On pseudo picture fuzzy cosets

Yuming Feng
School of Computer Science and Engineering
Chongqing Three Gorges University
Wanzhou, Chongqing, 404100
China
yumingfeng25928@163.com
ymfeng@sanxiau.edu.cn

Taiwo O. Sangodapo*
Department of Mathematics, Faculty of Science,
University of Ibadan, Ibadan
Nigeria
toewuola77@gmail.com
to.ewuola@ui.edu.ng

Abstract. In this paper, the concepts of pseudo picture fuzzy cosets, pseudo picture fuzzy double cosets and pseudo picture fuzzy middle cosets were introduced and some of their characteristics were established. In addition, we investigated the connections between pseudo picture fuzzy double cosets and picture fuzzy normal subgroup, also between pseudo picture fuzzy middle cosets and picture fuzzy normal subgroup.

Keywords: Picture fuzzy set, Picture fuzzy subgroup, Pseudo picture fuzzy cosets, Pseudo picture fuzzy double cosets, Pseudo picture fuzzy middle cosets.

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1. Introduction

The generalisation of theory of crisp sets into the theory of fuzzy sets was introduced by Zadeh [27]. This theory has become a vast and sprawling area of research in topology, algebra, engineering, convexity etc. The fuzzy sets only deal with the membership degree of an element in belonging to a set. In [1], Atanassov extended the theory of fuzzy sets into intuitionistic fuzzy sets which took care of both the membership and non-membership degrees of an element belonging to a set. Cuong and Kreinovich [14] generalised the notions of both fuzzy sets and intuitionistic fuzzy sets into picture fuzzy sets. In their work, one of the items needed to still determine the membership of an element in a set was added, and it is called the neutral membership degree. Thus, picture fuzzy sets theory comprises of positive membership, neutral membership and negative membership degrees.

∗. Corresponding author

Rosenfeld [23] put forward the notion of fuzzy group. As an extension of fuzzy group, Biswas [6] initiated the idea of intuitionistic fuzzy subgroup of a group. Zhan and Tan [28] also studied intuitionistic fuzzy subgroup. Sharma [24] established some properties of intuitionistic fuzzy subgroup of a group through cut set of intuitionistic fuzzy sets. In [25], Sharma introduced t-intuitionistic fuzzy sets and obtained some properties. Dogra and Pal [15] initiated the concept of picture fuzzy subring of a crisp ring and studied some related basic results. They also investigated some properties of picture fuzzy subring under classical ring homomorphism. Dogra and Pal [16] put forward the notion of picture fuzzy subspace of a crisp vector space and obtained some basic results related to it on the basis of some basic operations on picture fuzzy sets. Furthermore, direct sum of two picture fuzzy subspaces, isomorphism between two picture fuzzy subspaces, picture fuzzy linear transformation and picture fuzzy linearly independent set of vectors and some properties connected to these were established.

In [21], Mukherjee and Bhattacharya initiated fuzzy cosets. The extension to pseudo fuzzy cosets was studied by Nagarajan and Solaraiju [20] and some properties were established. This concept was later studied by Onasanya and Ilori [20] to obtain some independent proofs of the properties established in [20]. Sharma [25] introduced t-intuitionistic fuzzy left (right) cosets and investigated some of its properties. Dogra and Pal [17] introduced picture fuzzy subgroup of a crisp group, picture fuzzy left (right, middle) cosets, and some of their properties were obtained. In [26], Sharma and Sandhu initiated pseudo intuitionistic fuzzy cosets, pseudo intuitionistic fuzzy double cosets and pseudo intuitionistic fuzzy middle cosets of a group and established some of their properties. Since the notion of picture fuzzy set was a generalisation of both fuzzy sets and intuitionistic fuzzy sets [14], the idea of fuzzy cosets was extended to intuitionistic fuzzy cosets [25] and the pseudo fuzzy cosets was also extended to pseudo intuitionistic fuzzy cosets [26]. Thus, the concept of pseudo picture fuzzy cosets which is a generalisation of pseudo intuitionistic fuzzy cosets can be a research focus.

In this paper, the concepts of pseudo intuitionistic fuzzy cosets was generalised to pseudo picture fuzzy cosets. We have put forward the *pseudo picture fuzzy cosets* (PPFCs), *pseudo picture fuzzy double cosets* (PPFDs) and *pseudo picture fuzzy middle cosets* (PPFMs), and some of their characterisations were established. It was established that, this concept is a generalisation of the notion introduced by Sharma and Sandhu in [26]. The paper is organised as follows. In Section 2, we give some definitions, basic operations and preliminary results. In Section 3, we introduce PPFCs, PPFDs and PPFMs and establish some of their characterisations.

2. Preliminaries

In this section, some basic definitions, operations and preliminary results are stated.

Definition 2.1 ([27]). *Let Y be a nonempty set. A fuzzy set (FS) Q of Y is defined as*

$$Q = \{\langle y, \sigma_Q(y)\rangle | y \in Y\}$$

with a membership function

$$\sigma_Q : Y \longrightarrow [0, 1],$$

where the function $\sigma_Q(y)$ denotes the degree of membership of $y \in Q$.

Definition 2.2 ([1]). *Let a nonempty set Y be fixed. An intuitionistic fuzzy set (IFS) Q of Y is defined as*

$$Q = \{\langle y, \sigma_Q(y), \tau_Q(y)\rangle | y \in Y\},$$

where the functions

$$\sigma_Q : Y \to [0, 1] \text{ and } \tau_Q : Y \to [0, 1]$$

are called the membership and non-membership degrees, respectively, and for every $y \in Y$,

$$0 \leq \sigma_Q(y) + \tau_Q(y) \leq 1.$$

Definition 2.3 ([14]). *A picture fuzzy set Q of Y is defined as*

$$Q = \{(y, \sigma_Q(y), \tau_Q(y), \gamma_Q(y)) | y \in Y\},$$

where the functions

$$\sigma_Q : Y \to [0, 1],\ \tau_Q : Y \to [0, 1] \text{ and } \gamma_Q : Y \to [0, 1]$$

are called the positive, neutral and negative membership degrees, respectively, and $\sigma_Q, \tau_Q, \gamma_Q$ satisfy for any $y \in Y$,

$$0 \leq \sigma_Q(y) + \tau_Q(y) + \gamma_Q(y) \leq 1.$$

Then, $S_Q(y) = 1 - (\sigma_Q(y) + \tau_Q(y) + \gamma_Q(y))$ is called the refusal membership degree of $y \in Q$.

Definition 2.4 ([14]). *Let Q and R be two PFSs. Then, the inclusion, equality, union, intersection and complement are defined as follow:*

- *$Q \subseteq R$ if and only if for all $y \in Y$, $\sigma_Q(y) \leq \sigma_R(y)$, $\tau_Q(y) \leq \tau_R(y)$ and $\gamma_Q(y) \geq \gamma_R(y)$.*

- $Q = R$ *if and only if* $Q \subseteq R$ *and* $R \subseteq Q$.
- $Q \cup R = \{(y, \sigma_Q(y) \vee \sigma_R(y), \tau_Q(y) \wedge \tau_R(y)), \gamma_Q(y) \wedge \gamma_R(y))|y \in Y\}$.
- $Q \cap R = \{(y, \sigma_Q(y) \wedge \sigma_R(y), \tau_Q(y) \wedge \tau_R(y)), \gamma_Q(y) \vee \gamma_R(y))|y \in Y\}$.
- $coQ = \overline{Q} = \{(y, \gamma_Q(y), \tau_Q(y), \sigma_Q(y))|y \in Y\}$.

Definition 2.5 ([23])**.** *Let* $(G, *)$ *be a group and* $Q = \{(y, \sigma_Q(y)) \mid y \in G\}$ *be an FS in* G. *Then,* Q *is called a fuzzy subgroup (FSG) of* G *if* $\sigma_Q(a * b) \geq \sigma_Q(a) \wedge \sigma_Q(b)$ *and* $\sigma_Q(a^{-1}) \geq \sigma_Q(a)$ *for all* $a, b \in G$, *where* a^{-1} *is the inverse of* $a \in G$.

Definition 2.6 ([6, 24, 28])**.** *Let* $(G, *)$ *be a crisp group and*

$$Q = \{(y, \sigma_Q(y), \tau_Q(y)) \mid y \in G\}$$

be an IFS in G. *Then,* Q *is called intuitionistic fuzzy subgroup (IFSG) of* G *if*
(*i*) $\sigma_Q(a * b) \geq \sigma_Q(a) \wedge \sigma_Q(b),\ \tau_Q(a * b) \leq \tau_Q(a) \vee \tau_Q(b)$,
(*ii*) $\sigma_Q(a^{-1}) \geq \sigma_Q(a),\ \tau_Q(a^{-1}) \leq \tau_Q(a)$,
for all $a, b \in G$, *where* a^{-1} *is the inverse of* $a \in G$.

Definition 2.7 ([17])**.** *Let* $(G, *)$ *be a crisp group and*

$$Q = \{(y, \sigma_Q(y), \tau_Q(y), \eta_Q(y)) \mid y \in G\}$$

be a PFS in G. *Then,* Q *is called picture fuzzy subgroup (PFSG) of* G *if*
(*i*) $\sigma_Q(a * b) \geq \sigma_Q(a) \wedge \sigma_Q(b),\ \tau_Q(a * b) \geq \tau_Q(a) \wedge \tau_Q(b),\ \eta_Q(a * b) \leq \eta_Q(a) \vee \eta_Q(b)$,
(*ii*) $\sigma_Q(a^{-1}) \geq \sigma_Q(a),\ \tau_Q(a^{-1}) \geq \tau_Q(a),\ \eta_Q(a^{-1}) \leq \eta_Q(a)$,
for all $a, b \in G$, *where* a^{-1} *is the inverse of* $a \in G$.

Definition 2.8 ([17])**.** *Let* $(G, *)$ *be a crisp group and* $Q = (\sigma_Q, \tau_Q, \eta_Q)$ *be a PFSG of* G. *Then, for* $a \in G$, *the picture fuzzy left cosets (PFLCs) of* $Q \in G$ *is the PFS* $aQ = (\sigma_{aQ},\ \tau_{aQ},\ \eta_{aQ})$ *defined by*

$$\sigma_{aQ}(y) = \sigma_Q(a^{-1} * y),\ \tau_{aQ}(y) = \tau_Q(a^{-1} * y)\ \textit{and}\ \eta_{aQ}(y) = \eta_Q(a^{-1} * y),$$

for all $y \in G$.

Definition 2.9 ([17])**.** *Let* $(G, *)$ *be a crisp group and* $Q = (\sigma_Q, \tau_Q, \eta_Q)$ *be a PFSG of* G. *Then,* Q *is called a picture fuzzy normal subgroup (PFNSG) of* G *if*

$$\sigma_{Qa}(y) = \sigma_{aQ}(y),\ \tau_{Qa}(y) = \tau_{aQ}(y)\ \textit{and}\ \eta_{Qa}(y) = \eta_{aQ}(y),$$

for all $a,\ y \in G$.

Definition 2.10 ([17]). *Let $(G, *)$ be a crisp group and $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Then, for $a \in G$, the picture fuzzy right cosets (PFRCs) of $Q \in G$ is the PFS $Qa = (\sigma_{Qa},\ \tau_{Qa},\ \eta_{Qa})$ defined by*

$$\sigma_{Qa}(y) = \sigma_Q(y * a^{-1}),\ \tau_{Qa}(y) = \tau_Q(y * a^{-1}) \text{ and } \eta_{Qa}(y) = \eta_Q(y * a^{-1}),$$

for all $y \in G$.

Definition 2.11 ([17]). *Let $(G, *)$ be a crisp group and $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Then, for $a \in G$, the picture fuzzy middle cosets (PFMCs) of $Q \in G$ is the PFS $aQa^{-1} = (\sigma_{aQa^{-1}},\ \tau_{aQa^{-1}},\ \eta_{aQa^{-1}})$ defined by*

$$\sigma_{aQa^{-1}}(y) = \sigma_Q(a^{-1} * y * a),\ \tau_{aQa^{-1}}(y) = \tau_Q(a^{-1} * y * a)$$

and

$$\eta_{aQa^{-1}}(y) = \eta_Q(ya^{-1} * y * a),$$

for all $y \in G$.

3. Pseudo picture fuzzy sets

This section defines pseudo picture fuzzy cosets, pseudo picture fuzzy double cosets and pseudo picture fuzzy middle cosets were introduced and some of their charactristics are established.

Definition 3.1. *Let $(G, *)$ be a crisp group and $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Then, for any $a \in G$ the pseudo picture fuzzy left cosets (PPFLCs) of Q with respect to some fixed PFS y of G is a PFS*

$$(aQ)^y = (\sigma_{(aQ)^y}(x),\ \tau_{(aQ)^y}(x),\ \eta_{(aQ)^y}(x))$$

defined by

$$\sigma_{(aQ)^y}(x) = \sigma_y(a)\sigma_Q(x),$$
$$\tau_{(aQ)^y}(x) = \tau_y(a)\tau_Q(x)$$

and

$$\eta_{(aQ)^y}(x) = \eta_y(a)\eta_Q(x),$$

for all $x \in G$.

Definition 3.2. *Let $(G, *)$ be a crisp group and $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Then, for any $a \in G$ the pseudo picture fuzzy right cosets (PPFRCs) of Q with respect to some fixed PFS y of G is a PFS*

$$(Qa)^y = (\sigma_{(Qa)^y}(x),\ \tau_{(Qa)^y}(x),\ \eta_{(Qa)^y}(x))$$

defined by

$$\sigma_{(Qa)^y}(x) = \sigma_Q(x)\sigma_y(a),$$
$$\tau_{(Qa)^y}(x) = \tau_Q(x)\tau_y(a)$$

and

$$\eta_{(Qa)^y}(x) = \eta_Q(x)\eta_y(a),$$

for all $x \in G$*.*

Example 3.1. Let $G = \{1, w, w^2\}$ be a group. Let

$$Q = \{(1, 0.1, 0.15, 0.7), (w, 0.2, 0.3, 0.4), (w^2, 0.3, 0.4, 0.1)\}$$

be a PFSG of G. Let y be a PFS of G defined as

$$\sigma_y(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0.4, & \text{if } x = w \\ 0.2, & \text{if } x = w^2 \end{cases}$$

$$\tau_y(x) = \begin{cases} 0, & \text{if } x = 1 \\ 0.3, & \text{if } x = w \\ 0.35, & \text{if } x = w^2 \end{cases}$$

and

$$\eta_y(x) = \begin{cases} 0, & \text{if } x = 1 \\ 0.2, & \text{if } x = w \\ 0.4, & \text{if } x = w^2 \end{cases}$$

Thus, PPFLC of Q determined by an element w is $(wQ)^y = (\sigma_{(wQ)^y}, \tau_{(wQ)^y}, \eta_{(wQ)^y})$. Now,

$$\sigma_{(wQ)^y}(x) = \sigma_y(w)\sigma_Q(x),$$
$$\tau_{(wQ)^y}(x) = \tau_y(w)\tau_Q(x)$$

and

$$\eta_{(wQ)^y}(x) = \eta_y(w)\eta_Q(x).$$

Hence,

$$\sigma_{(wQ)^y}(x) = \begin{cases} 0.04, & \text{if } x = 1 \\ 0.08, & \text{if } x = w \\ 0.12, & \text{if } x = w^2 \end{cases}$$

$$\tau_{(wQ)^y}(x) = \begin{cases} 0.045, & \text{if } x = 1 \\ 0.09, & \text{if } x = w \\ 0.12, & \text{if } x = w^2 \end{cases}$$

and

$$\eta_{(wQ)^y}(x) = \begin{cases} 0.14, & \text{if } x = 1 \\ 0.08, & \text{if } x = w \\ 0.02, & \text{if } x = w^2 \end{cases}.$$

Proposition 3.1. *Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Then, PPFLC $(aQ)^y$ is a PFSG of crisp group G for any $a \in G$.*

Proof. Let

$$Q = (\sigma_Q, \tau_Q, \eta_Q)$$

be a PFSG of G and

$$(aQ)^y = (\sigma_{(aQ)^y}(x),\ \tau_{(aQ)^y}(x),\ \eta_{(aQ)^y}(x))$$

be a PPFLC of $Q \in G$ for $a, x \in G$.

Now, for every $g,\ h \in G$, we have

$$\begin{aligned} \sigma_{(aQ)^y}(g * h) &= \sigma_y(a)\sigma_Q(g * h) \\ &\geq \sigma_y(a)\left(\sigma_Q(g) \wedge \sigma_Q(h)\right) \\ &= (\sigma_y(a)\sigma_Q(g)) \wedge (\sigma_y(a)\sigma_Q(h)) \\ &= \sigma_{(aQ)^y}(g) \wedge \sigma_{(aQ)^y}(h), \end{aligned}$$

$$\begin{aligned} \tau_{(aQ)^y}(g * h) &= \tau_y(a)\tau_Q(g * h) \\ &\geq \tau_y(a)\left(\tau_Q(g) \wedge \tau_Q(h)\right) \\ &= (\tau_y(a)\tau_Q(g)) \wedge (\tau_y(a)\tau_Q(h)) \\ &= \tau_{(aQ)^y}(g) \wedge \tau_{(aQ)^y}(h) \end{aligned}$$

and

$$\begin{aligned} \eta_{(aQ)^y}(g * h) &= \eta_y(a)\eta_Q(g * h) \\ &\leq \eta_y(a)\left(\eta_Q(g) \vee \eta_Q(h)\right) \\ &= (\eta_y(a)\eta_Q(g)) \vee (\eta_y(a)\eta_Q(h)) \\ &= \eta_{(aQ)^y}(g) \vee \eta_{(aQ)^y}(h). \end{aligned}$$

Therefore, the PPFLC $(aQ)^y$ is a PFSG of G. □

Proposition 3.2. *Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Then, PPFRC $(aQ)^y$ is a PFSG of crisp group G for any $a \in G$.*

Proof. This is similar to the proof of Proposition 3.1. □

Proposition 3.3. *Any two pseudo picture fuzzy cosets of PFSG are either disjoint or identical.*

Proof. Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Let

$$(aQ)^y = (\sigma_{(aQ)^y}(x),\ \tau_{(aQ)^y}(x),\ \eta_{(aQ)^y}(x))$$

and

$$(bQ)^y = (\sigma_{(bQ)^y}(x),\ \tau_{(bQ)^y}(x),\ \eta_{(bQ)^y}(x))$$

be any two identical PPFCs for $a,\ b \in G$, then for all $g \in G$,

$$\sigma_{(aQ)^y}(g) = \sigma_{(bQ)^y}(g),\ \tau_{(aQ)^y}(g) = \tau_{(bQ)^y}(g) \text{ and } \eta_{(aQ)^y}(g) = \eta_{(bQ)^y}(g).$$

Suppose on the contrary that the PPFCs $(aQ)^y$ and $(bQ)^y$ are disjoint. Then, there is no such $y \in G$ such that

$$\sigma_{(aQ)^y}(h) \neq \sigma_{(bQ)^y}(h),\ \tau_{(aQ)^y}(h) \neq \tau_{(bQ)^y}(h) \text{ and } \eta_{(aQ)^y}(h) \neq \eta_{(bQ)^y}(h),$$

which means that $\sigma_y(a)\sigma_Q(h) \neq \sigma_y(b)\sigma_Q(h)$, $\tau_y(a)\tau_Q(h) \neq \tau_y(b)\tau_Q(h)$ and $\eta_y(a)\eta_Q(h) \neq \eta_y(b)\eta_Q(h)$ and we get

$$\sigma_y(a) \neq \sigma_y(b),\ \tau_y(a) \neq \tau_y(b) \text{ and } \eta_y(a) \neq \eta_y(b).$$

So, the assumption that

$$\sigma_{(aQ)^y}(g) = \sigma_{(bQ)^y}(g),\ \tau_{(aQ)^y}(g) = \tau_{(bQ)^y}(g),\ \eta_{(aQ)^y}(g) = \eta_{(bQ)^y}(g),\ \forall g \in G$$

is not true.

Conversely, let

$$(aQ)^y = (\sigma_{(aQ)^y},\ \tau_{(aQ)^y},\ \eta_{(aQ)^y})$$

and

$$(bQ)^y = (\sigma_{(bQ)^y},\ \tau_{(bQ)^y},\ \eta_{(bQ)^y})$$

be two disjoint PPFCs for every $a,\ b,\ g \in G$. Then,

$$\sigma_{(aQ)^y}(g) \neq \sigma_{(bQ)^y}(g),$$
$$\tau_{(aQ)^y}(g) \neq \tau_{(bQ)^y}(g)$$

and

$$\eta_{(aQ)^y}(g) \neq \eta_{(bQ)^y}(g),$$

which implies that

$$\sigma_y(a)\sigma_Q(g) \neq \sigma_y(b)\sigma_Q(g),$$
$$\tau_y(a)\tau_Q(g) \neq \tau_y(b)\tau_Q(g)$$

and

$$\eta_y(a)\eta_Q(g) \neq \eta_y(b)\eta_Q(g),$$

but if they are assumed to be identical, then

$$\sigma_y(a)\sigma_Q(g) = \sigma_y(b)\sigma_Q(g),$$
$$\tau_y(a)\tau_Q(g) = \tau_y(b)\tau_Q(g)$$

and

$$\eta_y(a)\eta_Q(g) = \eta_y(b)\eta_Q(g).$$

So,

$$\sigma_y(a) = \sigma_y(b),$$
$$\tau_y(a) = \tau_y(b)$$

and

$$\eta_y(a) = \eta_y(b).$$

Thus, this makes the assumption that

$$\sigma_y(a)\sigma_Q(g) \neq \sigma_y(b)\sigma_Q(g),$$
$$\tau_y(a)\tau_Q(g) \neq \tau_y(b)\tau_Q(g)$$

and

$$\eta_y(a)\eta_Q(g) \neq \eta_y(b)\eta_Q(g),$$

i.e., $\sigma_{(aQ)^y}(g) \neq \sigma_{(bQ)^y}(g)$, $\tau_{(aQ)^y}(g) \neq \tau_{(bQ)^y}(g)$ and $\eta_{(aQ)^y}(g) \neq \eta_{(bQ)^y}(g)$ are false. □

Proposition 3.4. *Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ and $R = (\sigma_R, \tau_R, \eta_R)$ be two PFSGs of G. Then $(aQ)^y \subseteq (aR)^y$ if and only if $Q \subseteq R$, for all $a \in G$ and $y \in Y$.*

Proof. Suppose that $(aQ)^y \subseteq (aR)^y$ and we get

$$\sigma_{(aQ)^y}(g) \leq \sigma_{(aR)^y}(g),$$
$$\tau_{(aQ)^y}(g) \leq \tau_{(aR)^y}(g)$$

and

$$\eta_{(aQ)^y}(g) \geq \eta_{(aR)^y}(g),$$

for all $g \in G$, which implies that

$$\sigma_y(a)\sigma_Q(g) \leq \sigma_y(a)\sigma_R(g),$$
$$\tau_y(a)\tau_Q(g) \leq \tau_y(a)\tau_R(g)$$

and

$$\eta_y(a)\eta_Q(g) \geq \eta_y(a)\eta_R(g),$$

for all $g \in G$. And we obtain

$$\sigma_Q(g) \leq \sigma_R(g),\ \tau_Q(g) \leq \tau_R(g) \text{ and } \eta_Q(g) \geq \eta_R(g),\ \forall\ g \in G.$$

Thus, $Q \subseteq R$.

Conversely, suppose that $Q \subseteq R$, and we get $\sigma_Q(g) \leq \sigma_R(g)$, $\tau_Q(g) \leq \tau_R(g)$ and $\eta_Q(g) \geq \eta_R(g),\ \forall\ g \in G$. So,

$$\sigma_y(a)\sigma_Q(g) \leq \sigma_y(a)\sigma_R(g),$$
$$\tau_y(a)\tau_Q(g) \leq \tau_y(a)\tau_R(g)$$

and

$$\eta_y(a)\eta_Q(g) \geq \eta_y(a)\eta_R(g),$$

for all g$\in G$. And we obtain

$$\sigma_{(aQ)^y}(g) \leq \sigma_{(aR)^y}(g),\ \tau_{(aQ)^y}(g) \leq \tau_{(aR)^y}(g) \text{ and } \eta_{(aQ)^y}(g) \geq \eta_{(aR)^y}(g),$$

for all $g \in G$. □

Definition 3.3. *Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ and $R = (\sigma_R, \tau_R, \eta_R)$ be two PFSG. Then, for any $a \in G$ the pseudo picture fuzzy double cosets (PPFDCs) of Q and R with respect to some fixed PFS y of G is the PFS*

$$(QaR)^y = (\sigma_{(QaR)^y},\ \tau_{(QaR)^y},\ \eta_{(QaR)^y})$$

of G, which is defined as

$$\sigma_{(QaR)^y}(g) = \sigma_{(Qa)^y}(g) \wedge \sigma_{(aR)^y}(g),$$
$$\tau_{(QaR)^y}(g) = \tau_{(Qa)^y}(g) \wedge \tau_{(aR)^y}(g)$$

and

$$\eta_{(QaR)^y}(g) = \eta_{(Qa)^y}(g) \vee \eta_{(aR)^y}(g)G,$$

for every $g \in G$.

Proposition 3.5. *Every PPFDC is a PFSG of G.*

Proof. Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ and $R = (\sigma_R, \tau_R, \eta_R)$ be two PFSGs of G. Let

$$(QaR)^y = (\sigma_{(QaR)^y},\ \tau_{(QaR)^y},\ \eta_{(QaR)^y})$$

where

$$\sigma_{(QaR)^y}(g) = \sigma_{(Qa)^y}(g) \wedge \sigma_{(aR)^y}(g),$$
$$\tau_{(QaR)^y}(g) = \tau_{(Qa)^y}(g) \wedge \tau_{(aR)^y}(g)$$

and

$$\eta_{(QaR)^y}(g) = \eta_{(Qa)^y}(g) \wedge \eta_{(aR)^y}(g)$$

$g \in G$ be PPFDC. Let $g, h \in G$ be any elements, then

$$\begin{aligned}
&\sigma_{(QaR)^y}(g*h)\\
&= \sigma_{(Qa)^y}(g*h) \wedge \sigma_{(aR)^y}(g*h)\\
&= \sigma_y(a)\sigma_{(Q)}(g*h) \wedge \sigma_y(a)\sigma_{(R)}(g*h)\\
&\geq \sigma_y(a)(\sigma_Q(g) \wedge \sigma_Q(h)) \wedge \sigma_y(a)(\sigma_R(g) \wedge \sigma_R(h))\\
&= [(\sigma_y(a)\sigma_Q(g)) \wedge (\sigma_y(a)\sigma_Q(h))] \wedge [(\sigma_y(a)\sigma_R(g)) \wedge (\sigma_y(a)\sigma_R(h))]\\
&= [(\sigma_y(a)\sigma_Q(g)) \wedge (\sigma_y(a)\sigma_R(g))] \wedge [(\sigma_y(a)\sigma_Q(h)) \wedge (\sigma_y(a)\sigma_R(h))]\\
&= \left[\sigma_{(Qa)^y}(g) \wedge \sigma_{(aR)^y}(g)\right] \wedge \left[\sigma_{(Qa)^y}(h) \wedge \sigma_{(aR)^y}(h)\right]\\
&= \sigma_{(QaR)^y}(g) \wedge \sigma_{(QaR)^y}(h),\\
&\tau_{(QaR)^y}(g*h)\\
&= \tau_{(Qa)^y}(g*h) \wedge \tau_{(aR)^y}(g*h)\\
&= \tau_y(a)\tau_{(Q)}(g*h) \wedge \tau_y(a)\tau_{(R)}(g*h)\\
&\geq \tau_y(a)(\tau_Q(g) \wedge \tau_Q(h)) \wedge \tau_y(a)(\tau_R(g) \wedge \tau_R(h))\\
&= [(\tau_y(a)\tau_Q(g)) \wedge (\tau_y(a)\tau_Q(h))] \wedge [(\tau_y(a)\tau_R(g)) \wedge (\tau_y(a)\tau_R(h))]\\
&= [(\tau_y(a)\tau_Q(g)) \wedge (\tau_y(a)\tau_R(g))] \wedge [(\tau_y(a)\tau_Q(h)) \wedge (\tau_y(a)\tau_R(h))]\\
&= \left[\tau_{(Qa)^y}(g) \wedge \tau_{(aR)^y}(g)\right] \wedge \left[\tau_{(Qa)^y}(h) \wedge \tau_{(aR)^y}(h)\right]\\
&= \tau_{(QaR)^y}(g) \wedge \tau_{(QaR)^y}(h)
\end{aligned}$$

and

$$\begin{aligned}
&\eta(QaR)^y(g*h)\\
&= \eta(Qa)^y(g*h) \vee \eta(aR)^y(g*h)\\
&= \eta y(a)\eta(Q)(g*h) \vee \eta_y(a)\eta(R)(g*h)\\
&\leq \eta_y(a)(\eta_Q(g) \vee \eta_Q(h)) \vee \eta_y(a)(\eta_R(g) \vee \eta_R(h))\\
&= [(\eta_y(a)\eta_Q(g)) \vee (\eta_y(a)\eta_Q(h))] \vee [(\eta_y(a)\eta_R(g)) \vee (\eta_y(a)\eta_R(h))]\\
&= [(\eta_y(a)\eta_Q(g)) \vee (\eta_y(a)\eta_R(g))] \vee [(\eta_y(a)\eta_Q(h)) \vee (\eta_y(a)\eta_R(h))]\\
&= \left[\eta_{(Qa)^y}(g) \vee \eta_{(aR)^y}(g)\right] \vee \left[\eta_{(Qa)^y}(h) \vee \eta_{(aR)^y}(h)\right]\\
&= \eta_{(QaR)^y}(g) \vee \eta_{(QaR)^y}(h).
\end{aligned}$$

Therefore, $(QaR)^y$ is a PFSG of G. □

Proposition 3.6. *Let Q and R be two PFSGs. If Q and R are PFNSGs, then the PPFDC $(QaR)^y$ is a PFNSG.*

Proof. Let $(QaR)^y = (\sigma_{(QaR)^y},\ \tau_{(QaR)^y},\ \eta_{(QaR)^y})$, where

$$\begin{aligned}
\sigma_{(QaR)^y}(g) &= \sigma_{(Qa)^y}(g) \wedge \sigma_{(aR)^y}(g),\\
\tau_{(QaR)^y}(g) &= \tau_{(Qa)^y}(g) \wedge \tau_{(aR)^y}(g)
\end{aligned}$$

and

$$\eta_{(QaR)^y}(g) = \eta_{(Qa)^y}(g) \wedge \eta_{(aR)^y}(g),$$

$g \in G$ be PPFDC where Q and R are PFNSGs of G. By Proposition 3.5, $(QaR)^y$ is PFSG of G. Let $g, h \in G$, then

$$\begin{aligned}
\sigma_{(QaR)^y}(g*h*g^{-1}) &= \left[\sigma_{(Qa)^y}(g*h*g^{-1})\right] \wedge \left[\sigma_{(aR)^y}(g*h*g^{-1})\right]\\
&= \left[\sigma_y(a)\sigma_Q(g*h*g^{-1})\right] \wedge \left[\sigma_y(a)\sigma_R(g*h*g^{-1})\right]\\
&= \left[\sigma_y(a)\sigma_Q((g*h)*g^{-1})\right] \wedge \left[\sigma_y(a)\sigma_R((g*h)*g^{-1})\right]\\
&= \left[\sigma_y(a)\sigma_Q((g*h)*g^{-1})\right] \wedge \left[\sigma_y(a)\sigma_R((g*h)*g^{-1})\right]\\
&= \left[\sigma_y(a)\sigma_Q(g^{-1}*(g*h))\right] \wedge \left[\sigma_y(a)\sigma_R(g^{-1}*(g*h))\right]\\
&= \left[\sigma_y(a)\sigma_Q(g^{-1}*g)*h)\right] \wedge \left[\sigma_y(a)\sigma_R(g^{-1}*g)*h)\right]\\
&= [\sigma_y(a)\sigma_Q(h))] \wedge [\sigma_y(a)\sigma_R(h)]\\
&= (\sigma_{(Qa)^y}(h)) \wedge (\sigma_{(aR)^y}(h))\\
&= \sigma_{(QaR)^y}(h),\\
\tau_{(QaR)^y}(g*h*g^{-1}) &= \left[\tau_{(Qa)^y}(g*h*g^{-1})\right] \wedge \left[\tau_{(aR)^y}(g*h*g^{-1})\right]\\
&= \left[\tau_y(a)\tau_Q(g*h*g^{-1})\right] \wedge \left[\tau_y(a)\tau_R(g*h*g^{-1})\right]\\
&= \left[\tau_y(a)\tau_Q((g*h)*g^{-1})\right] \wedge \left[\tau_y(a)\tau_R((g*h)*g^{-1})\right]\\
&= \left[\tau_y(a)\tau_Q((g*h)*g^{-1})\right] \wedge \left[\tau_y(a)\tau_R((g*h)*g^{-1})\right]\\
&= \left[\tau_y(a)\tau_Q(g^{-1}*(g*h))\right] \wedge \left[\tau_y(a)\tau_R(g^{-1}*(g*h))\right]\\
&= \left[\tau_y(a)\tau_Q(g^{-1}*g)*h)\right] \wedge \left[\tau_y(a)\tau_R(g^{-1}*g)*h)\right]\\
&= [\tau_y(a)\tau_Q(h))] \wedge [\tau_y(a)\tau_R(h)]\\
&= (\tau_{(Qa)^y}(h)) \wedge (\tau_{(aR)^y}(h))\\
&= \tau_{(QaR)^y}(h)
\end{aligned}$$

and

$$\begin{aligned}
\eta_{(QaR)^y}(g*h*g^{-1}) &= \left[\eta_{(Qa)^y}(g*h*g^{-1})\right] \wedge \left[\eta_{(aR)^y}(g*h*g^{-1})\right]\\
&= \left[\eta_y(a)\eta_Q(g*h*g^{-1})\right] \wedge \left[\eta_y(a)\eta_R(g*h*g^{-1})\right]\\
&= \left[\eta_y(a)\eta_Q((g*h)*g^{-1})\right] \wedge \left[\eta_y(a)\eta_R((g*h)*g^{-1})\right]\\
&= \left[\eta_y(a)\eta_Q((g*h)*g^{-1})\right] \wedge \left[\eta_y(a)\eta_R((g*h)*g^{-1})\right]\\
&= \left[\eta_y(a)\eta_Q(g^{-1}*(g*h))\right] \wedge \left[\eta_y(a)\eta_R(g^{-1}*(g*h))\right]\\
&= \left[\eta_y(a)\eta_Q(g^{-1}*g)*h)\right] \wedge \left[\eta_y(a)\eta_R(g^{-1}*g)*h)\right]\\
&= [\eta_y(a)\eta_Q(h))] \wedge [\eta_y(a)\eta_R(h)]\\
&= (\eta_{(Qa)^y}(h)) \wedge (\eta_{(aR)^y}(h))\\
&= \eta_{(QaR)^y}(h).
\end{aligned}$$

Hence, $(QaR)^y$ is a PFNSG of G. □

Definition 3.4. *Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be PFSG of G. Then, for any $a \in G$, pseudo picture fuzzy middle cosets (PPFMC) of Q is a PFS $(aQa^{-1})^y =$*

$(\sigma_{(aQa^{-1})^y},\ \tau_{(aQa^{-1})^y},\ \eta_{(aQa^{-1})^y})$ *of* G *defined by*

$$\sigma_{(aQa^{-1})^y}(g) = \sigma_y(a)\sigma_Q(a^{-1} * g * a)\ \sigma_y(a^{-1}),$$
$$\tau_{(aQa^{-1})^y}(g) = \tau_y(a)\tau_Q(a^{-1} * g * a)\ \tau_y(a^{-1})$$

and

$$\eta_{(aQa^{-1})^y}(g) = \eta_y(a)\eta_Q(a^{-1} * g * a)\ \eta_y(a^{-1}),$$

for all $g \in G$.

Proposition 3.7. *Let* $Q = (\sigma_Q, \tau_Q, \eta_Q)$ *be a PFNSG of* G. *Then for every* $a \in G$, *PPFMC* $(aQa^{-1})^y$ *is a PFNSG of* G.

Proof. Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFNSG of G and $a \in G$, let

$$(aQa^{-1})^y = (\sigma_{(aQa^{-1})^y},\ \tau_{(aQa^{-1})^y},\ \eta_{(aQa^{-1})^y}),$$

where $\sigma_{(aQa^{-1})^y}(g)$, $\tau_{(aQa^{-1})^y}(g)$, and $\eta_{(aQa^{-1})^y}(g)$ are as defined in Definition 3.4 for all $g \in G$. Let $g, h \in G$, then

$$\begin{aligned}
\sigma_{(aQa^{-1})^y}(g * h) &= \sigma_y(a)\sigma_Q(a^{-1} * (g * h) * a)\ \sigma_y(a^{-1}) \\
&= \sigma_y(a)\sigma_Q(a^{-1} * (g * h * a))\ \sigma_y(a^{-1}) \\
&= \sigma_y(a)\sigma_Q((g * h * a) * a^{-1})\ \sigma_y(a^{-1}) \\
&= \sigma_y(a)\sigma_Q((g * h) * (a * a^{-1}))\ \sigma_y(a^{-1}) \\
&= \sigma_y(a)\sigma_Q(g * h)\ \sigma_y(a^{-1}) \\
&= \sigma_y(a)\sigma_Q(h * g)\ \sigma_y(a^{-1}) \\
&= \sigma_{(aQa^{-1})^y}(h * g),
\end{aligned}$$

$$\begin{aligned}
\tau_{(aQa^{-1})^y}(g * h) &= \tau_y(a)\tau_Q(a^{-1} * (g * h) * a)\ \tau_y(a^{-1}) \\
&= \tau_y(a)\tau_Q(a^{-1} * (g * h * a))\ \tau_y(a^{-1}) \\
&= \tau_y(a)\tau_Q((g * h * a) * a^{-1})\ \tau_y(a^{-1}) \\
&= \tau_y(a)\tau_Q((g * h) * (a * a^{-1}))\ \tau_y(a^{-1}) \\
&= \tau_y(a)\tau_Q(g * h)\ \tau_y(a^{-1}) \\
&= \tau_y(a)\tau_Q(h * g)\ \tau_y(a^{-1}) \\
&= \tau_{(aQa^{-1})^y}(h * g)
\end{aligned}$$

and

$$\begin{aligned}
\eta_{(aQa^{-1})^y}(g * h) &= \eta_y(a)\eta_Q(a^{-1} * (g * h) * a)\ \eta_y(a^{-1}) \\
&= \eta_y(a)\eta_Q(a^{-1} * (g * h * a))\ \eta_y(a^{-1}) \\
&= \eta_y(a)\eta_Q((g * h * a) * a^{-1})\ \eta_y(a^{-1}) \\
&= \eta_y(a)\eta_Q((g * h) * (a * a^{-1}))\ \eta_y(a^{-1}) \\
&= \eta_y(a)\eta_Q(g * h)\ \eta_y(a^{-1}) \\
&= \eta_y(a)\eta_Q(h * g)\ \eta_y(a^{-1}) \\
&= \eta_{(aQa^{-1})^y}(h * g).
\end{aligned}$$

Hence, $(aQa^{-1})^y$ is a PFNSG of G. □

Conclusion and future scopes

In this paper, we have extended the concepts of pseudo fuzzy cosets and pseudo intuitionistc fuzzy cosets to pseudo picture fuzzy cosets (PPFCs), and established some of the properties related to pseudo picture fuzzy cosets, pseudo picture fuzzy double cosets (PPFDCs) and pseudo picture fuzzy middle cosets (PPFMCs). Furthermore, the connections between PPFDCs and PFNSG, and PPFMCs and PFNSG were obtained, respectively. In further research, it will be of interest to study the pseudo picture fuzzy cosets in more complicated uncertain environments like spherical fuzzy environment and establish the generalisation of these results.

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