$ is balanced. Note that, Σ is antibalanced if and only if it can be switched to all-negative.

We restate the Lemma 2.4 from [11] as follows.

Lemma 3.1 ([11])**.** *A signed graph Σ is antibalanced if and only if $\chi(\Sigma) \leq 2$.*

Theorem 3.1. *Let Σ be a signed graph and $M(\Sigma)$ be its Mycielskian. Then, $\chi(M(\Sigma)) \leq 2$ if and only if Σ is all-negative.*

Proof. If Σ is an all-negative signed graph with vertex set $\{v_1, v_2, \ldots v_p\}$, then the only positive edges of $M(\Sigma)$ are $v_i'w$, $1 \leq i \leq p$. Now, the switching function $\zeta_M' : V(M(\Sigma)) \to \{-1, 1\}$ defined by $\zeta_M'(v_i) = \zeta_M'(v_i') = 1$ for all $1 \leq i \leq p$ and $\zeta_M'(w) = -1$ switches $M(\Sigma)$ to all-negative. Therefore, $M(\Sigma)$ is antibalanced and hence $\chi(M(\Sigma)) \leq 2$, by Lemma 3.1. Conversely, if Σ is not all-negative, it contains at least one positive edge, say v_iv_j. Then, $v_iv_jv_i'wv_j'v_i$ forms a negative 5 - cycle in $-M(\Sigma)$, making it unbalanced. Thus, $M(\Sigma)$ is not antibalanced and therefore, by Lemma 3.1, $\chi(M(\Sigma)) > 2$. □

We have the following theorem in [1].

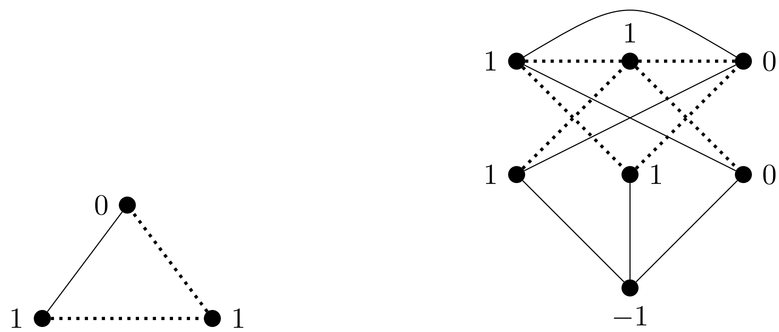
Theorem 3.2 ([7])**.** *Let $\chi(G)$ and $\chi(M(G))$ be the chromatic numbers of a graph G and its Mycielskian $M(G)$ respectively. Then $\chi(M(G)) = \chi(G) + 1$.*

Theorem 3.3. *Let $M(\Sigma)$ be the Mycielskian of a signed graph (Σ). Then, $\chi(\Sigma) \leq \chi(M(\Sigma)) \leq \chi(\Sigma) + 1$. Furthermore, $\chi(M(\Sigma)) = \chi(\Sigma)$ if Σ is all-negative and $\chi(M(\Sigma)) = \chi(\Sigma) + 1$ if Σ is all-positive.*

Proof. Let $\chi(\Sigma) = n$ and let $c : V \to M_n$ be an n - coloring for Σ. We extend c to an $(n+1)$ - coloring of $M(\Sigma)$. If $n = 2k$, we extend c to an $(n+1)$ - coloring of $M(\Sigma)$ by setting $c(v_i') = c(v_i)$ for all i and $c(w) = 0$. If $n = 2k+1$, we extend c to an $(n+1)$ - coloring of $M(\Sigma)$ as follows. Let v_t be any vertex in V with $c(v_t) = 0$. Then, for all $v_i \neq v_t$, set $c(v_i') = c(v_i)$, $c(v_t') = c(v_t) = k+1$ and $c(w) = -(k+1)$. Hence, $\chi(M(\Sigma)) \leq \chi(\Sigma) + 1$.

Now, if Σ is all-negative, it can be colored using just one color, namely -1. Let $c : V(\Sigma) \to \{\pm 1\}$ be the proper 2 - coloring for Σ. This can be extended to a proper 2 - coloring for $M(\Sigma)$ by setting $c(v_i') = c(v_i) = -1$ for all i and $c(w) = +1$. If Σ is all-positive, then $M(\Sigma)$ is all-positive. Thus, $\chi(M(\Sigma)) = \chi(M(G)) = \chi(G) + 1 = \chi(\Sigma) + 1$. □

Remark 3.1. Let Σ be a signed graph with $\chi(\Sigma) = n$ and let $c : V(\Sigma) \to M_n$ be an n - coloring of Σ. The *deficiency* of the coloring c is the number of unused colors from M_n (see, [6]). The existence of signed graphs satisfying $\chi(M(\Sigma)) = \chi(\Sigma)$ is a consequence of the deficiency of the coloring of Σ. Specifically, if the coloring of Σ has a deficiency of at least 1, then an unused color can be assigned to w, making the chromatic number of $M(\Sigma)$ and Σ equal. As an example, consider Σ as the balanced 3 - cycle shown in Figure 3a. Note that, $\chi(\Sigma) = 3$ and the color -1 in the color set $\{0, \pm 1\}$ is unused.



(a) Σ (b) $M(\Sigma)$

Figure 3: A signed graph Σ satisfying $\chi(M(\Sigma)) = \chi(\Sigma)$

We now establish some results on the balanced Mycielskian of signed graphs.

Proposition 3.2. *Let $\Sigma = (G, \sigma)$ be a balanced signed graph and $M_B(\Sigma) = (M(G), \sigma_B)$ be its balanced Mycielskian. Then, $\chi(M_B(\Sigma)) = \chi(\Sigma) + 1$.*

Proof. Since Σ and $M_B(\Sigma)$ are both balanced, $\chi(M_B(\Sigma)) = \chi(M(G))$ and $\chi(\Sigma) = \chi(G)$. The result then follows from Theorem 3.2. □

The following theorem was put forward by Mycielski in [7]

Theorem 3.4 ([7])**.** *For any positive integer n, there exists a triangle-free graph with chromatic number n.*

The next theorem is an analogous result for balanced signed graphs.

Theorem 3.5. *For any positive integer n, there exists a balanced triangle-free signed graph that is not all-positive, and having chromatic number n.*

Proof. The proof is based on mathematical induction. For $n = 1$ and $n = 2$, the signed graphs $\Sigma_1 = K_1$ and $\Sigma_2 = K_2^-$, where K_2^- is the all-negative signed complete graph on two vertices have the required property. Suppose that for $k > 2$, such a signed graph Σ_k satisfying the induction hypothesis exists. Then, $M_B(\Sigma_k)$ is a balanced signed graph that is not all-positive. Also, by Proposition 3.2, we have, $\chi(\Sigma_{k+1}) = \chi(\Sigma_k) + 1 = k + 1$. □

4. Matrices of the Mycielskian of signed graphs

Given a signed graph $\Sigma = (V, E, \sigma)$ where $V = \{v_1, v_2, \ldots, v_p\}$ is the vertex set, $E = \{e_1, e_2, \ldots, e_q\}$ is the edge set and $\sigma : E \to \{-1, 1\}$ is the sign function. Let $M(\Sigma)$ be the Mycielskian of Σ. In this section, we introduce the adjacency matrix, the incidence matrix and the Laplacian matrix of the Mycielskian $M(\Sigma)$ of Σ.

4.1 The adjacency matrix

The adjacency matrix of Σ, denoted by $\mathbf{A} = \mathbf{A}(\Sigma)$, is a $p \times p$ matrix (a_{ij}) in which $a_{ij} = \sigma(v_i v_j)$ if v_i and v_j are adjacent and 0 otherwise (see, [9, Section 3]).

Since v_i is adjacent to v_j' and v_i' is adjacent to v_j in $M(\Sigma)$ whenever v_i and v_j are adjacent in Σ, **the adjacency matrix $\mathbf{A_M}$** of the Mycielskian $M(\Sigma)$ takes the block form

$$\mathbf{A_M} = \mathbf{A}(M(\Sigma)) = \begin{bmatrix} \mathbf{A}(\Sigma) & \mathbf{A}(\Sigma) & \mathbf{0}_{p\times 1} \\ \mathbf{A}(\Sigma) & \mathbf{0}_{p\times p} & \mathbf{j}_{p\times 1} \\ \mathbf{0}^t_{1\times p} & \mathbf{j}^t_{1\times p} & 0 \end{bmatrix},$$

where $\mathbf{0}$ is a matrix of zeros and $\mathbf{j}$ is a matrix of ones of the specified order. $\mathbf{A_M}$ is a symmetric matrix of order $2p + 1$.

Given a graph G with adjacency matrix $A(G)$, the connection between the ranks of $A(G)$ and $A(M(G))$, the connection between the number of positive, negative and zero eigenvalues $A(G)$ and $A(M(G))$ were studied by Fisher *et al.* in [2]. We initiate a similar study in the case of signed graphs.

Let $\Sigma = (V, E, \sigma)$ be a given signed graph and let $t \notin V$. We denote the signed graph obtained by joining all the vertices of Σ to t with negative edges by Σ_{t^-}. That is, Σ_{t^-} is the negative join $\Sigma \vee_- K_1$. The adjacency matrix of Σ_t takes the block form

$$\mathbf{A}_{t^-} = \mathbf{A}(\Sigma_{t^-}) = \begin{bmatrix} \mathbf{A} & \textbf{-j} \\ \textbf{-j}^t & 0 \end{bmatrix}.$$

We now have the following theorem.

Theorem 4.1. *Let Σ be a signed graph and $\boldsymbol{A}(\Sigma)$ be the adjacency matrix of Σ. Let $r(\boldsymbol{A})$ denote the rank and $n_+(\boldsymbol{A})$, $n_-(\boldsymbol{A})$ and $n_0(\boldsymbol{A})$ respectively denote the number of positive, negative and zero eigenvalues of a symmetric matrix $\boldsymbol{A}$, then we have the following.*

(*i*) $r(\boldsymbol{A_M}) = r(\boldsymbol{A}) + r(\boldsymbol{A}_{t^-})$

(*ii*) $n_+(\boldsymbol{A_M}) = n_+(\boldsymbol{A}) + n_+(\boldsymbol{A}_{t^-})$

(*iii*) $n_-(\boldsymbol{A_M}) = n_-(\boldsymbol{A}) + n_-(\boldsymbol{A}_{t^-})$

(*iv*) $n_0(\boldsymbol{A_M}) = n_0(\boldsymbol{A}) + n_0(\boldsymbol{A}_{t^-})$

Proof. The adjacency matrix $\mathbf{A_M}$ can be factorized as

$$\mathbf{A_M} = \begin{bmatrix} \mathbf{A} & \mathbf{A} & \mathbf{0} \\ \mathbf{A} & \mathbf{0} & \mathbf{j} \\ \mathbf{0}^t & \mathbf{j}^t & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \textbf{-I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}^t & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \textbf{-A} & \textbf{-j} \\ \mathbf{0}^t & \textbf{-j}^t & 0 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \textbf{-I} & \mathbf{0} \\ \mathbf{0}^t & \mathbf{0}^t & 1 \end{bmatrix} = \mathbf{P}\,\mathbf{B}\,\mathbf{P}^t$$

where $\mathbf{P} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \textbf{-I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}^t & 1 \end{bmatrix}$ is an invertible matrix and $\mathbf{B} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{t^-} \end{bmatrix}$.

Thus, the matrices $\mathbf{A_M}$ and $\mathbf{B}$ are congruent, and hence by Sylvester's law of inertia, they have the same rank and the same number of positive, negative and zero eigenvalues. □

4.2 The incidence matrix

The incidence matrix of Σ, denoted by $\mathbf{H} = \mathbf{H}(\Sigma)$, is the $p \times q$ matrix

$$\mathbf{H}(\Sigma) = \begin{bmatrix} \mathbf{x}(e_1) & \mathbf{x}(e_2) & \cdots & \mathbf{x}(e_q) \end{bmatrix},$$

where for each edge $e_k = v_i v_j$, $1 \le k \le q$, the vector $\mathbf{x}(e_k) = \begin{pmatrix} x_{1k} \\ \vdots \\ x_{pk} \end{pmatrix} \in \mathbb{R}^{p\times 1}$ has its i^{th} and j^{th} entries as $x_{ik} = \pm 1$ and $x_{jk} = \mp\sigma(e_k)$ respectively and all other entries as 0 (see, [9, Section 3]).

Let us denote the vertex set V_M and the edge set E_M of $M(\Sigma)$ as

$$V_M = \{v_1, v_2, \ldots, v_p, v_1', v_2', \ldots, v_p', w\},$$
$$E_M = \{e_1, e_2, \ldots, e_q, e_1', e_1'', e_2', e_2'', \ldots, e_q', e_q'', f_1, f_2, \cdots, f_p\}$$

respectively, where, for each $1 \le k \le q$, the edges e_k' and e_k'' of $M(\Sigma)$ are defined by $e_k' = v_i v_j'$ and $e_k'' = v_i' v_j$ whenever $e_k = v_i v_j$ is an edge of Σ with $1 \le i < j \le q$ and f_i is defined by $f_i = v_i' w$ for $1 \le i \le p$. Then, the **incidence matrix** $\mathbf{H_M} = \mathbf{H}(M(\Sigma))$ takes the block form

$$\mathbf{H_M} = \mathbf{H}(M(\Sigma)) = \left[\begin{array}{c|c|c|c|c|c|c|c|c} \mathbf{H}(\Sigma)_{p\times q} & \mathbf{x_1} & \mathbf{y_1} & \mathbf{x_2} & \mathbf{y_2} & \cdots & \mathbf{x_p} & \mathbf{y_p} & \mathbf{0}_{p\times p} \\ \hline \mathbf{0}_{p\times q} & \mathbf{y_1} & \mathbf{x_1} & \mathbf{y_2} & \mathbf{x_2} & \cdots & \mathbf{y_p} & \mathbf{x_p} & \mathbf{I}_{p\times p} \\ \hline \multicolumn{8}{c|}{\mathbf{0}_{1\times 3q}} & \textbf{-j}_{1\times p} \end{array}\right].$$

Here, $\mathbf{H}(\Sigma)$ is the incidence matrix of Σ, $\mathbf{I}$ is the identity matrix, $\mathbf{0}$ is the zero matrix and **-j** is the matrix with all entries -1 of the specified order. $\mathbf{x_i}$'s and $\mathbf{y_i}$'s are matrices of order $p \times 1$ and satisfies the condition $\mathbf{x_i} + \mathbf{y_i} = \mathbf{x}(e_i)$ for all $1 \le i \le q$.

4.3 The Laplacian matrix

The Laplacian matrix of Σ, denoted by $\mathbf{L} = \mathbf{L}(\Sigma)$ is the $p \times p$ matrix

$$\mathbf{L}(\Sigma) = \mathbf{D}(|\Sigma|) - \mathbf{A}(\Sigma),$$

where $\mathbf{A}(\Sigma)$ is the adjacency matrix of Σ and $\mathbf{D}(|\Sigma|)$ is the degree matrix of the underlying graph $|\Sigma|$ (see, [9, Section 3]).

Accordingly, we define the **Laplacian matrix for the Mycielskian** of Σ as

$$\mathbf{L_M} = \mathbf{L}(M(\Sigma)) = \mathbf{D}(|M(\Sigma)|) - \mathbf{A}(M(\Sigma)) = \mathbf{D_M} - \mathbf{A_M},$$

where $\mathbf{A_M}$ is the adjacency matrix and $\mathbf{D_M}$ is the diagonal degree matrix of the Mycielskian of Σ. Now, $\mathbf{D_M}$ takes the block form

$$\mathbf{D_M} = \begin{bmatrix} 2\mathbf{D}(|\Sigma|)_{p\times p} & \mathbf{0}_{p\times p} & \mathbf{0}_{p\times 1} \\ \mathbf{0}_{p\times p} & (\mathbf{D}(|\Sigma|) + \mathbf{I})_{p\times p} & \mathbf{0}_{p\times 1} \\ \mathbf{0}^t_{1\times p} & \mathbf{0}^t_{1\times p} & p \end{bmatrix},$$

where $p = |V|$, $\mathbf{D}(\Sigma)$ is the diagonal degree matrix of Σ, $\mathbf{I}$ is the identity matrix and $\mathbf{0}$ is the zero matrix of the specified order.

Consequently, the Laplacian matrix $\mathbf{L_M} = \mathbf{L}(M(\Sigma))$ takes the block form

$$\mathbf{L_M} = \begin{bmatrix} (2\mathbf{D}(|\Sigma|) - \mathbf{A}(\Sigma))_{p\times p} & \text{-}\mathbf{A}(\Sigma)_{p\times p} & \mathbf{0}_{p\times 1} \\ \text{-}\mathbf{A}(\Sigma)_{p\times p} & (\mathbf{D}(|\Sigma|) + \mathbf{I})_{p\times p} & \text{-}\mathbf{j}_{p\times 1} \\ \mathbf{0}^t_{1\times p} & \text{-}\mathbf{j}^t_{1\times p} & p \end{bmatrix}.$$

5. Conclusion and scope

In this paper, we have defined the Mycielskian of a signed graph and discussed some of its properties. We have seen that the Mycielskian of a balanced signed graph need not be balanced and hence we provide an alternate construction in which the Mycielskian of Σ is balanced whenever Σ is balanced, This paper also discusses the chromatic number of the Mycielskian of a signed graph and established that the chromatic number of a signed graph and its Mycielskian are related. We also established the block forms of various matrices of the Mycielskian of a signed graph such as the adjacency matrix, the incidence matrix, and the Laplacian matrix.

This work finds its application in many areas, especially in sociology, where social systems can be represented by signed graphs. Triangle-free signed graphs are important for balanced social systems, and our construction creates larger triangle-free signed graphs from a given triangle-free signed graph. The balanced Mycielskian construction provides a method to extend a balanced system to a much larger system without affecting balance. Developing another balance preserving, switching preserving constructions for the Mycielskian of signed graphs, and computing the spectra of various matrices of the Mycielskian of signed graphs are some exciting areas for further investigation.

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