Area integral characterizations and Φ-Carleson measures for harmonic Bergman-Orlicz spaces

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Abstract. Let Φ be a growth function. In this paper, we define a harmonic Bergman-Orlicz space $\mathcal{B}_\alpha^\Phi$ and characterize it in terms of area integral functions. Furthermore, we define Φ-Carleson measures and then discuss Φ-Carleson measures for harmonic Bergman-Orlicz spaces.
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1. Introduction

Let $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$ be two vectors in the n-dimensional real vector space $\mathbb{R}^n$. We write

$$\langle x, y\rangle = x_1y_1 + ... + x_ny_n \quad \text{and} \quad |x| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + ... + x_n^2}.$$

For $a \in \mathbb{R}^n$, let $\mathbb{B}(a, r) = \{x : |x - a| < r\}$, $\mathbb{S}(a, r) = \partial\mathbb{B}(a, r)$ and $\overline{\mathbb{B}(a, r)} = \mathbb{B}(a, r) \cup \mathbb{S}(a, r)$. In particular, let $\mathbb{B} = \mathbb{B}(0, 1)$, $\mathbb{S} = \partial\mathbb{B}(0, 1)$ and $\overline{\mathbb{B}} = \mathbb{B} \cup \mathbb{S}$ the closure of $\mathbb{B}$. We denote by dv the normalized volume measure on $\mathbb{B}$ and $h(\mathbb{B})$ the class of all harmonic functions on $\mathbb{B}$. For each $\alpha > -1$, the weighted normalized volume measure $dv_\alpha(x) = c_\alpha(1 - |x|^2)^\alpha dv(x)$ and c_α is a positive constant so that $v_\alpha(\mathbb{B}) = 1$.

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A function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ is called a *growth function* if it is continuous and non-decreasing. The growth function Φ satisfies the Δ_2-condition if there exists a constant $K > 1$ such that

$$\Phi(2t) \leq K\Phi(t), \quad t \in [0,\infty).$$

For $\alpha > -1$ and a growth function Φ satisfying Δ_2-condition, the *Orlicz space* $L^\Phi(\mathbb{B}, dv_\alpha)$ is the set of all measurable functions f such that

$$\|f\|_{\alpha,\Phi} = \int_{\mathbb{B}} \Phi(|f(x)|)dv_\alpha(x) < \infty.$$

The *harmonic Bergman-Orlicz* space $\mathcal{B}^\Phi_\alpha$ is the subspace of $L^\Phi(\mathbb{B}, dv_\alpha)$ consisting of all $f \in h(\mathbb{B})$. The Luxembourg gauge on $\mathcal{B}^\Phi_\alpha$ is defined by

$$\|f\|^{lux}_{\alpha,\Phi} = \inf\{\lambda > 0 : \int_{\mathbb{B}} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dv_\alpha(x) \leq 1\}.$$

We observe that $\Phi(t) = t^p$, the associated harmonic Bergman-Orlicz space is the classical weighted harmonic Bergman space $\mathcal{B}^p_\alpha$ (cf. [1, 9]).

For $f \in h(\mathbb{B})$, recall that the radial derivative $\mathcal{R}$ of f is given by

$$\mathcal{R}f(x) = x \cdot \nabla f(x) = \frac{\partial}{\partial t}(f(tx))_{t=1} = \sum_{m=1}^{\infty} m f_m(x),$$

where ∇ is the usual gradient and the last form is the homogeneous expansion of f. The fundamental theorem of calculus shows that

$$f(x) - f(0) = \int_0^1 (\mathcal{R}f)(tx)\frac{dt}{t}.$$

For $a \in \mathbb{B}$, we denote by φ_a the Möbius transformation in $\mathbb{B}$. It's an involution of $\mathbb{B}$ such that $\varphi_a(0) = a$ and $\varphi_a(a) = 0$, which is of the form

$$\varphi_a(x) = \frac{|x-a|^2 a - (1-|a|^2)(x-a)}{[x,a]^2},\ x \in \mathbb{B},$$

where $[x,a] = \sqrt{1 - 2\langle x,a\rangle + |x|^2|a|^2}$.

Let $a \in \mathbb{B}$ and $r \in (0,1)$, the *pseudo-hyperbolic ball* with center a and radius r is denoted by

$$E(a,r) = \{x \in \mathbb{B} : |\varphi_a(x)| < r\}.$$

Indeed, $E(a,r)$ is a Euclidean ball with center c_a and radius r_a given by

$$c_a = \frac{(1-r^2)a}{1-|a|^2r^2} \quad \text{and} \quad r_a = \frac{r(1-|a|^2)}{1-|a|^2r^2}, \tag{1}$$

respectively (cf. [16]). It is well known that for $\alpha > -1$ and any $x \in E(a,r)$,

$$1-|a|^2 \approx 1-|x|^2 \approx [a,x] \quad \text{and} \quad v_\alpha(E(a,r)) \approx (1-|a|^2)^{n+\alpha}. \tag{2}$$

For fixed $0 < s < \infty$ and $0 < r < \frac{1}{2}$, we consider the following area integral functions which were introduced by Chen and Ouyang (see [3, 4])

- $A^s_{\mathcal{R}}(f)(x) = \Big(\int_{E(x,r)} |(1-|y|^2)\mathcal{R}f(y)|^s d\tau(y)\Big)^{1/s}$,
- $A^s_{\nabla}(f)(x) = \Big(\int_{E(x,r)} |(1-|y|^2)\nabla f(y)|^s d\tau(y)\Big)^{1/s}$,
- $A^s(f)(x) = \Big(\int_{E(x,r)} |f(y)|^s d\tau(y)\Big)^{1/s}$,

where $d\tau(x) = (1-|x|^2)^{-n}dv(x)$ is the *invariant measure* on $\mathbb{B}$.

Let $\mathbf{B}_n$ be the unit ball of the n-dimensional complex vector space $\mathbb{C}^n$. For $0 < p < \infty$ and $\alpha > -1$, the standard weighted Bergman space $\mathcal{A}^p_\alpha(\mathbf{B}_n)$ consists of all holomorphic functions g on $\mathbf{B}_n$ such that

$$\int_{\mathbf{B}_n} |g(z)|^p dv_\alpha(z) < \infty.$$

It is well known that a holomorphic function $g \in \mathcal{A}^p_\alpha(\mathbf{B}_n)$ if and only if $(1-|z|^2)\nabla g(z) \in L^p(\mathbf{B}_n, dv_\alpha)$. In [18], B. Sehba extended this characterization to the holomorphic Bergman-Orlicz space. By adding the restriction $s > 1$, Chen and Ouyang [3, 4] proved that $g \in \mathcal{A}^p_\alpha(\mathbf{B}_n)$ is equivalent to one (and hence all) of the conditions $A^s_{\mathcal{R}}(g) \in L^p(\mathbf{B}_n, dv_\alpha)$, $A^s_{\nabla}(g) \in L^p(\mathbf{B}_n, dv_\alpha)$, $A^s(g) \in L^p(\mathbf{B}_n, dv_\alpha)$. As a consequence, they obtained some new maximal and area integral characterizations for Besov spaces. For the further discussions on this topic, we refer to [12].

Motivated by [3, 4, 18], our first aim in this paper is to extend Chen and Ouyang's result to the setting of harmonic Bergman-Orlicz space $\mathcal{B}^\Phi_\alpha$. In order to state our results, we need some more definitions on the growth function Φ.

We say that a growth function Φ is of upper type $q \geq 1$ if there exists $C > 0$ such that, for $s > 0$ and $t \geq 1$,

$$\Phi(st) \leq Ct^q\Phi(s). \tag{3}$$

Denote by $\mathcal{U}^q$ the set of growth functions Φ of upper type q, (for some $q \geq 1$), such that the function $t \to \frac{\Phi(t)}{t}$ is non-decreasing.

We say that Φ is of lower type $p > 0$ if there exists $C > 0$ such that, for $s > 0$ and $0 < t \leq 1$,

$$\Phi(st) \leq Ct^p\Phi(s). \tag{4}$$

Denote by $\mathcal{L}_p$ the set of growth functions Φ of lower type p, (for some $p \leq 1$), such that the function $t \to \frac{\Phi(t)}{t}$ is non-increasing.

Let

$$\mathcal{U} = \bigcup_{q\geq 1} \mathcal{U}^q \qquad and \qquad \mathcal{L} = \bigcup_{0<p\leq 1} \mathcal{L}_p.$$

From the above definitions on Φ, we may always suppose that any $\Phi \in \mathcal{U}$ (resp. $\mathcal{L}$), is convex (resp. concave) and that Φ is a $\mathcal{C}^1$ function with derivative $\Phi'(t) \approx \frac{\Phi(t)}{t}$ (cf. [17, 18]).

Recall that the complementary function Ψ of a convex growth function Φ, is the function defined from $\mathbb{R}_+$ onto itself by

$$\Psi(s) = \sup_{t\in\mathbb{R}_+} \{ts - \Phi(t)\}.$$

A growth function Φ is said to satisfy the ∇_2-*condition* whenever both Φ and its complementary function Ψ satisfy the Δ_2-condition. See [15, 18] for more details on the complementary function Ψ.

Theorem 1.1. *Let $\alpha > -1$, $f \in h(\mathbb{B})$. Assume that Φ is a growth function satisfying one of the following conditions:*

- *(i) $\Phi \in \mathcal{U}^q$ and satisfies the ∇_2-condition;*
- *(ii) $\Phi \in \mathcal{L}_p$ and the function $\Phi_p(t) = \Phi(t^{\frac{1}{p}})$ satisfies the ∇_2-condition.*

Then the following statements are equivalent.

- *(a) $f \in \mathcal{B}^\Phi_\alpha$;*
- *(b) $A^s_{\mathcal{R}}(f) \in L^\Phi(\mathbb{B}, dv_\alpha)$;*
- *(c) $A^s_{\nabla}(f) \in L^\Phi(\mathbb{B}, dv_\alpha)$;*
- *(d) $A^s(f) \in L^\Phi(\mathbb{B}, dv_\alpha)$.*

For $a \in \mathbb{B} \setminus \{0\}$ and $\delta > 0$, the Carleson cone is defined as

$$\mathcal{C}_\delta(a) = \Big\{x \in \mathbb{B} : \Big|x - \frac{a}{|a|}\Big| < \delta\Big\}.$$

Let μ be a positive Borel measure on $\mathbb{B}$ and $s > 0$. We say that μ is an s-Carleson measure on $\mathbb{B}$ if there exists a constant C such that for any $a \in \mathbb{B}\setminus\{0\}$ and any $0 < \delta < 2$ such that

$$\mu(\mathcal{C}_\delta(a)) \leq C\delta^{(n-1)s}.$$

When $s = 1$, the above measure is called a Carleson measure. Carleson measures were first introduced in the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ by Carleson [2]. These measures are pretty adapted to the studies of various questions on function spaces.

Given $0 < p, q < \infty$, the question of the characterization of the positive measures μ on $\mathbf{B}_n$ such that the embedding $I_\mu : \mathcal{A}^p_\alpha(\mathbf{B}_n) \to L^q(\mathbf{B}_n, d\mu)$ is continuous has attracted much attention. In the setting of Bergman spaces of the unit disk $\mathbb{D}$, this question was answered due to Hastings and Luecking [10, 13] by using Carleson measures. For the extensions of these results to the unit ball $\mathbf{B}_n$, see [5, 13, 14]. In [19], Ueki established the boundedness and compactness of composition operators between weighted Bergman spaces in $\mathbf{B}_n$ in terms of s-Carleson measures.

Our second aim of this paper is to investigate the Φ-Carleson measure in the real unit ball $\mathbb{B}$ whose definition is given as follows.

Definition 1.1. *Let Φ be a growth function. A positive Borel measure μ on $\mathbb{B}$ is called a Φ-Carleson measure if there exists a constant $C > 0$ such that for any $a \in \mathbb{B} \setminus \{0\}$ and any $0 < \delta < 2$,*

$$\mu(\mathcal{C}_\delta(a)) \le \frac{C}{\Phi(\frac{1}{\delta^{n-1}})}.$$

Obviously, when $\Phi(t) = t^s$, the Φ-Carleson measure is the usual s-Carleson measure on $\mathbb{B}$.

The following result provides an equivalent definition of the Φ-Carleson measure.

Theorem 1.2. *Let $\tau > 0$, $\Phi \in \mathcal{U} \cup \mathcal{L}$ and μ be a positive measure on $\mathbb{B}$. Then μ is a Φ-Carleson measure if and only if*

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \Phi\Big(\frac{(1-|a|^2)^\tau}{[a,x]^{(n-1)+\tau}}\Big) d\mu(x) < \infty. \tag{5}$$

Let Φ_1, Φ_2 be two growth functions. A positive measure μ on $\mathbb{B}$ is called a Φ_2-*Carleson measure for* $\mathcal{B}^{\Phi_1}_\alpha$ if there is a constant C such that

$$\int_{\mathbb{B}} \Phi_2\Big(\frac{|f(x)|}{C\|f\|^{lux}_{\alpha,\Phi_1}}\Big) d\mu(x) \le 1,$$

for all $f \in \mathcal{B}^{\Phi_1}_\alpha$ with $\|f\|^{lux}_{\alpha,\Phi_1} \neq 0$.

In our final result, we discuss the Φ-Carleson measure for harmonic Bergman-Orlicz spaces.

Theorem 1.3. *Let $\alpha > -1$, $\Phi_1, \Phi_2 \in \mathcal{U} \cup \mathcal{L}_{(\frac{1}{2})}$ ($\mathcal{L}_{(\frac{1}{2})} = \cup_{\frac{1}{2}<p\le 1}\mathcal{L}_p$) and μ be a positive measure on $\mathbb{B}$. If Φ_2/Φ_1 is non-decreasing, then the following statements are equivalent.*

(*a*) *There exists a constant $C_1 > 0$ such that for any $a \in \mathbb{B} \setminus \{0\}$ and any $0 < \delta < 1$,*

$$\mu(\mathcal{C}_\delta(a)) \le \frac{C_1}{\Phi_2 \circ \Phi_1^{-1}(\frac{1}{\delta^{n+\alpha}})}; \tag{6}$$

(*b*) *μ is a Φ_2-Carleson measure for $\mathcal{B}_\alpha^{\Phi_1}$;*

(*c*) *There exists a constant $C_3 > 0$ such that*

$$(7) \quad \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \Phi_2\Big(\Phi_1^{-1}\big(\frac{1}{(1-|a|^2)^{n+\alpha}}\big)\frac{(1-|a|^2)^{2(n+\alpha)}}{[a,x]^{2(n+\alpha)}}\Big) d\mu(x) \le C_3.$$

The organization of this paper is as follows. In Section 2, some necessary terminologies are introduced and several known results are recalled. Sections 3 and 4 are devoted to the proofs of Theorems 1.1 $\sim$ 1.3. Throughout this paper, we always assume without loss of generality that our growth functions Φ are satisfying $\Phi(1) = 1$. The constants are denoted by C, they are positive and may differ from one occurrence to the other. For nonnegative quantities X and Y, $X \lesssim Y$ means that X is dominated by Y times some inessential positive constant. We write $X \approx Y$ if $Y \lesssim X \lesssim Y$.

2. Preliminaries

In this section, we introduce notations and collect some preliminary results that we will need later.

2.1 Operators on Orlicz spaces

Let Φ be a $\mathcal{C}^1$ growth function. Recall that the lower and the upper indices of Φ are respectively defined by

$$a_\Phi = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)} \quad \text{and} \quad b_\Phi = \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)}.$$

It is known that when Φ is convex, then $1 \le a_\Phi \le b_\Phi < \infty$ and, if Φ is concave, then $0 \le a_\Phi \le b_\Phi \le 1$. Note that a convex growth function satisfies the ∇_2-condition if and only if $1 < a_\Phi \le b_\Phi < \infty$ (cf. [6], Lemma 2.1).

Definition 2.1. *Let Φ be a growth function. A linear operator T defined on $L^\Phi(\mathbb{B}, dv_\alpha)$ is said to be of mean strong type $(\Phi,\Phi)_\alpha$ if*

$$\int_{\mathbb{B}} \Phi(|Tf|) dv_\alpha(x) \le C \int_{\mathbb{B}} \Phi(|f|) dv_\alpha(x),$$

for any $f \in L^\Phi(\mathbb{B}, dv_\alpha)$, and T is said to be mean weak type $(\Phi,\Phi)_\alpha$ if

$$\sup_{t>0} \Phi(t) v_\alpha(\{x \in \mathbb{B} : |Tf(x)| > t\}) \le C \int_{\mathbb{B}} \Phi(|f|) dv_\alpha(x),$$

for any $f \in L^\Phi(\mathbb{B}, dv_\alpha)$, where C is independent of f.

We remark that if $\Phi(t) = t^p$, then the mean strong type $(t^p, t^p)_\alpha$ is the usual strong type (p, p). The following interpolation result comes from [7, Theorem 4.3].

Lemma 2.1. *Let Φ_0, Φ_1 and Φ_2 be three convex growth functions. Sup- pose that their upper and lower indices satisfy the following condition*

$$1 \le a_{\Phi_0} \le b_{\Phi_0} < a_{\Phi_2} \le b_{\Phi_2} < a_{\Phi_1} \le b_{\Phi_1} < \infty.$$

If T is of mean weak types $(\Phi_0, \Phi_0)_\alpha$ and $(\Phi_1, \Phi_1)_\alpha$, then it is of mean strong type $(\Phi_2, \Phi_2)_\alpha$.

Let $\beta \in \mathbb{R}$ and consider the operator E_β defined for functions f on $\mathbb{B}$ by

$$E_\beta f(x) = \int_{\mathbb{B}} f(y) \frac{(1-|y|^2)^\beta}{[x,y]^{n+\beta}} dv(y).$$

For a proof of the following lemma, see [9, Theorem 1.6].

Lemma 2.2. *Let $1 \le p < \infty$ and $\alpha, \beta > -1$. The operator $E_\beta : L^p(\mathbb{B}, dv_\alpha) \to L^p(\mathbb{B}, dv_\alpha)$ is bounded if and only if $\alpha + 1 < p(\beta + 1)$.*

Combining Lemmas 2.1 and 2.2, the following result can be easily derived, see [18, Theorem 2.5].

Lemma 2.3. *Let $\alpha, \beta > -1$ and Φ be a $\mathcal{C}^1$ convex growth function with its lower indice a_Φ. If $1 < p < a_\Phi$ and $\alpha + 1 < p(\beta + 1)$, then E_β is of mean strong type $(\Phi, \Phi)_\alpha$.*

2.2 Harmonic functions

It is well-known that the weighted harmonic Bergman spaces $\mathcal{B}^2_\alpha$ for $\alpha > -1$ is a reproducing kernel Hilbert space with reproducing kernel $R_\alpha(x, y)$:

$$f(x) = \int_{\mathbb{B}} f(y) R_\alpha(x,y) dv_\alpha(y), \ f \in \mathcal{B}^2_\alpha. \tag{8}$$

From [7], we know that (8) is also true for all $f \in \mathcal{B}^1_\alpha$.

The reproducing kernels $R_\alpha(x, y)$ can be expressed in terms of zonal harmonics as

$$R_\alpha(x,y) = \sum_{k=0}^{\infty} \frac{(1 + \frac{n}{2} + \alpha)_k}{(\frac{n}{2})_k} Z_k(x,y) = \sum_{k=0}^{\infty} \gamma_k(\alpha) Z_k(x,y),$$

where the series absolutely and uniformly converges on $K \times \mathbb{B}$ for any compact subset K of $\mathbb{B}$ and $(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}$. A straightforward computation gives that

$$|R_\alpha(x,y)| \lesssim \frac{1}{[x,y]^{n+\alpha}}. \tag{9}$$

Note that $R_\alpha(x, y)$ is real-valued, symmetric in the variables x and y and harmonic with respect to each variable since the same is true for all $Z_k(x, y)$. For the extension of reproducing kernels $R_\alpha(x, y)$ to all $\alpha \in \mathbb{R}$, see [7, 9].

We recall some useful inequalities concerning harmonic functions which are useful for our investigations.

Lemma 2.4 ([7, 16])**.** *Let $0 < p < \infty$, $0 < r < 1$ and $f, g \in h(\mathbb{B})$. Then there exists some positive constant C such that*

(1) $|f(x)|^p \le C \int_{E(x,r)} |f(y)|^p d\tau(y)$;

(2) $|\nabla f(x)|^p \le \frac{C}{(1-|x|^2)^p} \int_{E(x,r)} |f(y)|^p d\tau(y)$.

Moreover, if $0 < p \le 1$ and $\alpha > -1$, then there exists a positive constant C such that

(3)

$$\int_{\mathbb{B}} |f(x)g(x)|(1-|x|^2)^{(n+\alpha)/p-n} dv(x) \le C\Big(\int_{\mathbb{B}} |f(x)g(x)|^p dv_\alpha(x)\Big)^{1/p}.$$

The following standard estimate will be needed in the sequel.

Lemma 2.5 ([16])**.** *Let $\alpha > -1$ and $\beta \in \mathbb{R}$. Then for any $x \in \mathbb{B}$,*

$$\int_{\mathbb{B}} \frac{(1-|y|^2)^\alpha}{[x,y]^{n+\alpha+\beta}} dv(y) \approx \begin{cases} (1-|x|^2)^{-\beta}, & \beta > 0, \\ \log \dfrac{1}{1-|x|^2}, & \beta = 0, \\ 1, & \beta < 0. \end{cases}$$

3. Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. Before the proof, we need some preparation.

Lemma 3.1 ([8])**.** *Let $\Phi \in \mathcal{L}_p$. Then the growth function Φ_p, defined by $\Phi_p(t) = \Phi(t^{\frac{1}{p}})$ is in $\mathcal{U}^q$ for some $q \ge 1$. Moreover, for $s > 0$ and $t \ge 1$,*

$$\Phi_p(ts) \le t^{\frac{1}{p}} \Phi_p(s).$$

By Lemmas 2.4 and Lemma 3.1, we can obtain the following useful integral estimates.

Lemma 3.2. *Let $f \in h(\mathbb{B})$ and $\Phi \in \mathcal{U}^q \cup \mathcal{L}_p$. Then for $0 < r < 1$ and $x \in \mathbb{B}$,*

(1) $\Phi\big((1-|x|^2)|\nabla f(x)|\big) \lesssim \int_{E(x,r)} \Phi(|f(y))|d\tau(y)$;

(2) $\Phi(|f(x)|) \lesssim \int_{E(x,r)} \Phi(|f(y)|)d\tau(y)$.

Proof. Let

$$p_\Phi = \begin{cases} 1, & \text{if} \quad \Phi \in \mathcal{U}^q, \\ p, & \text{if} \quad \Phi \in \mathcal{L}_p. \end{cases}$$

By Lemma 2.4, for each $x \in \mathbb{B}$,

$$\big((1-|x|^2)|\nabla f(x)|\big)^{p_\Phi} \lesssim \int_{E(x,r)} |f(y)|^{p_\Phi} d\tau(y).$$

Set

$$\Phi_p(t) = \begin{cases} \Phi(t), & \text{if } \Phi \in \mathcal{U}^q, \\ \Phi(t^{\frac{1}{p}}), & \text{if } \Phi \in \mathcal{L}_p. \end{cases}$$

It follows from Lemma 3.1 and the convexity of $\Phi_p(t)$ that

$$\Phi((1-|x|^2)|\nabla f(x)|) \lesssim \int_{E(x,r)} \Phi(|f(y)|) d\tau(y).$$

This proves (1).

By Lemma 2.4 and an argument similar to the above, the assertion of (2) follows. □

Lemma 3.3. *Assume that Φ is a growth function satisfying one of the following conditions:*

(*i*) $\Phi \in \mathcal{U}^q$ *and satisfies the ∇_2-condition;*

(*ii*) $\Phi \in \mathcal{L}_p$ *and the function $\Phi_p(t) = \Phi(t^{\frac{1}{p}})$ satisfies the ∇_2-condition.*

If $\alpha > -1$ and $f \in h(\mathbb{B})$, then

$$\int_{\mathbb{B}} \Phi(|f(x)-f(0)|) dv_\alpha(x) \lesssim \int_{\mathbb{B}} \Phi((1-|x|^2)|\mathcal{R}f(x)|) dv_\alpha(x); \tag{10}$$

and

$$\int_{\mathbb{B}} \Phi((1-|x|^2)|\nabla f(x)|) dv_\alpha(x) \lesssim \int_{\mathbb{B}} \Phi(|f(x)|) dv_\alpha(x). \tag{11}$$

Proof. We first prove (10). Let $f \in h(\mathbb{B})$. Then for $s > -1$,

$$\mathcal{R}f(x) = \int_{\mathbb{B}} \mathcal{R}f(y) R_s(x,y) dv_s(y).$$

Since $\int_{\mathbb{B}} \mathcal{R}f(y) dv_s(y) = 0$, subtracting this from the previous equation yields

$$\mathcal{R}f(x) = \int_{\mathbb{B}} \mathcal{R}f(y)(R_s(x,y)-1) dv_s(y).$$

Consequently,

$$\begin{aligned} |f(x)-f(0)| &= \Big|\int_0^1 \int_{\mathbb{B}} \mathcal{R}f(y)(R_s(tx,y)-1) dv_s(y) \frac{dt}{t}\Big| \\ &= \Big|\int_{\mathbb{B}} \mathcal{R}f(y) \int_0^1 \frac{R_s(tx,y)-1}{t} dt dv_s(y)\Big|. \end{aligned}$$

Set

$$G(x,y)=\int_0^1 \frac{R_s(tx,y)-1}{t}dt.$$

From the proof of [9, Lemma12.1], it deduces that

$$|G(x,y)|\le\int_0^1\Big|\frac{R_s(tx,y)-1}{t}\Big|dt\lesssim\int_0^1\frac{dt}{[tx,y]^{n+s}}\lesssim\frac{1}{[x,y]^{n+s-1}}.$$

Therefore,

$$|f(x)-f(0)| \lesssim \int_{\mathbb{B}}(1-|y|^2)|\mathcal{R}f(y)|\frac{1}{[x,y]^{n+s-1}}dv_{s-1}(y).$$

We first consider the case Φ satisfies the condition (i) of the lemma. Fix p so that $1<p<a_\Phi$. By taking s large enough so that $\alpha+1<ps$, we conclude from Lemma 2.3 that

$$\int_{\mathbb{B}}\Phi(|f(x)-f(0)|)dv_\alpha(x)\lesssim\int_{\mathbb{B}}\Phi((1-|x|^2)|\mathcal{R}f(x)|)dv_\alpha(x).$$

We next consider the case of $\Phi\in\mathcal{L}_p$ and $\Phi_p(t)=\Phi(t^{\frac{1}{p}})$ satisfies the ∇_2-condition. Set $s=(n+\alpha')/p-n$ and $\alpha'>\alpha+p$. By Lemma 2.4, it deduces that

$$\begin{aligned}|f(x)-f(0)|^p &\lesssim \int_{\mathbb{B}}|\mathcal{R}f(y)|^p|G(x,y)|^p dv_{\alpha'}(y)\\ &\lesssim \int_{\mathbb{B}}\frac{|\mathcal{R}f(y)|^p}{[x,y]^{p(n+s-1)}}dv_{\alpha'}(y)\\ &\lesssim \int_{\mathbb{B}}\frac{|(1-|y|^2)\mathcal{R}f(y)|^p}{[x,y]^{n+\alpha'-p}}dv_{\alpha'-p}(y).\end{aligned}$$

As the growth function $t\to\Phi_p(t)=\Phi(t^{\frac{1}{p}})$ is in $\mathcal{U}^q$ and satisfies the ∇_2-condition, proceeding as in the first part of this proof yields that

$$\begin{aligned}\int_{\mathbb{B}}\Phi(|f(x)-f(0)|)dv_\alpha(x) &= \int_{\mathbb{B}}\Phi_p(|f(x)-f(0)|^p)dv_\alpha(x)\\ &\lesssim \int_{\mathbb{B}}\Phi_p((1-|x|^2)|\mathcal{R}f(x)|)^p)dv_\alpha(x)\\ &= \int_{\mathbb{B}}\Phi((1-|x|^2)|\mathcal{R}f(x)|)dv_\alpha(x).\end{aligned}$$

We now come to prove (11). By Lemma 3.2, we have

$$\Phi((1-|x|^2)|\nabla f(x)|)\lesssim\int_{E(x,r)}\Phi(|f(y)|)d\tau(y),\quad x\in\mathbb{B}.$$

Integrating both sides of the above inequality over $\mathbb{B}$ with respect to $dv_\alpha(x)$ and applying Fubini's theorem, we get

$$\int_{\mathbb{B}} \Phi((1-|x|^2)|\nabla f(x)|)dv_\alpha(x) \lesssim \int_{\mathbb{B}} \Phi(|f(x)|)dv_\alpha(x).$$

This completes the proof. □

Proof of Theorem 1.1. We only prove $(a) \Leftrightarrow (b)$. Similar discussions can be applied to prove $(a) \Leftrightarrow (c)$ and $(a) \Leftrightarrow (d)$.

We first assume that $A^s_{\mathcal{R}}(f) \in L^\Phi(\mathbb{B}, dv_\alpha)$. By Lemma 2.4, for each $x \in \mathbb{B}$, we have

$$|(1-|x|^2)\mathcal{R}f(x)| \lesssim A^s_{\mathcal{R}}(f)(x).$$

Then $(b) \Rightarrow (a)$ follows from Lemma 3.3.

For the converse, we assume that $f \in \mathcal{B}^\Phi_\alpha$. For each fixed $x \in \mathbb{B}$, let

$$h(x) = \sup\{(1-|\zeta|^2)|\mathcal{R}f(\zeta)| : \zeta \in E(x, \frac{1}{2})\}.$$

From (1), we can find r' such that $0 < \frac{1}{2} < r' < 1$ and $E(\xi, \frac{1}{2}) \subset E(x, r')$ for every $\xi \in E(x, \frac{1}{2})$. It follows from Lemma 3.2 that

$$\Phi\big(|A^s_{\mathcal{R}}(f)(x)\big)| \lesssim \Phi(h(x)) \lesssim \int_{E(x,r')} \Phi(|f(y)|)d\tau(y)$$

Hence by Fubini's theorem and (2),

$$\begin{aligned}
\int_{\mathbb{B}} \Phi\big(|A^s_{\mathcal{R}}(f)(x)\big)|dv_\alpha(x) &\lesssim \int_{\mathbb{B}} (1-|x|^2)^\alpha \int_{E(x,r')} \Phi(|f(y)|)d\tau(y)dv(x) \\
&\lesssim \int_{\mathbb{B}} \Phi(|f(y)|)d\tau(y) \int_{E(y,r')} (1-|x|^2)^\alpha dv(x) \\
&\lesssim \int_{\mathbb{B}} \Phi(|f(y)|)dv_\alpha(y).
\end{aligned}$$

This completes the proof. □

4. Proofs of Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2. Assume first that (5) holds. For each $a \in \mathbb{B} \setminus \{0\}$, set $\delta = 1 - |a|$. A simple computation gives that

$$[a, x] \leq 1 - |a|^2 \leq 2\delta,$$

for $x \in \mathcal{C}_\delta(a)$. Therefore

$$\begin{aligned}
\mu(\mathcal{C}_\delta(a))\Phi(\frac{1}{\delta^{n-1}}) &= \int_{\mathcal{C}_\delta(a)} \Phi(\frac{1}{\delta^{n-1}}) d\mu(x) \\
&\lesssim \int_{\mathcal{C}_\delta(a)} \Phi(\frac{2^{n-1}}{[a,x]^{n-1}}) d\mu(x) \\
&\lesssim \int_{\mathcal{C}_\delta(a)} \Phi(\frac{2^{n-1}(1-|a|^2)^\tau}{[a,x]^{n-1+\tau}}) d\mu(x) \\
&\lesssim \int_{\mathbb{B}} \Phi\Big(\frac{(1-|a|^2)^\tau}{[a,x]^{(n-1)+\tau}}\Big) d\mu(x),
\end{aligned}$$

where the last inequality follows from the monotonicity of Φ or $\frac{\Phi(t)}{t}$.

Conversely, assume that μ is a Φ-Carleson measure. The proof is based on a standard slicing trick, see [11, Lemma 2.2]. Without loss of generality, let $\frac{1}{2} < |a| < 1$. Denote $Q_0(a) = \emptyset$ and

$$Q_k(a) = \Big\{x \in \mathbb{B} : \Big|x - \frac{a}{|a|}\Big| < 2^{k-1}(1-|a|)\Big\},\ k = 1, 2, ..., N,$$

where N is the smallest integer such that $2^{N-1}(1-|a|) \geq 2$.

Since for each $x \in Q_k(a) \backslash Q_{k-1}(a)$, $[a,x] \geq |a| 2^{(k-2)}(1-|a|)$, we have

$$\begin{aligned}
&\int_{\mathbb{B}} \Phi\Big(\frac{(1-|a|^2)^\tau}{[a,x]^{(n-1)+\tau}}\Big) d\mu(x) \\
&\lesssim \sum_{k=1}^{N} \int_{Q_k(a)\backslash Q_{k-1}(a)} \Phi\Big(\frac{(1-|a|^2)^\tau}{2^{(k-2)(n-1+\tau)}(1-|a|)^{(n-1)+\tau}}\Big) d\mu(x) \\
&\lesssim \sum_{k=1}^{N} \frac{\Phi\Big(\frac{1}{2^{(k-2)(n-1+\tau)}(1-|a|)^{n-1}}\Big)}{\Phi\Big(\frac{1}{2^{(k-1)(n-1)}(1-|a|)^{n-1}}\Big)} \\
&\lesssim \sum_{k=1}^{N} \frac{1}{2^{k\tau\varsigma}} < \infty,
\end{aligned}$$

where $\varsigma = 1$ if $\Phi \in \mathcal{U}$ and $\varsigma = p$ if $\Phi \in \mathcal{L}$ is of lower type $0 < p \leq 1$. The proof is complete.

In order to prove Theorem 1.3, we need the following two lemmas.

Lemma 4.1. *Let $\alpha > -1$, $\Phi \in \mathcal{U} \cup \mathcal{L}$ and $f \in \mathcal{B}_\alpha^\Phi$. Then there exists a positive constant C such that for each $a \in \mathbb{B}$,*

$$|f(a)| \leq C\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big)\|f\|_{\alpha,\Phi}^{lux}. \tag{12}$$

Proof. If $\|f\|_{\alpha,\Phi}^{lux} = 0$, then $f = 0$ a.e. on $\mathbb{B}$ so that (12) obviously holds. Suppose that $\|f\|_{\alpha,\Phi}^{lux} \neq 0$. In view of (2) and Lemma 2.4, we see that for $a \in \mathbb{B}$

and $0 < p < \infty$,

$$|f(a)|^p \lesssim \int_{E(a,r)} |f(x)|^p \Big(\frac{(1-|a|^2)}{[x,a]^2}\Big)^{n+\alpha} dv_\alpha(x).$$

It follows a similar discussion in the proof of Lemma 3.2,

$$\begin{aligned}\Phi\Big(\frac{|f(a)|}{\|f\|^{lux}_{\alpha,\Phi}}\Big) &\lesssim \int_{E(a,r)} \Phi\Big(\frac{|f(x)|}{\|f\|^{lux}_{\alpha,\Phi}}\Big)\Big(\frac{(1-|a|^2)}{[x,a]^2}\Big)^{n+\alpha} dv_\alpha(x)\\ &\lesssim \frac{1}{(1-|a|^2)^{n+\alpha}},\end{aligned}$$

which gives (12). □

Lemma 4.2. *Let $\alpha > -1$, $\frac{1}{2} < p \leq 1$ and $\Phi \in \mathcal{U} \cup \mathcal{L}_p$. Then each $a \in \mathbb{B}$, the following function*

$$f_a(x) = \Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big) R_{n+2\alpha}(x,a)(1-|a|^2)^{2(n+\alpha)}$$

belongs to $\mathcal{B}^\Phi_\alpha$.

Proof. Let

$$h_a(x) = \frac{(1-|a|^2)^{2(n+\alpha)}}{[x,a]^{2(n+\alpha)}}.$$

Since $\alpha > -1$, from (8),

$$\begin{aligned}&\int_{\mathbb{B}} \Phi(|f_a(x)|) dv_\alpha(x)\\ &= \int_{\mathbb{B}} \Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big)|R_{n+2\alpha}(x,a)|(1-|a|^2)^{2(n+\alpha)}\Big) dv_\alpha(x)\\ &\lesssim \int_{\mathbb{B}} \Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big) h_a(x)\Big) dv_\alpha(x)\\ &= I_1 + I_2,\end{aligned}$$

where

$$I_1 = \int_{\{x\in\mathbb{B}: h_a(x)\leq 1\}} \Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big) h_a(x)\Big) dv_\alpha(x)$$

and

$$I_2 = \int_{\{x\in\mathbb{B}: h_a(x)\geq 1\}} \Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big) h_a(x)\Big) dv_\alpha(x).$$

We now divide the remainder of the proof into the following two cases.

Case I. $\Phi \in \mathcal{U}$. By the monotonicity of $\frac{\Phi(t)}{t}$ and Lemma 2.5,

$$\begin{aligned} I_1 &\lesssim \int_{\{x\in\mathbb{B}:h_a(x)\le 1\}} h_a(x)\Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big)\Big)dv_\alpha(x) \\ &\lesssim \int_{\mathbb{B}} \frac{(1-|a|^2)^{(n+\alpha)}}{[x,a]^{2(n+\alpha)}}dv_\alpha(x) \lesssim 1. \end{aligned}$$

Using (3), there exists some $q \geq 1$ such that

$$\begin{aligned} I_2 &= \int_{\{x\in\mathbb{B}:h_a(x)\ge 1\}} \Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big)h_a(x)\Big)dv_\alpha(x). \\ &\lesssim \int_{\mathbb{B}} \frac{(1-|a|^2)^{(2q-1)(n+\alpha)}}{[x,a]^{2q(n+\alpha)}}dv_\alpha(x) \lesssim 1. \end{aligned}$$

Case II. $\Phi \in \mathcal{L}_p$ with $p > \frac{1}{2}$. Using (4) and Lemma 2.5, we have

$$\begin{aligned} I_1 &\lesssim \int_{\{x\in\mathbb{B}:h_a(x)\le 1\}} h_a(x)^p\Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big)\Big)dv_\alpha(x) \\ &\lesssim \int_{\mathbb{B}} \frac{(1-|a|^2)^{(2p-1)(n+\alpha)}}{[x,a]^{2p(n+\alpha)}}dv_\alpha(x) \lesssim 1. \end{aligned}$$

By the monotonicity of $\frac{\Phi(t)}{t}$ and Lemma 2.5 again,

$$\begin{aligned} I_2 &= \int_{\{x\in\mathbb{B}:h_a(x)\ge 1\}} \Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big)h_a(x)\Big)dv_\alpha(x) \\ &\lesssim \int_{\mathbb{B}} \frac{(1-|a|^2)^{(n+\alpha)}}{[x,a]^{2(n+\alpha)}}dv_\alpha(x) \lesssim 1. \end{aligned}$$

Combining the above two cases, the assertion of this lemma follows. □

Now we are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. The proof will follow by the routes $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.

We first prove $(a) \Rightarrow (b)$. For $y \in \mathbb{B} \setminus \{0\}$ and $\frac{1}{4} < r < 1$. By (1) and (2), we see that for large enough k, $E(y, r) \subset Q_k(y)$ and

$$\mu(E(y,r)) \le \mu(Q_k(y)) \lesssim \frac{1}{\Phi_2 \circ \Phi_1^{-1}\big(\frac{1}{2^{(k-1)(n+\alpha)}(1-|y|)^{n+\alpha}}\big)}. \tag{13}$$

Let $f \in \mathcal{B}_\alpha^{\Phi_1}$ with $\|f\|_{\alpha,\Phi_1}^{lux} \neq 0$. Note that $\Phi_2 \in \mathcal{U} \cup \mathcal{L}_{(\frac{1}{2})}$, then

$$\Phi_2\Big(\frac{|f(x)|}{\|f\|_{\alpha,\Phi_1}^{lux}}\Big) \lesssim \int_{E(x,\frac{1}{4})} \Phi_2\Big(\frac{|f(y)|}{\|f\|_{\alpha,\Phi_1}^{lux}}\Big)(1-|y|^2)^{-(n+\alpha)}dv_\alpha(y)$$

by Lemma 3.2. Thus

$$\begin{aligned} L &= \int_{\mathbb{B}} \Phi_2\Big(\frac{|f(x)|}{\|f\|^{lux}_{\alpha,\Phi_1}}\Big) d\mu(x) \\ &\lesssim \int_{\mathbb{B}} d\mu(x) \int_{E(x,\frac{1}{4})} \Phi_2\Big(\frac{|f(y)|}{\|f\|^{lux}_{\alpha,\Phi_1}}\Big)(1-|y|^2)^{-(n+\alpha)} dv_\alpha(y) \\ &\lesssim \int_{\mathbb{B}} \Big(\int_{\mathbb{B}} \chi_{E(y,\frac{1}{4})}(x) d\mu(x)\Big) \Phi_2\Big(\frac{|f(y)|}{\|f\|^{lux}_{\alpha,\Phi_1}}\Big)(1-|y|^2)^{-n} dv(y). \end{aligned}$$

From (1), we can find an integer k such that and $E(x,\frac{1}{4}) \subset Q_k(y)$ for every $x \in E(y,\frac{1}{4})$. It follows from Lemma 3.2 and (13) that

$$L \lesssim \int_{\mathbb{B}} \Phi_2\Big(\frac{|f(y)|}{\|f\|^{lux}_{\alpha,\Phi_1}}\Big)\mu(Q_k(y))(1-|y|^2)^{-n} dv(y).$$

By the assumption Φ_2/Φ_1 is non-decreasing and (12),

$$\begin{aligned} L &\lesssim \int_{\mathbb{B}} \Phi_1\Big(\frac{|f(y)|}{\|f\|^{lux}_{\alpha,\Phi_1}}\Big) \frac{\Phi_2\circ\Phi_1^{-1}\big(\frac{1}{(1-|y|^2)^{n+\alpha}}\big)}{\Phi_1\circ\Phi_1^{-1}\big(\frac{1}{(1-|y|^2)^{n+\alpha}}\big)} (1-|y|^2)^{-n}\mu(Q_k(y)) dv(y) \\ &\lesssim \int_{\mathbb{B}} \Phi_1\Big(\frac{|f(y)|}{\|f\|^{lux}_{\alpha,\Phi_1}}\Big) dv_\alpha(y) \le 1. \end{aligned}$$

This implies that we can find a constant $C_2 > 0$ such that

$$\int_{\mathbb{B}} \Phi_2\Big(\frac{|f(x)|}{C_2\|f\|^{lux}_{\alpha,\Phi_1}}\Big) d\mu(x) \le 1.$$

$(b) \Rightarrow (c)$. For $a \in \mathbb{B}$, recall that

$$f_a(x) = \Phi_1^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big) R_{n+2\alpha}(x,a)(1-|a|^2)^{2(n+\alpha)} \in \mathcal{B}^{\Phi_1}_\alpha$$

from Lemma 4.2. Thus, the implication easily follows by testing f_a and using the monotonicity of Φ_2 or the monotonicity of the function $\frac{\Phi_2(t)}{t}$.
$(c) \Rightarrow (a)$. The implication $(c) \Rightarrow (a)$ follows the same way as in the proof of Theorem 1.2. We omit the details here. □

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