Tight partitions for packing circles in a circle

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Abstract. We develop a new strategy for proving optimal packing densities for N congruent circles in a circle. Specifically, we introduce tight partitions, which generalize filled rings of circles, and show that for the densest packing, the union of tight partitions forms a connected graph containing the center of every circle, except for possibly rattlers on the container boundary. We then apply this to the case of $N = 14$ to reduce the list of potentially optimal solutions to one basic shape, which in turn admits a one-parameter family of configurations with two local extrema, one of which is the global optimal.

Keywords: circle packing, rings of circles, tight partitions.

1. Introduction

Circle packing problems with various containers and radii arise in applications to factory layouts [2, 5], communications networks [1, 3, 8], circular cutting [16], cylinder packing [6], container loading [13], and social distancing [24], but in general are considered to be NP-hard [4, 7]. For packing N congruent circles of unit radius in a circle, minimum container radii (or equivalently maximum densities) have been proved only for $N \leq 14$ and $N = 19$ [9, 10, 11, 12, 19, 21]. For general N, only heuristic methods have been proposed to find approximate solutions [15, 17, 20]; the best known solutions up to $N = 2647$ can be found at [23]. Our goal in the current paper is to provide a new strategy for proving optimal density which we hypothesize can be systematically applied to increasing N. We demonstrate the utility of this new approach by providing an independent proof for the case of $N = 14$.

Specifically, we geometrically reduce the number of basic configurations for circles using a new tool that we refer to as *tight partitions*, which generalize filled rings of circles, and which characterize global ring structure that must exist for potentially optimal configurations. For the case of $N = 14$, we use tight

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partitions to geometrically reduce the problem to one basic shape. This basic shape admits a one-parameter family of geometric configurations that have as endpoints a symmetric arrangement and an extreme one, where no further deformation of the basic shape is possible. We then show that the container radius is monotone decreasing from the symmetric arrangement to the extreme one, which therefore yields the optimal solution. With a similar analysis, we believe it is possible to establish that for any N, and for any given feasible configuration, the distinct local minima occur either in a symmetric or extreme arrangement. A proof of such a conjecture, along with the identification of a finite configuration list using tight partitions, will lead to a tractable combinatorial optimization problem for increasing N.

The outline of the paper is as follows. In Section 2 we define tight partitions, and prove Theorem 2.1 that the union of tight partitions forms a connected graph containing the center of every circle, except for possibly outermost rattlers. In Section 3 we apply this to $N = 14$, determining the basic shape of the optimal solution in Theorem 3.1. In Section 4 we then state and prove Theorem 4.1 which establishes the densest packing.

2. Partitions and tight partitions

Consider a packing of circles $C_1, \ldots, C_N$ of unit radius into a circular container of radius r centered at O. As introduced in [8], there is a set of *rings*, $R_1, \ldots, R_n$, that are concentric circles with center O and corresponding radii $0 \leq r_1 < \ldots < r_n = r - 1$, such that each circle C_i has its center on some ring R_j. A *filled ring* is one for which consecutive circles along that ring are mutually tangent, so there are no gaps. Since filled rings cannot be assumed to be present, our goal in this section is to provide a more general notion of well-defined layers without gaps. The observations in this section are basic, yet will lead to useful conceptual organization of subsequent sections.

We will refer to the complex of rings as $\mathcal{R}$; we will assume in this section that $r_i > 0$, but observe at the outset that all results hold for $N > 13$ even if $r_1 = 0$, since in that case there are still at least two rings with $r_i > 0$, which will suffice for all proofs. Given two circles $C_i, C_j \in \mathcal{R}$, there is a central angle $\theta_{C_iC_j}$ formed by line segments joining the centers of C_i, C_j with O.

Definition 2.1 (Partition)**.** *A partition P is a piecewise linear simple closed loop whose segments connect centers of circles in $\mathcal{R}$, such that if there are m segments, then the corresponding central angles θ_i have measures $0 < \theta_i < \pi$, $i \in \{1, \ldots, m\}$, with $\sum_{i=1}^{m} \theta_i = 2\pi$.*

A partition is thus an edge-path which connects centers of circles, and proceeds strictly monotonically once around the center O of the ring complex. We use the word *partition* because the central angles partition 2π. We will assume our packing is optimal at minimum radius r, so that we may assume the following three conditions:

A. r_n is the minimum outer radius for ring complexes $\mathcal{R}$ with N circles.
B. Given Condition A, every other radius r_i for $1 \leq i < n$ is maximized.
C. Given Conditions A and B, the total number of rings n is maximized.

That Condition A holds is obvious. Conditions B and C then guarantee that with Condition A in place, no circle may be moved further outward from O; in particular *rattlers*, which have local freedom of movement, are pushed as far outward from O as possible. We prove three initial lemmas that show partitions exist through every circle.

Lemma 2.1. *Let R_i, R_{i+1} be successive rings with respective circles C_i, C_{i+1}. Then the centers of C_i, C_{i+1} are not on the same radial ray extending from O.*

Proof. Suppose, for contradiction, that C_i and C_{i+1} have centers on a common radial ray extending from O. Then C_i and C_{i+1} must be tangent, with $r_i + 2 = r_{i+1}$, and in fact any point where circles of R_i and R_{i+1} intersect must also be a point of tangency on some radial line. Thus, all R_j for $i+1 \leq j \leq n$ can be rotated simultaneously such that all circles of R_i are disjoint from all circles of R_{i+1}, and we can increase r_i, contradicting Condition B. □

Lemma 2.2. *There exists a partition P for $\mathcal{R}$.*

Proof. Let U be the convex hull of all centers of circles in $\mathcal{R}$, and note that U is not a line segment due to Lemma 2.1 and the fact that $N > 2$. We observe that if $O \in \operatorname{Int} U$ we are done, for then $\operatorname{Bd} U$ is our desired partition P. If O is not initially contained in $\operatorname{Int} U$ we will show that the circles in $\mathcal{R}$ admit a perturbation within their circular container so that either $O \in \operatorname{Int} U$, or r_n can be reduced, contradicting Condition A.

To that end, if $O \notin \operatorname{Int} U$, since U is convex there is a diameter ℓ of the circular container for $\mathcal{R}$ that is disjoint from $\operatorname{Int} U$, so that $\operatorname{Int} U$ is entirely contained on one side of ℓ, as shown in part (a) of Figure 1. We call H the side disjoint from $\operatorname{Int} U$. We need to consider the cases when O is on $\operatorname{Bd} U$, or when O is disjoint from U altogether.

If O is on $\operatorname{Bd} U$, it is possible that O is a vertex of $\operatorname{Bd} U$, meaning a center of a circle is at O. If so, since U is convex we can translate that vertex and corresponding circle slightly into H so as to obtain $O \in \operatorname{Int} U$. The other possibility is that O lies on an edge of $\operatorname{Bd} U$. Then both endpoints of that edge are centers of circles that lie on ℓ on opposite sides of O, and we can rotate one of those circles slightly into H, so as to obtain $O \in \operatorname{Int} U$.

Finally, we consider the case where O is disjoint from U, as depicted in part (a) of Figure 1. Then all circles in $\mathcal{R}$ admit a translation within the circular container perpendicular to ℓ, eliminating all points of tangency between circles in $\mathcal{R}$ and the container as in part (b) of Figure 1. Thus, the container radius, and hence r_n, can be reduced, contradicting Condition A. □

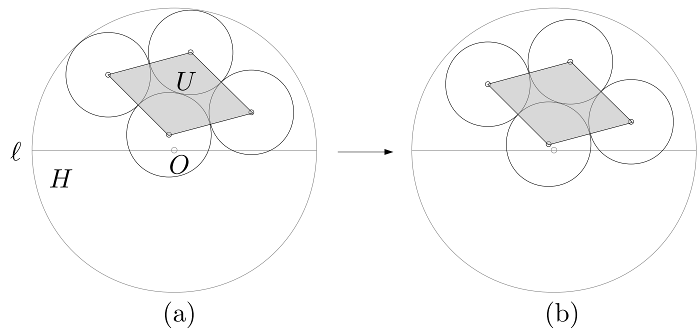

Figure 1: Figure for Lemma 2.2.

Lemma 2.3. *If $C \in \mathcal{R}$, there is a partition P_C that contains the center of C.*

Proof. Consider the radial ray extending from O through the center p of C. Since we know there is at least one partition P, this ray must intersect P. If it intersects a vertex of P at a center p_1 of a circle C_1, we can replace p_1 with p to obtain a new partition P_C which has an edge-path now through p. If the ray intersects an edge of P joining two centers p_1 and p_2 of circles C_1 and C_2, then we can obtain a new partition P_C which has an edge-path going from p_1 to p then to p_2. □

We now present our primary definition.

Definition 2.2 (Tight partition)**.** *A tight partition P for $\mathcal{R}$ is a partition where all segments have length 2.*

A tight partition is an edge-path which connects successive centers of tangent circles strictly monotonically once around O, and generalizes filled rings. Note that every packing $\mathcal{R}$ comes equipped with a *tangency graph*, where centers of circles are vertices, and edges between two vertices indicate tangency between those two circles. Therefore, a tight partition is a particular kind of loop in the tangency graph which proceeds monotonically around O. We also note that every optimal packing must have edges in its tangency graph, since if no tangencies exist then all circles have freedom of movement, and we may reduce r_n.

Before proceeding to the existence of tight partitions, we need two definitions. Consider two circles C, C' with radii $r_C, r_{C'}$; if C, C' are tangent, then their central angle is $\theta_{CC'} = \cos^{-1}\left((r_C^2 + r_{C'}^2 - 4)/(2r_C r_{C'})\right)$, which may be acute, right or obtuse.

Definition 2.3 (Angular defect between C and C')**.** *The angular defect between C and C' is defined as*

$$\delta_{CC'} = \begin{cases} \theta_{CC'} - \cos^{-1}\left((r_C^2 + r_{C'}^2 - 4)/(2r_C r_{C'})\right), & \text{if } |r_C - r_{C'}| < 2, \\ \theta_{CC'}, & \text{otherwise.} \end{cases}$$

The angle $\delta_{CC'} \geq 0$, since it is the angle needed to rotate C along its ring until it is either tangent to C' (the first case) or along the same radial ray (the second case).

Definition 2.4 (Angular defect for $\mathcal{R}$)**.** *The angular defect for $\mathcal{R}$ is defined as*

$$\delta = \min\{\ \delta_{CC'}\ \mid\ C, C' \in \mathcal{R} \text{ and } \delta_{CC'} > 0\}.$$

We now can prove the existence of tight partitions.

Proposition 2.1. *There exists a tight partition P for $\mathcal{R}$.*

Proof. Suppose for contradiction that there does not exist a tight partition. By Lemma 2.2, let $\mathcal{P}$ be the non-empty set of all partitions for $\mathcal{R}$. For each $P \in \mathcal{P}$, there is at least one edge e of length greater than 2, from circle C to C', with an angular defect $\delta_e = \delta_{CC'} > 0$. We know that $\delta_e \geq \delta > 0$. Throughout the proof we will be rotating circles along their rings, and we consider the counterclockwise direction to be the forward direction of rotation around O.

We label the circles of $\mathcal{R}$ randomly as $C_1, \ldots, C_N$. For any circle C_i, by Lemma 2.1 any points of tangency with other circles will either occur before C_i's radial ray, or after. This will be seen in the tangency graph at the vertex C_i as adjacent edges which extend backward in the counterclockwise direction (which we term *backward edges*), or adjacent edges which extend forward (which we term *forward edges*). Note that if C_i had a backward edge to C_j, that edge acts as a forward edge for C_j.

With this in mind, we rotate circles forward along their rings in the following manner: First, we rotate C_1 by $\delta/2$. If C_1 has forward edges connecting it to adjacent circles, its rotation will force those circles to rotate by $\delta/2$, and this rotation may propagate forward via connections in the tangency graph. However, no new edges in the tangency graph, and hence no new tight partitions, will be created in $\mathcal{R}$, since $\delta/2 < \delta$. Moreover, any circles originally connected by backward edges to C_1 will stay fixed, since if they moved along with C_1, this would imply a monotonic loop around O in the tangency graph, and hence a tight partition. Thus, all backward edges connected to C_1 will be eliminated from the tangency graph. The new angular defect is at least $\delta/2$.

For $i > 1$ we then rotate each C_i forward, one at a time in orderly succession, by $\delta/2^i$, where prior to each rotation the angular defect is at least $\delta/2^{i-1}$. As above, this may force other circles to rotate forward by connections in the tangency graph, but no new edges in the tangency graph will be created since $\delta/2^i < \delta/2^{i-1}$. Moreover, since no tight partitions exist, each C_i's backward edges will be eliminated. After the final rotation of C_N, all edges in the tangency graph are eliminated, and thus r_n can be reduced, contradicting Condition A. Therefore, tight partitions must exist. □

We now show how tight partitions relate to the specific rings in $\mathcal{R}$.

Lemma 2.4. *For each ring R_i, there exists at least one tight partition P which contains centers of circles from R_i.*

Proof. Suppose for contradiction that no tight partition contains circles from R_i. We argue exactly as in Proposition 2.1, but now in the presence of existing tight partitions. Specifically, when we rotate each C_j, if it is in an existing tight partition, it will stay in that tight partition, since the whole partition will be forced to move via forward edges in the tangency graph. However, no new tight partitions will be created, and any backward edges not in a tight partition will be eliminated. After the N rotations, all edges in the tangency graph that were not originally in a tight partition will therefore be eliminated. As a result, since no circle in R_i was in a tight partition, the circles in R_i will have no adjacent edges in the tangency graph. If $i = n$, then r_n can be reduced, contradicting Condition A; if $i < n$ then r_i can be increased, contradicting Condition B. □

Recalling that a tight partition is a particular loop in the tangency graph, let $\mathcal{T}$ be the subgraph of the tangency graph obtained by letting $\mathcal{T}$ be the union of all tight partitions.

Theorem 2.1. *$\mathcal{T}$ forms a connected graph, and includes every circle in $\mathcal{R}$ except possibly a proper subset of the circles in R_n, which are rattlers.*

Proof. Suppose for contradiction that there are at least two distinct components of $\mathcal{T}$, which we call $\mathcal{T}_1$ and $\mathcal{T}_2$. We begin with some topological observations. First, because $\mathcal{T}_1$ and $\mathcal{T}_2$ are each connected unions of loops around O, they are each contained in topological annuli which we call $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, which are disjoint from O and which are separated by a topological circle $\mathcal{C}$ in the plane. $\mathcal{C}$ also separates the plane into a topological disc and its complement. Without loss of generality we may assume $\mathcal{A}_1$ is contained in the disc, and $\mathcal{A}_2$ in its complement. Since $\mathcal{A}_1$ contains at least one tight partition which is a closed loop around O, the center O must also be contained in the disc. Therefore, $\mathcal{C}$ must be a topological circle containing O with $\mathcal{T}_1$ and its annulus $\mathcal{A}_1$, with $\mathcal{T}_2$ and its annulus $\mathcal{A}_2$ enclosing all of these. A schematic for this basic topological configuration is shown in Figure 2. Moreover, we can assume that $\mathcal{T}_2$ is the outermost component of $\mathcal{T}$ from O, if there are more than two components. Thus, of all the components of $\mathcal{T}$, only $\mathcal{T}_1$ contains circles of R_1, and only $\mathcal{T}_2$ contains circles from R_n, for no other components of $\mathcal{T}$ can intersect the annuli in Figure 2.

We now turn our attention to the entire tangency graph. $\mathcal{T}_1$ may be connected by an edge in the tangency graph to another circle $C \notin \mathcal{T}_1$, meaning there is no tight partition in $\mathcal{T}_1$ that also contains C. The same holds true for $\mathcal{T}_2$. As in Proposition 2.1 and Lemma 2.4 we use the angular defect to rotate all circles in $\mathcal{R}$, and observe that because $\mathcal{T}_1$ and $\mathcal{T}_2$ are connected unions of tight partitions, they will remain connected after the rotation of all circles. However, the edges in the tangency graph between each of them and other circles are

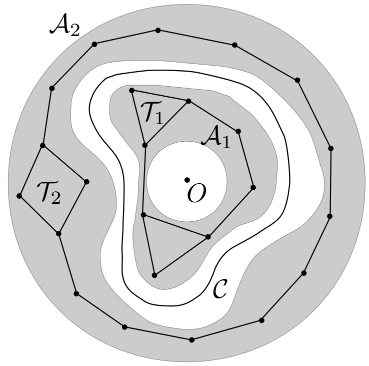

Figure 2: Figure for Theorem 2.1.

eliminated. Therefore, $\mathcal{T}_1$ and $\mathcal{T}_2$ are now disjoint components of the overall tangency graph, and any other circles are in components of the tangency graph disjoint from the $\mathcal{T}_i$. As a result, we can uniformly increase radii of all circles in $\mathcal{T}_1$, including all circles of R_1 and thus increasing r_1, without moving any circle in $\mathcal{T}_2$, and in particular not increasing r_n. This increase in r_1 contradicts Condition B, so that $\mathcal{T}$ must be a connected graph.

Finally, to see that $\mathcal{T}$ contains every circle in $\mathcal{R}$ except perhaps isolated rattlers in R_n, observe that the angular defect rotation ensures that any circles not in $\mathcal{T}$ are disconnected vertices in the tangency graph. Any such circles that are not in R_n can have their radii increased, contradicting either Conditions B or C, depending on whether an entire ring can increase, or just a subset of a ring. Thus, such disconnected circles must only be rattlers in R_n, and cannot include all of R_n, for otherwise r_n could be decreased. $\square$

We conclude with a useful corollary and definition.

Corollary 2.1. *For the connected graph $\mathcal{T}$ there is an outermost tight partition which contains every non-rattler circle in R_n, and an innermost tight partition which contains every circle in R_1.*

Proof. Since $\mathcal{T}$ is the union of tight partitions, which are loops in the tangency graph that proceed monotonically around O, there will be an innermost such loop closest to O, which is the innermost tight partition. Observe that this innermost tight partition bounds a disc containing O and no other vertex of $\mathcal{T}$. By Theorem 2.1, all circles in R_1 are vertices of $\mathcal{T}$, and hence must be in this innermost tight partition. Likewise there will be an outermost loop farthest from O, which is the outermost tight partition, and outside it can be no vertices of $\mathcal{T}$, so that by Theorem 2.1 it must contain every non-rattler circle of R_n. A priori these two tight partitions may be identical if $\mathcal{T}$ consists of only one tight partition, or they could possibly intersect along vertices or edges. $\square$

Definition 2.5 (P_{out}, P_{in}, gap chains)**.** $P_{\mathbf{out}}$ *is the outermost tight partition and* $P_{\mathbf{in}}$ *is the innermost tight partition. A gap chain* $C_1, \ldots, C_k$ *is a maximal sequence of consecutive circles in* P_{out} *from the inner rings* $R_1, \ldots, R_{n-1}$.

3. The basic shape of the optimal packing for $N = 14$

Our new proof for $N = 14$ leverages the fact that all minimum container radii for $1 \le N \le 13$ are known. More specifically, the case of $N = 13$ is known to have R_n a filled ring of 10 circles, yielding a radius of $A = 3.23606798$ [11], and the best known packing for $N = 14$ has r_n of $B = 3.32842855$, known since 1971 [14]. The inequality $A \le r_n \le B$ will be sufficient for us to hone in on the basic shape for $N = 14$ in this section, via the subsections below. We denote the number of circles in R_n by $|R_n|$.

3.1 $7 \le |R_n| \le 10$ and all circles in $R_1, \ldots, R_{n-1}$ touch R_n

We will assume that for minimum r_n, the number $|R_n|$ is maximized.

Lemma 3.1. *If a gap occurs between two consecutive circles* $C_1, C_2 \in R_n$*, then the central angle* $\theta_{C_1C_2} < 4\pi/10$.

Proof. The radius A is for a filled ring of 10 circles, so that any one circle in R_n has angular support at most $2\pi/10$, for $A \le r_n \le B$. If $\theta_{C_1C_2} \ge 4\pi/10$, there is enough angle in R_n for another circle; it remains to show that we can move another circle into R_n, contradicting the maximized $|R_n|$. We first assume both $C_1, C_2 \in P_{\text{out}}$; see part (a) of Figure 3, which shows C_1, C_2 connected by a gap chain $C_3, \ldots, C_k \in P_{\text{out}}$, where it is possible $k = 3$. Let p_i be the center of C_i. The polygon formed by the line segment $\overline{p_1p_2}$ and the portion of P_{out} from p_1 to p_2 must be convex, for otherwise at least one of the circles $C_3, \ldots, C_k$ could move freely out to R_n. We thus can assume that p_3 is closest to $\overline{p_1p_2}$ compared to $p_4, \ldots, p_k$, and as in part (a) of Figure 3 we can reflect C_3 through $\overline{p_1p_4}$ without obstruction, and rotate the resulting circle along C_1 to move it out to R_n. We conclude the proof by observing that if C_1 is a rattler, the only change will be that the line segment $\overline{p_1p_3}$ will have length greater than 2, but this does not affect the ability to reflect and rotate C_3 out to R_n. $\square$

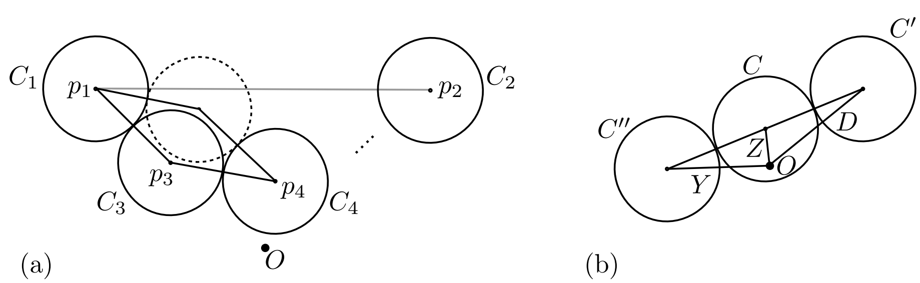

Figure 3: Figures for Lemmas 3.1, 3.4 and Proposition 3.1.

This then allows us to begin to narrow down possibilities for $|R_n|$.

Lemma 3.2. $6 \leq |R_n| \leq 10$*, and the sum of* $|R_n|$ *plus gaps in* R_n *is at least 11.*

Proof. A filled ring of 11 circles easily fits 4 inside, and so $|R_n| \leq 10$. If we let j be the number of gaps in R_n then $j \leq |R_n|$, and by Lemma 3.1 we require that the angular support around R_n is

$$2\pi < j \cdot \frac{4\pi}{10} + (|R_n| - j) \cdot \frac{2\pi}{10} = (|R_n| + j) \cdot \frac{2\pi}{10}$$

yielding $|R_n| + j > 10$. Since $j \leq |R_n|$ this forces $|R_n| \geq 6$. □

Denote by D *the maximum value for* r_{n-1}, which occurs when a single circle C_3 between C_1, C_2 could be reflected out to $r_n = B$ as in Lemma 3.1. This is when the angular gap is $2\sin^{-1}(1/B) = \cos^{-1}\left((B^2 + D^2 - 4)/(2BD)\right)$ which yields $D = 2.126660$.

Let E be the distance from O for a circle that forms an equilateral triangle with two tangent circles from R_n, when $r_n = B$. Then $E = \sqrt{B^2 - 1} - \sqrt{3} = 1.442605$, and we call any circle C with distance greater than E a gap circle, since it forces a gap in R_n.

Lemma 3.3. $|R_n| \geq 7$.

Proof. If $|R_n| = 6$, then the other 8 circles fit in a container of radius $D + 1 = 3.126660$, but the minimum container radius for $N = 8$ is 3.304765 [21]. □

We now work toward showing that all circles in $R_1, \ldots, R_{n-1}$ touch R_n.

Lemma 3.4. *There is at most one circle* $C \in R_1, \ldots, R_{n-1}$ *which does not touch* R_n*, and the centers of all circles in* $R_1, \ldots, R_{n-1}$ *that touch* R_n *form a convex partition* P*.*

Proof. If C is disjoint from R_n, then C is prevented from moving out to R_n by two circles $C', C'' \in R_1, \ldots, R_{n-1}$, so that the maximum distance for C from O is if C'', C, C' have centers collinear with C', C'' at distance D from O; see part (b) of Figure 3 setting $Y = D$. This yields a maximum distance of $\sqrt{D^2 - 4}$ for C which we call $F = 0.722969$. Since $F < 1$, there is at most one such C. It also follows that the centers of all circles $C \in R_1, \ldots, R_{n-1}$ that touch R_n form a convex partition, since if not, one of them would likewise be forced to include O by a similar calculation. □

We now conclude this subsection, but first observe that the central angle between the centers of two circles in a ring R_i is at least $2\sin^{-1}(1/r_i)$, but if they have a circle of at least radius Y between them, the angle is at least $2\cos^{-1}\left((r_i^2 + Y^2 - 4)/(2r_iY)\right)$. We denote the angular support of P_{in} as Θ_{in}, and the angular support of P_{out} as Θ_{out}.

Proposition 3.1. *All circles in* $R_1, \ldots, R_{n-1}$ *touch* R_n*.*

Proof. If a circle C does not touch R_n, we seek a contradiction through two cases:

$|R_n| = 7, 8$: If $|R_n| = 7$, there are 7 circles not in R_n. Since the 6 that touch R_n form a convex partiton, their total angular contribution is minimized when their distances from O are maximized. Thus, the farthest C can be from O is when these other 6 circles in $R_1, \ldots, R_{n-1}$ are at a maximal distance D from O, and all 7 inner circles form P_{in}; this opens up the most room for C to move a distance Z away from O. Then Θ_{in} must be $2\pi = 10\sin^{-1}(1/D) + 2\cos^{-1}\big((D^2+Z^2-4)/(2DZ)\big)$ which yields $Z = .168539$. Since this is the maximum value for Z, the closest the remaining 6 circles in $R_1, \ldots, R_{n-1}$ can be to O is $Y = 2 - Z = 1.831461$, which is greater than E, so they must be gap circles. Since the minimum container radius for both $N = 6, 7$ is 3 [21], we know two of these gap circles have distance from O of at least 2. Since Θ_{out} is minimized when $r_n = B$, it must be at least

$$2\sin^{-1}(1/B) + 4\cos^{-1}(B/4) + 8\cos^{-1}\left(\frac{B^2+Y^2-4}{2BY}\right) \approx 7.312832 > 2\pi,$$

contradicting the fact that it must equal 2π. Thus, when $|R_n| = 7$, C must touch R_n.

For $|R_n| = 8$, there are 6 circles not in R_n, so C can be further from O. Θ_{in} is $2\pi = 8\sin^{-1}(1/D) + 2\cos^{-1}\big((D^2+Z^2-4)/(2DZ)\big)$ which yields $Z = .453080$. All 5 remaining circles in $R_1, \ldots, R_{n-1}$ have distance at least $Y = 2 - Z = 1.546920$ which is greater than E, so they are gap circles. Since the minimum container radius for $N = 5$ is $G = 1.701302$ [21], Θ_{out} must be at least

$$\begin{aligned} 6\sin^{-1}(1/B) \quad &+ \quad 2\cos^{-1}(B/4) + 2\cos^{-1}\left(\frac{B^2+G^2-4}{2BG}\right) \\ &+ \quad 6\cos^{-1}\left(\frac{B^2+Y^2-4}{2BY}\right) \approx 6.414042 > 2\pi, \end{aligned}$$

contradicting the fact that it must equal 2π. This concludes the proof for $|R_n| = 7, 8$.

$|R_n| = 9, 10$: We use a different argument. In order for C not to touch R_n, it must be constrained by two circles $C', C'' \in R_1, \ldots, R_{n-1}$; we call Z, Y the distances of C, C'' from O, respectively. For a given Z, Y is minimized when the centers of C'', C, C' are collinear with C' at maximal distance D in part (b) of Figure 3. For $0.130750 \le Z \le F$, we have $Y = \sqrt{4 + Z^2 - 4Z\cos\left(\pi - \cos^{-1}(\frac{Z^2+4-D^2}{4Z})\right)}$, which is minimized when $Y = 1.873902$ at the left endpoint of its domain. Thus, Θ_{out} is at least

$$14\sin^{-1}(1/B) + 4\cos^{-1}\left(\frac{B^2+Y^2-4}{2BY}\right) \approx 6.499456 > 2\pi,$$

for $|R_n| = 9$, contradicting the fact that it must equal 2π. Since the Θ_{out} calculation for $|R_n| = 10$ adds a $2\sin^{-1}(1/B)$, it too is greater than 2π and the lemma is proved. □

3.2 $|R_n| = 10$ and all circles in $R_1, \ldots, R_{n-1}$ form P_{in}

We now define P to be the convex partition of all circles in $R_1, \ldots, R_{n-1}$, which follows from Lemma 3.4 and Proposition 3.1.

Lemma 3.5. *If two circles $C, C' \in P$ touch a circle $C_1 \in R_n$, then $|R_n| < 9$.*

Proof. Refer to part (a) of Figure 4, where we have $C, C' \in P$ touching a single circle C_1 from R_n between them; the distances x, y from O are for C, C', respectively. Observe that given x, then y is minimized when the centers of the circles form an equilateral triangle. When $r_n = B$, y is a function of x via $s = \cos^{-1}\left((B^2 + 4 - x^2)/(4B)\right)$ and $y = \sqrt{4 + B^2 - 4B\cos(\pi/3 - s)}$. Then for $|R_n| \geq 9$, we know Θ_{out} is at least

$$\begin{aligned} 14\sin^{-1}(1/B) + 2\cos^{-1}\left(\frac{B^2 + y^2 - 4}{2By}\right) \quad &+ \quad 2\cos^{-1}\left(\frac{B^2 + x^2 - 4}{2Bx}\right) \\ &\geq \quad 6.445686 > 2\pi \end{aligned}$$

where the minimum is when $x = D$ and $y \approx 1.665517$ is minimized. □

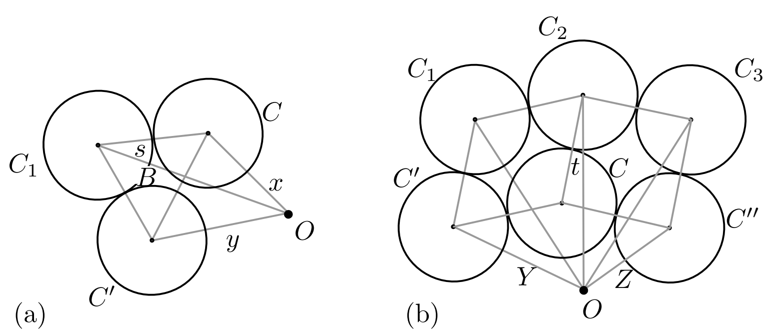

Figure 4: Figures for Lemmas 3.5 and 3.6.

If a sequence of circles $C_1, C_2, \ldots, C_k \in R_n$ proceeds from one gap circle to the next, we will call this sequence in R_n an overpass of length k. Note that we may assume that none of the C_i in an overpass are rattlers, since in maximizing the gaps for the two gap circles the $C_1, \ldots, C_k$ will rotate to form a path in the tangency graph.

Lemma 3.6. *Let $C \in P$ be a non-gap circle which touches an overpass of R_n. Then if $|R_n| \geq 9$, the overpass is at least length 4.*

Proof. If C touches an overpass of length 3, there are 3 circles $C_1, C_2, C_3 \in R_n$ between two gap circles $C', C'' \in P$ with distances Y, Z as in part (b) of Figure

4. The angle t can vary between $0 \leq t \leq \cos^{-1}(1/B) - \pi/3$, and given t then Y, Z are minimized when there are no gaps between C', C, C''. Then when $r_n = B$,

$$Y = \sqrt{B^2 + 4 - 4B\cos(\pi + t - 2\cos^{-1}(1/B))},$$
$$Z = \sqrt{B^2 + 4 - 4B\cos(\pi - t - 2\cos^{-1}(1/B))},$$

and we have that Θ_{out} is at least

$$\begin{aligned} 14\sin^{-1}(1/B) + 2\cos^{-1}\left(\frac{B^2 + Y^2 - 4}{2BY}\right) \quad &+ \quad 2\cos^{-1}\left(\frac{B^2 + Z^2 - 4}{2BZ}\right) \\ &\geq \quad 6.650365 > 2\pi, \end{aligned}$$

where the minimum value is achieved at $t = .04502$ when $Y = D$. We then observe that for gaps between C', C, C'', or shorter overpasses, Θ_{out} will be even greater, thus proving the lemma. $\square$

We can now prove the main results of this subsection.

Proposition 3.2. *$|R_n| = 10$.*

Proof. If $|R_n| = 7$, the inner convex partition P has 7 circles, whose minimum container is a filled ring with radius $1 + 1/\sin(\pi/7) \approx 3.304765 > D + 1$.

If $|R_n| = 8$, suppose first that there is a non-gap circle $C \in P$. Then its maximum distance is E. Let $C' \in P$ with distance Y, where $C' \neq C$. Then Y is minimized when the other four circles in P are at maximal distance D, and the angular support of P is $2\pi = 2\cos^{-1}\left((D^2 + E^2 - 4)/(2DE)\right) + 2\cos^{-1}\left((D^2 + Y^2 - 4)/(2DY)\right) + 4\sin^{-1}(1/D)$ yielding $Y \approx 1.917185$. Thus, all circles in P besides C are gap circles, and as in Proposition 3.1 we have $\Theta_{\text{out}} > 2\pi$, since $Y > 1.546920$, the value used in that proposition. Thus, there are six gap circles when $|R_n| = 8$ and by counting gaps, at most one of these gap circles avoids the situation of Lemma 3.5, where consecutive gap circles touched a common circle from R_n. Therefore, $|R_n| > 8$ since Θ_{out} is at least

$$\begin{aligned} 6\sin^{-1}(1/B) \quad &+ \quad 2\cos^{-1}(B/4) + 2\cos^{-1}\left(\frac{B^2 + G^2 - 4}{2BG}\right) \\ &+ \quad 6\cos^{-1}\left(\frac{B^2 + y^2 - 4}{2By}\right) \approx 6.852784 > 2\pi, \end{aligned}$$

where $y \approx 1.665517$ is minimized from Lemma 3.5.

For $|R_n| = 9$, we have at most 3 gap circles, since with 5 gap circles we could not avoid the situation in Lemma 3.5, and with 4 gap circles we could not avoid an overpass of length 3, contradicting Lemma 3.6. We then note that the equation $2\pi = (18 - 2k)\sin^{-1}(1/B) + 2k\cos^{-1}\left((B^2 + y^2 - 4)/(2By)\right)$ has solutions $W = 1.721602$ for $k = 2$, and $V = 1.595722$ for $k = 3$, meaning there cannot be 2 gap circles of distance greater than W, nor 3 gap circles of distance greater than V, for otherwise $\Theta_{\text{out}} > 2\pi$. Then with 3 gap circles at distances V, W, D, we would have Θ_{in} is at least

$$\begin{aligned} 2\cos^{-1}\left(\frac{E^2+D^2-4}{2ED}\right) &\quad + \quad \cos^{-1}\left(\frac{V^2+W^2-4}{2VW}\right) \\ +\cos^{-1}\left(\frac{E^2+V^2-4}{2EV}\right) &\quad + \quad \cos^{-1}\left(\frac{E^2+W^2-4}{2EW}\right) \approx 6.350338 > 2\pi, \end{aligned}$$

where no distances of circles can be increased in order to decrease Θ_{in}. This proves the proposition. □

Proposition 3.3. *All circles in $R_1, \ldots, R_{n-1}$ form P_{in}.*

Proof. We first show that P is tight. Suppose for contradiction that P has a gap, so that there are two circles $C \in P$ on either side of that gap. C is prevented from moving inward by a circle $C' \in P$ and a circle $C_1 \in R_n$. For a given r_n, the closest distance Y for C is when the centers of C_1, C, C' are collinear and C' is at minimal distance $r_n - 2$. Thus, $Y = \sqrt{r_n^2 - 2r_n - 2}$ using Laws of Cosines, and graphing $Y(r_n)$ yields $Y > \sqrt{r_n^2 - 1} - \sqrt{3}$ for $A \le r_n \le B$, so that the C's are gap circles. Then Θ_{out} is at least

$$16\sin^{-1}(1/r_n) + 4\cos^{-1}\left(\frac{r_n^2 + Y^2 - 4}{2r_nY}\right) \ge 6.522939 > 2\pi,$$

where the angular support is minimized at $r_n = B$. Thus, P is tight.

Finally, if a subset of P formed a tight partition, then since all circles in P are at least distance $A-2$ from O, at least one circle in P would be at distance at least $\sqrt{(A-2)^2 - 1} + \sqrt{3} = 2.458593 > D$, which cannot happen. □

3.3 The basic optimal shape for $N = 14$

We begin with a definition and two lemmas.

Definition 3.1. *A minimal polygon is formed by joining centers of circles, so that all sides are length 2, and no subset of the sides forms a polygon.*

Lemma 3.7. *The only minimal rhombus is P_{in}.*

Proof. By Proposition 3.1 any minimal rhombus different from P_{in} must have two circles $C_1, C_2 \in R_n$ and two circles $C, C' \in P_{\text{in}}$; see part (a) of Figure 5. At $r_n = B$, for varying angle θ where $\pi/2 \le \theta \le 2\pi/3$, the distances x, y of C, C' are

$$\begin{aligned} x &= \sqrt{4 + B^2 - 4B\cos(\pi - (\theta + \cos^{-1}(1/B)))}, \\ y &= \sqrt{4 + B^2 - 4B\cos(\theta - \cos^{-1}(1/B))}. \end{aligned}$$

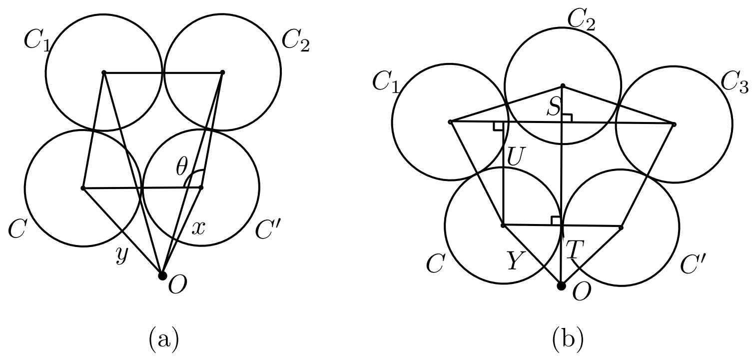

Figure 5: Figures for Lemmas 3.7 and 3.8.

Solving for $x = E$ yields two solutions $\theta = 1.657510, 2\pi/3$ with $y(1.657510) = Y \approx 1.665517$. Graphing Θ_{out} as a function of θ shows it is minimized at Y, namely

$$\begin{aligned} 16\sin^{-1}(1/B) + 2\cos^{-1}\left(\frac{B^2+E^2-4}{2BE}\right) \quad &+ \quad 2\cos^{-1}\left(\frac{B^2+Y^2-4}{2BY}\right) \\ &\approx \quad 6.445686 > 2\pi. \end{aligned}$$

This proves the lemma. □

Lemma 3.8. *Any minimal pentagon has two circles from P_{in}, at most one of which is a gap circle.*

Proof. Since the distance between the centers of 4 consecutive circles on R_n is at least $5.236068 > 4$, we cannot have just one circle from P_{in} in a minimal pentagon. Thus, we have two circles $C, C' \in P_{\text{in}}$ and $C_1, C_2, C_3 \in R_n$, where the symmetric configuration is shown in part (b) of Figure 5. Now for general r_n, $|\overline{C_1C_3}| = \sqrt{2r_n^2 - 2r_n^2\cos(4\sin^{-1}(1/r_n))}$ using ΔOC_1C_3. Thus, $S = \sqrt{4-(|\overline{C_1C_3}|/2)^2}$ and $U = \sqrt{4-((|\overline{C_1C_3}|-2)/2)^2}$, with $T = r_n - (S+U)$ and $Y = \sqrt{1+T^2}$. Graphing Y for $A \le r_n \le B$ yields $Y(r_n) < \sqrt{r_n^2-1} - \sqrt{3}$ and thus neither of C, C' are gap circles in the symmetric configuration. Now in order for C to be pushed out to be a gap circle, C' would be pushed inward, so at most one of them is a gap circle. □

We have two lemmas whose proofs we defer until after our main theorem for this section.

Lemma 3.9. *If there are no rattlers on R_n, then a minimal polygon has at most 5 sides.*

Lemma 3.10. *No rattlers exist on R_n.*

We can now prove our theorem, which refers ahead to Figure 9.

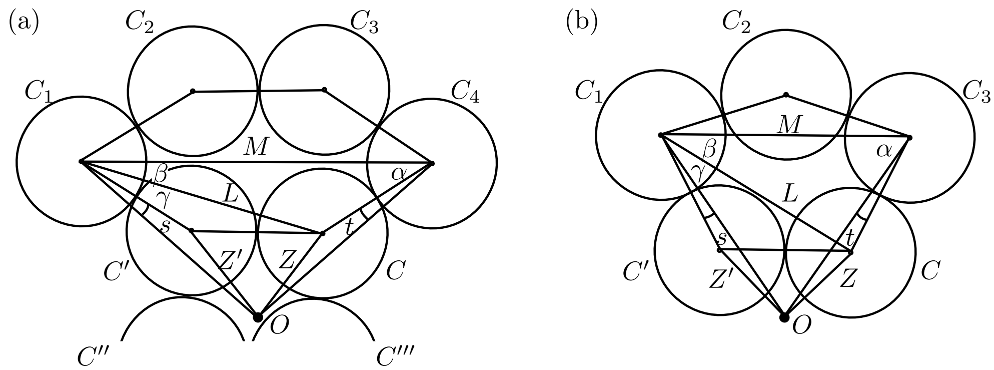

Figure 6: Hexagon and pentagon for Lemmas 3.9 and 3.10.

Theorem 3.1. *The basic shape of the optimal packing of 14 equal circles in a circle is in Figure 9, with the following features not mentioned previously:*

1. *Only the two circles in P_{in} with centers on ℓ touch two circles in R_n;*
2. *The packing has reflective symmetry across the vertical line ℓ;*
3. *The top or bottom triangles may be minimal.*

Proof. Since only minimal triangles or pentagons are possible along R_n, two circles in P_{in} are forced to touch R_n twice, and these cannot be consecutive on P_{in}; the scheme in part (a) of Figure 7, where points are circles and arcs are tangencies, is useful to verify this. The remainder of the packing must be 4 minimal pentagons. By symmetry of the rhombus and the 5 circles in R_n on either side of the rhombus, the theorem follows. □

For the two remaining proofs we set notation that $C, C', C'', C''' \in P_{\text{in}}$ with respective distances Z, Z', Z'', Z''' from O.

Proof of Lemma 3.9. Since the distance between 5 consecutive circles on R_n is at least $6.155367 > 6$, we cannot have minimal polygons with more than 6 sides. Thus, we consider a hexagon with $C_1, C_2, C_3, C_4 \in R_n$ and $C, C' \in P_{\text{in}}$; see part (a) of Figure 6, where we indicate $C'', C''' \in P_{\text{in}}$ for context. We first describe dependencies for the next lemma. For a given value of r_n, the angle t is our variable, which may be positive or negative depending on whether it is to the right or left of the radial line through the center of C_4. Everything else is determined by t as follows in part (a) of Figure 6:

$$M = \sqrt{2r_n^2 - 2r_n^2\cos(6\sin^{-1}(1/r_n))}, \qquad Z = \sqrt{r_n^2 + 4 - 4r_n\cos t},$$
$$\alpha = \cos^{-1}(1/r_n) + t - \cos^{-1}((M-2)/4), \qquad L = \sqrt{M^2 + 4 - 4M\cos\alpha},$$
$$\beta = \cos^{-1}((M^2 + L^2 - 4)/(2LM)), \qquad \gamma = \cos^{-1}(L/4),$$
$$s = \cos^{-1}(1/r_n) - \cos^{-1}((M-2)/4) - \beta - \gamma, \qquad Z' = \sqrt{r_n^2 + 4 - 4r_n\cos s}.$$

We observe for the next lemma that similar equations hold for the pentagon in part (b) of Figure 6 provided $6\sin^{-1}(1/r_n)$ is replaced by $4\sin^{-1}(1/r_n)$ in the formula for M, and where $M-2$ is used in α and s, then M is used instead.

Referring back to part (a) of Figure 6, a priori C, C' could be gap circles, forcing gaps to the right of C_4 and left of C_1, respectively. To see that in fact neither of C, C' are gap circles, first observe that the farthest C' can be rolled along C_1 to the right is when $Z' = \sqrt{r_n^2-1}-\sqrt{3}$ and the hexagon becomes a pentagon. For $A \leq r_n \leq B$ we thus fix this Z', and graphing Z shows $Z < \sqrt{r_n^2-1}-\sqrt{3}$; thus C, C' are not gap circles. But since we are assuming no rattlers on R_n, we must have at least one gap circle by Lemma 3.2, which without loss of generality is C'''. We consider part (a) of Figure 7, where points of tangency between circles in our hexagon are indicated by black line segments, with the curvature of the segments giving the direction of tangency. In order to avoid rhombuses the three solid gray lines must be positioned exactly where they are. But then since all three of the pentagons are minimal, the positions of all circles are determined, meaning the dashed gray line from C' to C_2 must be present as well and C' must be a gap circle. Thus, in fact this is our optimal shape shown in Figure 9 and we conclude that no hexagons exist, provided there are no rattlers. $\square$

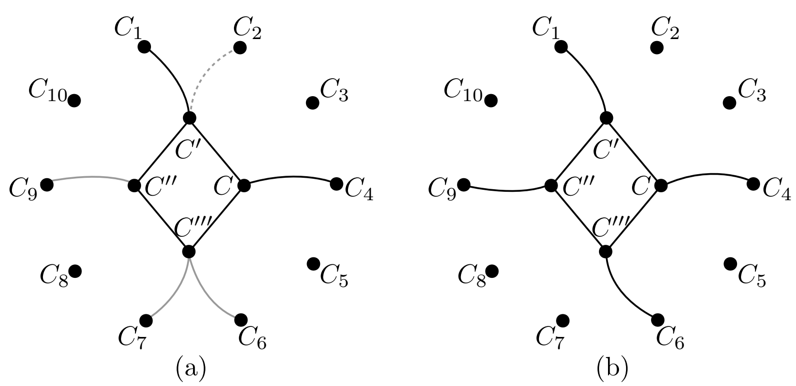

Figure 7: Tangencies for Lemmas 3.9 and 3.10.

Proof of Lemma 3.10. We now show there are no rattlers. Any consecutive rattlers $C_2, \ldots, C_{k-1}$ must occur between two circles $C_1, C_k \in R_n$ with a gap chain of exactly two circles $C, C' \in P_{\text{in}}$, where the centers of C_1, C', C, C_k are in clockwise order. The first sentence in the proof of Lemma 3.9 shows that $k = 3, 4$, meaning we have a non-minimal pentagon or hexagon. The maximum total angular gap on R_n is $2\pi - 20\sin^{-1}(1/B) \approx .180063 \equiv \Phi$. If we had a pentagon with a rattler, the minimum angle between C_1, C_3 is when the centers of C', C_1 and C, C_3 share radial rays, yielding an angle of $2\sin^{-1}(1/(B-2)) \approx 1.704517$. But subtracting $4\sin^{-1}(1/B)$ from this for C_1, C_2, C_3 yields $.483893 > \Phi$. Thus, we may assume only hexagons have rattlers.

If we have one hexagon with rattlers, we have two cases. First, if we have no minimal hexagons, then as in part (a) of Figure 7, to avoid rhombuses we may assume that C''' must be a gap circle. The pentagons are minimal, and the exact same argument holds as in Lemma 3.9, showing that we have the basic optimal shape with no rattlers. Second, if we have a minimal hexagon then we also have two minimal pentagons. These are all adjacent in some order and via parts (a) and (b) of Figure 6 with possible relabeling, all of Z, Z', Z'', Z''' are functions of one variable t for the minimal hexagon in part (a). For $r_n = B$ we have $-0.21844 \leq t \leq 0.05348$, where the left endpoint is when $Z = E$ and the right is when $Z' = E$. If the minimal hexagon has a pentagon on either side, graphing Θ_{in} in part (a) of Figure 8 shows it attains a minimum of 2π at either endpoint where the hexagon becomes a pentagon. But this is the optimal shape as in Lemma 3.9. Likewise, if the two pentagons are to the left of the minimal hexagon, graphing Z'', Z''' shows that one of C'', C''' is always a gap circle, and graphing Θ_{out} in part (b) of Figure 8 shows it has a minimum of 2π where $Z' = E$, again realizing the optimal shape. If the two pentagons are to the right, the minimum is 2π when $Z = E$. This eliminates the case of one hexagon with rattlers.

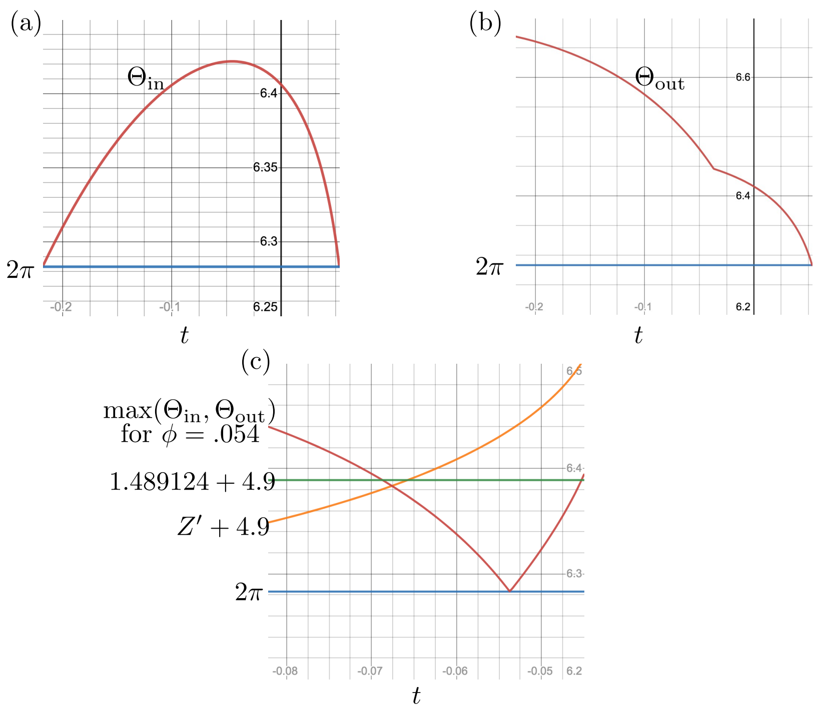

Figure 8: Graphs for Lemma 3.10 generated in Desmos.

If there are two hexagons with rattlers, then the tangencies are in part (b) of Figure 7, since a minimal pentagon must prevent C_1, C_4, C_6, C_9 from moving outward. Thus, each circle C, C', C'', C''' is in a minimal pentagon, and the direction of the tangencies require that $s \leq 0$ and $t \geq 0$ in part (b) of Figure 6. In particular the maximum value for Z (and so also Z', Z'', Z''') must be when $s = 0$, which yields 1.489124. Since the total angular gap Φ is shared by the two non-minimal hexagons, the hexagon with gap chain C, C' has total angular gap ϕ of at most $\Phi/2 \approx .0900315$. As ϕ increases, M in part (a) of Figure 6 can increase for this hexagon, but the dependent quantities change accordingly, so that we can still calculate $\max(\Theta_{\text{in}}, \Theta_{\text{out}})$ as a function of $t \leq 0$ for the hexagon, but now with graphs parametrized by $0 \leq \phi \leq \Phi/2$. Graphing these show that they attain a minimum near 2π at $t = -.05375$ when $\phi = .054$; this is shown in part (c) of Figure 8 where the red curve is tangential to the blue line at 2π. But at $t = -.05375$, we clearly have $Z' > 1.489124$, as indicated by the orange curve $(Z' + 4.9)$ being above the green line $(1.489124 + 4.9)$; this is true for a neighborhood of (t, ϕ) values and violates the constraint $Z' \leq 1.489124$. This proves the lemma. □

4. The optimal solution

We can now determine the optimal packing. We refer the reader to Figure 9 which shows the basic shape of the optimal packing, with the center O placed at the origin, and having reflective symmetry across the y-axis.

We may assume that $\beta_2 \leq \beta_1$, with Figure 9 showing the case $\beta_2 = \beta_1$ which has reflective symmetry over the x-axis. For convenience of notation we have used r to denote the radius of R_n. The tight partition P_{out} applied to the left side of the packing yields the equation

$$\gamma_1 + \gamma_2 + 8\sin^{-1}(1/r) = \pi \tag{1}$$

The quantities L_1 and L_2 denote the distance from O to the centers of C' and C'', respectively.

With this notation, we can now prove our main theorem.

Theorem 4.1. *The optimal packing for 14 circles occurs when $\beta_2 = \pi/6$ in Figure 9, meaning there is no gap between C_5 and C_6 in P_{out}.*

Proof. We show that if $\beta_2 > \pi/6$, then r can be reduced and is not optimal. The conclusion is then that the optimal solution occurs when $\beta_2 = \pi/6$ and there is no gap between C_5 and C_6 in P_{out}.

We therefore consider a value of β_2 satisfying $\pi/6 < \beta_2 \leq \beta_1 < \pi/2$, and for the moment fix the outer radius r associated with that packing. We will also for the moment assume that the rhombus formed by P_{in} is rigid, meaning that the quantity $L_1 + L_2$ is fixed. We will, however, be examining vertical translations of this rigid P_{in}, with the result that if L_2 is decreased, then L_1 must be increased by the same amount.

Since $\beta_2 > \pi/6$, we can rotate C_1 through C_5 counterclockwise along the container boundary by some positive angle $\epsilon > 0$, and likewise C_{10} through C_6

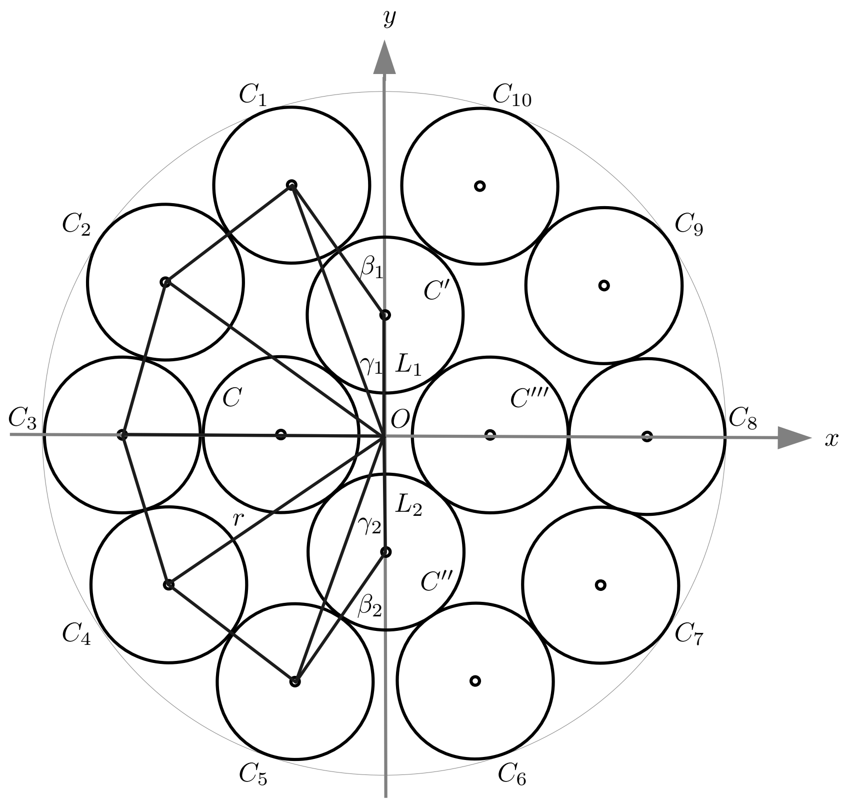

Figure 9: Quantities needed for the proof of Theorem 4.1.

clockwise by the same positive angle $\epsilon > 0$, to decrease β_2. This will force the rhombus P_{in} upward. Since $\beta_2 \leq \beta_1$, the points of tangency between C and C_3, and C''' and C_8, have nonnegative y-coordinate, so that these present no obstruction to the upward translation of the rhombus P_{in}.

It therefore only remains to show that the decreasing of the gap between C_5 and C_6 results in an increasing of the gap between C_1 and C_{10} that is large enough to accommodate the upward translation of C'. This can be formalized by considering γ_2 and γ_1, and first observing that by differentiating Equation 1, the rotation of circles in P_{out} results in

$$\frac{d\gamma_1}{d\gamma_2} = -1. \tag{2}$$

Now we need to compare this with the effect the upward translation of P_{in} has on γ_1. The Law of Cosines for the triangle having vertices O and the centers of C_1 and C' is

$$L_1^2 + r^2 - 2rL_1 \cos\gamma_1 = 4,$$

and implicitly differentiating this yields the positive derivative

$$\frac{d\gamma_1}{dL_1} = \frac{2r\cos\gamma_1 - 2L_1}{2rL_1\sin\gamma_1}.$$

Applying a similar Law of Cosines calculation for the triangle having vertices O and the centers of C_5 and C'', we obtain the positive derivative

$$\frac{dL_2}{d\gamma_2} = \frac{2rL_2\sin\gamma_2}{2r\cos\gamma_2 - 2L_2}.$$

Since $L_1 + L_2$ is constant we know $\frac{dL_1}{dL_2} = -1$ so that by the chain rule

$$\begin{aligned}
\frac{d\gamma_1}{d\gamma_2} &= \frac{d\gamma_1}{dL_1}\cdot\frac{dL_1}{dL_2}\cdot\frac{dL_2}{d\gamma_2} \\
&= \frac{2r\cos\gamma_1 - 2L_1}{2rL_1\sin\gamma_1}\cdot -1\cdot\frac{2rL_2\sin\gamma_2}{2r\cos\gamma_2 - 2L_2} \\
&= \frac{2r\cos\gamma_1 - 2L_1}{2r\cos\gamma_2 - 2L_2}\cdot -1\cdot\frac{2rL_2\sin\gamma_2}{2rL_1\sin\gamma_1}.
\end{aligned}$$

Since $\beta_2 \leq \beta_1$ we also have $\gamma_2 \leq \gamma_1$, and since $L_2 \leq L_1$ as well, we know that the first and third factors in the last expression are both positive values at most one. The result is that the upward translation of the rhombus P_{in} yields

$$\frac{d\gamma_1}{d\gamma_2} \leq -1. \tag{3}$$

Comparing Equation 2 with Inequality 3 shows that the rotation of circles in P_{out} will open up γ_1 enough to translate P_{in} upward. The result is that both C and C''' will no longer touch the outer ring, and thus have just two points of tangency with the circles in P_{in}. Both C and C''' can therefore be perturbed to be rattlers, and r can then be decreased. This establishes the theorem. □

We conclude the paper by observing that the optimal configuration established in Theorem 4.1 is indeed that conjectured by Pirl [21]. This is shown in Figure 10, where the global optimal, rotated clockwise by $\pi/2$, is obtained with container radius 4.328 (accurate up to four decimal places) using the trust-region Dogleg algorithm with the MATLAB *fsolve* function.

Acknowledgements

The authors would like to thank the anonymous referee whose helpful comments greatly improved the focus and exposition of this paper, in particular Section 4.

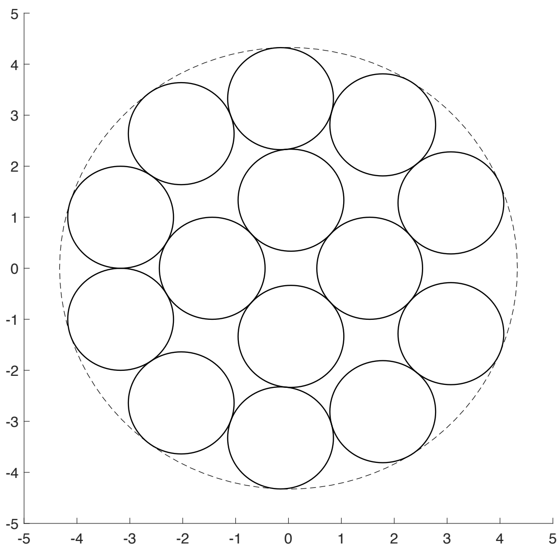

Figure 10: The optimal packing with container radius 4.328, plotted with MATLAB.

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Accepted: March 3, 2023