On the completion of symmetric metric spaces

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Abstract. In this work, we investigate particular properties on the completion of symmetric spaces. Symmetric spaces are metric spaces and, naturally, question arises as to whether their completions are also symmetric. In this work, we provide an affirmative response to this question. More precisely, we prove that every metric space is isometrically a subset of a symmetric space. In addition, we prove that the completion of a symmetric metric space is likewise symmetric. Some additional functorial properties are established along with some other results. Additionally, generic examples of symmetric spaces will be provided in this manuscript.

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1. Introduction

The purpose of this section is to recall some standard terminology and nomenclature related to metric spaces [1] and symmetric metric spaces [2]. To start with, recall that a **metric space** is a pair of the form (X, d_X), where X is a nonempty set and $d_X : X \times X \to \mathbb{R}$ is a function satisfying the following properties:

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(i) if $x, y \in X$, then $d_X(x,y) \geq 0$,

(ii) if $x, y \in X$, then $d_X(x,y) = 0$ if and only if $x = y$,

(iii) $d_X(x,y) = d_X(y,x)$, for any $x, y \in X$, and

(iv) $d_X(x,y) \leq d_X(x,z) + d_X(z,y)$, for any $x, y, z \in X$.

If (X, d_X) and (Y, d_Y) are metric spaces, then a function $f : X \to Y$ is an **isometry** if $d_Y(f(x), f(y)) = d_X(x,y)$, for all $x, y \in X$. Obviously, any isometry is an injective and continuous function. If f is a surjective isometry, then we say that (X, d_X) and (Y, d_Y) and **isometric** spaces. In such case, f^{-1} is likewise an isometry. Evidently, the relation of being isometric is an equivalence relation in the class of metric spaces.

Let (X, d_X) be a metric space, $x_0 \in X$ and $f : X \to X$ a surjective isometry. Then (X, d_X) is **x_0-symmetric with respect to** f if, for each $x \in X$,

$$d_X(x, x_0) = d_X(f(x), x_0) = \frac{1}{2} d_X(x, f(x)).$$

If there is no ambiguity, then (X, d_X) is simply called **x_0-symmetric**. As an example, if $X = [-1,1]$ with the metric d of $\mathbb{R}$ and $f : X \to X$ is given by $f(x) = -x$, then (X, d) is 0-symmetric with respect to f. Also, if $(X, \|\cdot\|)$ is a real Banach space with the norm $\|\cdot\| : X \to \mathbb{R}$, d_X is the respective induced norm and $a \in X$, then (X, d_X) is a-symmetric with respect to $f(x) = 2a - x$.

The following are some properties satisfied by symmetric metric spaces.

Proposition 1. *Let (X, d_X) be a metric space, $x_0 \in X$ and $f : X \to X$ a surjective isometry. If (X, d_X) is x_0-symmetric with respect to f, then it is also x_0-symmetric with respect to f^{-1}.*

Proof. Beforehand, notice that f^{-1} is also a surjective isometry. Let $y \in X$, and take $x \in X$ such that $y = f(x)$. It follows that

$$\begin{aligned} d_X(y, x_0) &= d_X(f(x), x_0) = d_X(x, x_0) = d_X(f^{-1}(y), x_0) \\ &= \frac{1}{2} d_X(x, f(x)) = \frac{1}{2} d_X(f^{-1}(y), y). \end{aligned}$$

We conclude that (X, d_X) is x_0-symmetric with respect to f^{-1}. □

Proposition 2. *Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$ and suppose that $f : X \to X$ and $\phi : X \to Y$ are surjective isometries. If (X, d_X) is a x_0-symmetric metric space with respect to f, then (Y, d_Y) is $\phi(x_0)$-symmetric with respect to $g = \phi \circ f \circ \phi^{-1}$.*

Proof. Being the composition of surjective isometries, g itself is a surjective isometry. On the other hand, if $y \in Y$, then the x_0-symmetry of (X, d_X) with respect to f and isometry properties of f, ϕ and ϕ^{-1} assure that

$$d_Y(y, \phi(x_0)) = d_X(\phi^{-1}(y), x_0) = d_X(f(\phi^{-1}(y)), x_0) = d_Y(g(y), \phi(x_0))$$

and
$$\begin{aligned}\frac{1}{2}d_Y(y,g(y)) &= \frac{1}{2}d_X(\phi^{-1}(y),f(\phi^{-1}(y))) = d_X(f(\phi^{-1}(y)),x_0)\\ &= d_Y(g(y),\phi(x_0)).\end{aligned}$$

These facts establish that (Y,d_Y) is $\phi(x_0)$-symmetric with respect to g. □

Let (X,d_X) be a metric space, $x_0 \in X$ and $f,g : X \to X$ to surjective isometries. In general, it is not true that (X,d_X) is a x_0-symmetric metric space with respect to $g \circ f$ when it is x_0-symmetric with respect to f and g. Indeed, let (X,d_X) be the real numbers with its usual distance, and let us define $f(x) = g(x) = -x$, for each $x \in X$. It is obvious that (X,d_X) is 0-symmetric with respect to f and g, but it is not 0-symmetric with respect to $g \circ f$. In fact, notice that $d_X(x,(g\circ f)(x)) = 0$, for each $x \in X$.

2. Main results

This section is devoted to providing additional properties and ways to construct symmetric metric spaces. In the remainder and unless we mention something different, we will assume that (X,d_X) is a metric space, $x_0 \in X$ and $f : X \to X$ will be a surjective isometry.

To start with, we recall some standard definitions. If (X,d_X) is a metric space, $x \in X$ and $A \subseteq X$ is nonempty, then we define

$$d_X(x,A) = \inf_{y\in A} d_X(x,y).$$

In addition, if $B \subseteq X$ is also nonempty, then we define the number $d_X(A,B)$ alternatively (and equivalently) in the following way:

$$d_X(A,B) = \inf_{\substack{x\in A\\ y\in B}} d_X(x,y) = \inf_{x\in A} d_X(x,B) = \inf_{y\in B} d_X(y,A).$$

Proposition 3. *Suppose that (X,d_X) is an x_0-symmetric metric space with respect to f, and let $A \subseteq X$ be nonempty. Then*

$$d_X(x_0,A) = d_X(x_0,f(A)) \geq \frac{1}{2}d_X(A,f(A)).$$

Proof. Observe that the following inequalities hold:

$$\begin{aligned}d_X(x_0,A) &= \inf_{x\in A} d_X(x_0,x) = \inf_{x\in A} d_X(x_0,f(x)) = \inf_{y\in f(A)} d_X(x_0,y)\\ &= d_X(x_0,f(A)) = \inf_{x\in A} d_X(x_0,f(x)) = \frac{1}{2}\inf_{x\in A} d_X(x,f(X))\\ &\geq \frac{1}{2}\inf_{\substack{x\in A\\ y\in f(A)}} d_X(x,y) = \frac{1}{2}d_X(A,f(A)),\end{aligned}$$

which yields the conclusion of this result. □

The following result is motivated by the reduced cone CX defined in [3].

Theorem 1. *Every metric space is isometrically a subset of a symmetric space.*

Proof. Let (X, d_X) be any metric space, and fix $x_0 \in X$ arbitrarily. Throughout, we will let $Y = (X \times \{x_0\}) \cup (\{x_0\} \times X)$. Obviously, Y is a subset of $X \times X$. Define the function $d_Y : Y \times Y \to \mathbb{R}$ as

$$d_Y((x_1, x_2), (y_1, y_2)) = d_X(x_1, y_1) + d_X(x_2, y_2),$$

for each (x_1, x_2) and (y_1, y_2) in Y. It is easy to check then that (Y, d_Y) is a metric space. Let $\phi : X \to Y$ be given by $\phi(x) = (x, x_0)$, for each $x \in X$. Notice firstly that ϕ is an isometry by virtue of the fact that

$$d_Y(\phi(x), \phi(y)) = d_Y((x, x_0), (y, x_0)) = d_X(x, y), \quad \forall x, y \in X.$$

Let us define $f : Y \to Y$ by $f(x, x_0) = (x_0, x)$ and $f(x_0, x) = (x, x_0)$, for each $x \in X$. Evidently, f is a surjective function. Moreover, f is also an isometry. To check this fact, various cases need to be considered. Indeed, observe that

$$\begin{aligned}
&d_Y(f(x, x_0), f(y, x_0)) = d_X(x, y) = d_Y((x, x_0), (y, x_0)), \\
&d_Y(f(x_0, x), f(x_0, y)) = d_X(x, y) = d_Y((x_0, x), (x_0, y)), \\
&d_Y(f(x, x_0), f(x_0, y)) = d_X(y, x_0) + d_X(x_0, x) = d_Y((x, x_0), (y, x_0)),
\end{aligned}$$

for each $x, y \in X$. We claim now that Y is x^*-symmetric with respect to f, where $x^* = (x_0, x_0) \in Y$. To show that, notice firstly that, for each $x \in X$,

$$\begin{aligned}
d_Y(f(x, x_0), x^*) &= d_Y((x, x_0), x^*) = d_X(x, x_0) \\
&= \frac{1}{2}\left[d_X(x, x_0) + d_X(x, x_0)\right] = \frac{1}{2} d_Y((x, x_0), f(x, x_0)).
\end{aligned}$$

In similar fashion, we can prove also that

$$d_Y(f(x_0, x), x^*) = d_Y((x_0, x), x^*) = \frac{1}{2} d_Y((x_0, x), f(x_0, x)), \quad \forall x \in X.$$

We conclude that (Y, d_Y) is x^*-symmetric with respect to f, and that (X, d_X) is isometric to a subset of (Y, d_Y), as desired. □

It is well known that every metric space (X, d_X) can be extended to be a complete metric space. Moreover, the metric space (X, d_X) is dense in its completion. Our last result establishes that the completion is symmetric if the space (X, d_X) is symmetric. Before proving the theorem, we recall some of the details in the construction of the proof for the completion of a metric space. Let $\mathcal{S}(X)$ be the set of all Cauchy sequences in (X, d_X), and define a relation of X as follows: if (x_n) and (y_n) are members of $\mathcal{S}(X)$, we say that they are equivalent if $\lim_{n \to \infty} d_X(x_n, y_n) = 0$. This is an equivalence relation on $\mathcal{S}(X)$, and the set of equivalence classes is denoted by $\mathcal{C}(X, d_X)$ or, simply, by $\mathcal{C}(X)$. For the sake of briefness, the equivalence class determined by the Cauchy sequence $(x_n) \in \mathcal{S}(X)$ will be denoted also by (x_n).

Define next the function $d_{\mathcal{C}(X)} : \mathcal{C}(X) \times \mathcal{C}(X) \to \mathbb{R}$ by

$$d_{\mathcal{C}(X)}((x_n),(y_n)) = \lim_{n\to\infty} d_X(x_n, y_n),$$

for any two equivalence classes (x_n) and (y_n) in $\mathcal{C}(X)$. This function is well defined on $\mathcal{C}(X)$ and, moreover, it is a metric. The space $(\mathcal{C}(X), d_{\mathcal{C}(X)})$ is a complete metric space. In addition, if $\iota_X : X \to \mathcal{C}(X)$ is the function that assigns to each $x \in X$ the constant sequence whose nth term is x, then ι_X is an isometry and $\iota_X(X)$ is dense in $\mathcal{C}(X)$. The space $(\mathcal{C}(X), d_{\mathcal{C}(X)})$ constructed in this way is called the *completion* of the metric space (X, d_X).

Interestingly, if (X, d_X) is a metric space, $(\mathcal{C}(X), d_{\mathcal{C}(X)})$ is its completion, (Y, d_Y) and complete metric space and $f : X \to Y$ an isometry, then there exists a unique isometry $\overline{f} : \mathcal{C}(X) \to Y$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\iota_X"] \arrow[dr, "f"'] & \mathcal{C}(X) \arrow[d, "\overline{f}"] \\ & Y \end{tikzcd}$$

The uniqueness of completions up to isometries is a consequence of this property. Moreover, if (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$ is an isometry, then there exists a unique isometry $\mathcal{C}(f) : \mathcal{C}(X) \to \mathcal{C}(Y)$ which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\iota_X"] \arrow[d, "f"'] & \mathcal{C}(X) \arrow[d, "\mathcal{C}(f)"] \\ Y \arrow[r, "\iota_Y"'] & \mathcal{C}(Y) \end{tikzcd}$$

In addition, recall that $\mathcal{C}$ preserves compositions of isometries and identity mappings. This implies that $\mathcal{C}$ is a functor from the category of metric spaces with isometries, into the category of complete metric spaces. With these conventions, the following proposition shows that if (X, d_X) is an x_0-symmetric metric space with respect to the isometry $f : X \to X$, then $(\mathcal{C}(X), d_{\mathcal{C}(X)})$ is $\iota_X(x_0)$-symmetric with respect to $\mathcal{C}(f)$. The statement is summarized as follows.

Theorem 2. *The completion of a symmetric metric space is likewise symmetric.*

Proof. We will use the notation preceding the theorem. Since $f : X \to X$ is a surjective isometry, then $\mathcal{C}(f) : \mathcal{C}(X) \to \mathcal{C}(X)$ is likewise a surjective isometry. For the sake of convenience, let $\hat{f} = \mathcal{C}(f)$ and $x_0^* = \iota_X(x_0)$. To show that $(\mathcal{C}(X), d_{\mathcal{C}(X)})$ is x_0^*-symmetric with respect to $\hat{f}$, it remains to check that, for each $x^* \in \mathcal{C}(X)$, the following identities are satisfied:

$$d_{\mathcal{C}(X)}(x^*, x_0^*) = d_{\mathcal{C}(X)}(\hat{f}(x^*), x_0^*) = \frac{1}{2} d_{\mathcal{C}(X)}(x^*, \hat{f}(x^*)). \tag{1}$$

Let us assume firstly that $x^* \in \iota_X(X)$. So, there exists $x \in X$ with the property that $x^* = \iota_X(x)$. As a consequence of this, the fact that ι_X is an isometry, the functorial properties of the completion and the x_0-symmetry of (X, d_X) with respect to f, we obtain

$$\begin{aligned} d_{\mathcal{C}(X)}(x^*, x_0^*) &= d_{\mathcal{C}(X)}(\iota_X(x), \iota_X(x_0)) = d_X(x, x_0) = d_X(f(x), x_0) \\ &= d_{\mathcal{C}(X)}(\iota_X(f(x)), \iota_X(x_0)) = d_{\mathcal{C}(X)}(\hat{f}(\iota_X(x)), \iota_X(x_0)) \\ &= d_{\mathcal{C}(X)}(\hat{f}(x^*), x_0^*). \end{aligned}$$

Similarly, notice that

$$\begin{aligned} d_{\mathcal{C}(X)}(x^*, x_0^*) = d_X(x, x_0) &= \frac{1}{2} d_X(x, f(x)) = \frac{1}{2} d_{\mathcal{C}(X)}(\iota_X(x), \iota_X(f(x))) \\ &= \frac{1}{2} d_{\mathcal{C}(X)}(x^*, \hat{f}(\iota_X(x))) = \frac{1}{2} d_{\mathcal{C}(X)}(x^*, \hat{f}(x^*)). \end{aligned}$$

As a consequence, we have proved that (1) holds for each $x^* \in \iota_X(X)$. To show that the conclusion is also valid for all $x^* \in \mathcal{C}(X)$, recall that the closure of $\iota_X(X)$ is equal to $\mathcal{C}(X)$, and let (x_n^*) be any sequence in $\iota_X(X)$ which converges to x^*. Thus, if $n \in \mathbb{N}$, then

$$d_{\mathcal{C}(X)}(x_n^*, x_0^*) = d_{\mathcal{C}(X)}(\hat{f}(x_n^*), x_0^*) = \frac{1}{2} d_{\mathcal{C}(X)}(x_n^*, \hat{f}(x_n^*))$$

Taking now the limit when $n \to \infty$, using that the metric $d_{\mathcal{C}(X)}$ and $\hat{f}$ are both continuous functions, we prove that (1) is satisfied for all $x^* \in \mathcal{C}(X)$. We conclude that $(\mathcal{C}(X), d_{\mathcal{C}(X)})$ is x_0^*-symmetric with respect to $\hat{f}$. □

3. Examples

In this section, we provide some constructions of symmetric spaces. Various examples will be provided at this stage of our work. In the first of them, we will show that some products of symmetric spaces are likewise symmetric.

Example 1. Let (X_i, d_{X_i}) be metric spaces, $x_i^* \in X_i$ and $f_i : X_i \to X_i$ surjective isometries, and assume that (X_i, d_{X_i}) is x_i^*-symmetric with respect to f_i, for each $i = 1, 2$. Let $X = X_1 \times X_2$, fix $x^* = (x_1^*, x_2^*)$, and agree that $x = (x_1, x_2)$ and $y = (y_1, y_2)$, for each $x, y \in X$. Let us define $d_X : X \times X \to \mathbb{R}$ by means of the equation

$$d_X(x, y) = d_{X_1}(x_1, y_1) + d_{X_2}(x_2, y_2),$$

for each $x, y \in X$. It is obvious that (X, d_X) is a metric space. Let $f : X \to X$ be defined as $f(x) = (f_1(x_1), f_2(x_2))$, for each $x \in X$. Then f is surjective and, moreover, it is an isometry by virtue that

$$\begin{aligned} d_X(f(x), f(y)) &= d_X((f_1(x_1), f_2(x_2)), (f_1(y_1), f_2(y_2))) \\ &= d_{X_1}(f_1(x_1), f_1(y_1)) + d_{X_2}(f_2(x_2), f_2(y_2)) \\ &= d_{X_1}(x_1, y_1) + d_{X_2}(x_2, y_2) = d_X(x, y). \end{aligned}$$

Additionally,

$$\begin{aligned}
d_X(x,x^*) &= d_{X_1}(x_1,x_1^*) + d_{X_2}(x_2,x_2^*) = d_{X_1}(f_1(x_1),x_1^*) + d_{X_2}(f_2(x_2),x_2^*) \\
&= d_X(f(x),x^*) = \frac{1}{2}\left[d_{X_1}(x_1,f_1(x_1)) + d_{X_2}(x_2,f_1(x_2))\right] \\
&= \frac{1}{2} d_X(x,f(x)).
\end{aligned}$$

We conclude that (X,d_X) is x^*-symmetric with respect to f. □

It is worth pointing out that the last example can be generalized to the product of a finite number of symmetric metric spaces. Moreover, the example can be extended to account for different metrics, including the infinity metric and the Euclidean metric induced in d_{X_1} and d_{X_2}.

To state our next result, recall that if (X,d_X) is a metric space and $E \subseteq X$ is nonempty, we say that E is **bounded** if there exists $K \in \mathbb{R}$ such that $d_X(x,y) \le K$, for each $x,y \in E$. If that is the case, then we let

$$\operatorname{diam} E = \sup\{d_X(x,y) : x,y \in E\}.$$

Theorem 3. *Let (X,d_X) be x_0-symmetric with respect to f, and let $E \neq \emptyset$. Let $B = \{g : E \to X : \operatorname{diam} g(E) < \infty\}$, and $d_B : B \times B \to \mathbb{R}$ be given by*

$$d_B(g,h) = \sup_{e\in E} d_X(g(e),h(e)), \quad \forall g,h \in B.$$

Let $\Phi : B \to B$ be given by $\Phi(g) = f \circ g$, for each $g \in B$. Then B is g_{x_0}-symmetric with respect to Φ, where $g_{x_0} : E \to X$ is the constant $g_{x_0} \equiv x_0$.

Proof. To start with, observe that (B,d_B) is indeed a metric space. To show that Φ is surjective, let $h : E \to X$ be such that $\operatorname{diam} h(E) < \infty$, and let $g = f^{-1} \circ h$. The fact that f is an isometry assures that $\operatorname{diam} g(E) = \operatorname{diam} h(E) < \infty$, which means that $g \in B$ and, moreover, $\Phi(g) = h$. The fact that Φ is an isometry is a consequence of the fact that f is an isometry, so

$$d_B(\Phi(g),\Phi(h)) = \sup_{e\in E} d_X(f(g(e)),f(h(e))) = \sup_{e\in E} d_X(g(e),h(e)) = d_B(g,h),$$

for each $g,h \in B$. Finally, observe that

$$\begin{aligned}
d_B(g,g_{x_0}) &= \sup_{e\in E} d_X(g(e),x_0) = \sup_{e\in E} d_X(f(g(e)),x_0) \\
&= \sup_{e\in E} d_X((\Phi(g))(e),x_0) = d_B(\Phi(g),g_{x_0}) \\
&= \frac{1}{2}\sup_{e\in E} d_X(f(g(e)),g(e)) = \frac{1}{2} d_B(\Phi(g),g),
\end{aligned}$$

for each $g \in B$. We conclude that B is g_{x_0}-symmetric with respect to Φ. □

Theorem 4. *Let* (X, d_X) *be a compact metric space, assume that* (Y, d_Y) *is* y_0*-symmetric with respect to* f*, and let* $C = \{g : X \to Y : g \text{ is continuous}\}$*. Let* $d_C : C \times C \to \mathbb{R}$ *be defined by*

$$d_C(g, h) = \sup_{x \in X} d_Y(g(x), h(x)), \quad \forall g, h \in C.$$

Then (C, d_C) *is* g_{y_0}*-symmetric with respect to* $\Phi(g) = f \circ g$*. Here,* $g_{y_0} : X \to Y$ *is the constant function* $g_{y_0} \equiv y_0$*.*

Proof. The proof is similar to that of the previous theorem. We just need to point out here that the function d_C is well defined in this case, in view of the compactness of the metric space (X, d_X). $\square$

It is worth pointing out that the compactness assumption on the metric space (X, d_X) can be omitted in the last theorem. To that end, we require to redefine the set C as

$$C = \{g : X \to Y : g \text{ is continuous and } \operatorname{diam} g(E) < \infty\}.$$

Using all the remaining assumptions in Theorem 4, we can readily reach the same conclusion.

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