Flow-selfdual curves in a geometric surface

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Abstract. For a natural parametrization of a curve γ in an orientable two-dimensional Riemannian manifold, we compare two differential equations associated to γ. The main tool of our study is the geodesic curvature k of γ and when these equations coincide we call γ as being flow-selfdual since the second equation corresponds to the flow-curvature k_f of γ in the same manner as the first equation involves k. We obtain that these curves have a constant geodesic curvature and then we discuss four examples. Also, we generalize this type of differential equations to vector fields on Riemannian manifolds of arbitrary dimension.

Keywords: two-dimensional Riemannian manifold, geodesic curvature, flow-selfdual curve.

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1. Flow-selfdual curves and tangential vector fields

The framework of this study is a *geometric surface* i.e. ([3]) a smooth, orientable two-dimensional Riemannian manifold (M^2, g). Being orientable M supports an almost complex structure J; in fact J is integrable and for an arbitrary point $p \in M$ we consider $J_p : T_pM \to T_pM$ as being the multiplication with the complex unit $i \in \mathbb{C}$. Let ∇ be the Levi-Civita connection of g.

Fix also a smooth curve $\gamma : I \subseteq \mathbb{R} \to M$ which we suppose to be *regular*: $\gamma'(t) \in T_{\gamma(t)}M \setminus \{0\}$. Let $\mathfrak{X}(\gamma)$ be the $C^\infty(I)$-module of vector fields along γ i.e. smooth maps $X : I \to TM$ with $X(t) \in T_{\gamma(t)}M$ for all $t \in I$. It follows the *unit tangent vector field* $T \in \mathfrak{X}(\gamma)$ with:

$$T(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}, \tag{1.1}$$

where $\|\cdot\|$ denotes the norm induced by g on the tangent spaces. Therefore, the *Frenet frame* of γ is $\mathcal{F} := \begin{pmatrix} T \\ N := J(T) \end{pmatrix} \in \mathfrak{X}(\gamma) \times \mathfrak{X}(\gamma)$.

The Riemannian geometry of γ is described by its *geodesic curvature* $k : I \to \mathbb{R}$ provided by the *Frenet equations*:

$$\nabla_{T(t)}\mathcal{F}(t) = \begin{pmatrix} 0 & k(t) \\ -k(t) & 0 \end{pmatrix} \mathcal{F}(t) = k(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathcal{F}(t) \tag{1.2}$$

which means:

$$k(t) := \frac{g(\nabla_{\gamma'(t)}T(t), N(t))}{\|\gamma'(t)\|} = \frac{g(\nabla_{\gamma'(t)}\gamma'(t), J(\gamma'(t)))}{\|\gamma'(t)\|^3}. \tag{1.3}$$

Recall also the pair (g, J) yields *the symplectic form* $\Omega(\cdot, \cdot) := g(\cdot, J\cdot)$ and whence:

$$k(t) := \frac{\Omega(\nabla_{\gamma'(t)}\gamma'(t), \gamma'(t))}{\|\gamma'(t)\|^3}. \tag{1.4}$$

The starting point of this short note is the remark that under the hypothesis of γ being parametrized by arc-length the second covariant derivative applied to the Frenet equations yields the following differential equation:

$$\mathcal{E} : (\nabla_{\gamma'}\nabla_{\gamma'}\gamma')(t) - \frac{k'(t)}{k(t)}(\nabla_{\gamma'}\gamma')(t) + k^2(t)\gamma'(t) = 0. \tag{1.5}$$

The pair (g, J) being a Kähler structure (since $\dim M = 2$) it follows that ∇ commutes with N and then N satisfies the same differential equation. For curves parametrized by arc-length the vector field $\nabla_{\gamma'}\gamma'$ is called *the curvature vector field* of γ.

In the very recent paper [3] we introduce a modification of the curvature k called *flow-curvature* and denoted k^f. For a general parametrization of γ it holds:

$$k^f(t) := k(t) - \frac{1}{\|\gamma'(t)\|} < k(t). \tag{1.6}$$

Since k^f is obtained exactly in the same manner as k i.e. through the Frenet equation of the flow-frame:

$$\mathcal{F}^f(t) := \begin{pmatrix} E_1^f \\ E_2^f \end{pmatrix}(t) = Rotation(t)\mathcal{F}(t) = \begin{pmatrix} \cos tT(t) - \sin tN(t) \\ \sin tT(t) + \cos tN(t) \end{pmatrix} \tag{1.7}$$

it follows a second differential equation of third order satisfied by non-flow-flat curves:

$$\mathcal{E}^f : (\nabla_{\gamma'}\nabla_{\gamma'}E_1^f)(t) - \frac{(k^f)'(t)}{k^f(t)}(\nabla_{\gamma'}E_1^f)(t) + (k^f)^2(t)E_1^f(t) = 0. \tag{1.8}$$

It is natural to connect the differential equations $\mathcal{E}$ and $\mathcal{E}^f$ and this leads to our new type of curves:

Definition 1.1. The non-flow-flat curve γ, parametrized by arc-length, is called *flow-selfdual* it it satisfies also the differential equation:

$$\mathcal{E}_f : (\nabla_{\gamma'}\nabla_{\gamma'}\gamma')(t) - \frac{(k^f)'(t)}{k^f(t)}(\nabla_{\gamma'}\gamma')(t) + (k^f)^2(t)\gamma'(t) = 0. \tag{1.9}$$

Our main theoretical result is the following:

Proposition 1.2. *The non-flow-flat curve γ is a flow-selfdual one if and only if $k = \frac{1}{2} = -k^f$ which means that all four unit vector fields γ', N, E_1^f, E_2^f satisfy the same differential equation of Schrödinger type*:

$$\mathcal{E} = \mathcal{E}^f = \mathcal{E}_f : (\nabla_{\gamma'}\nabla_{\gamma'}U)(t) + \frac{1}{4}U(t) = 0, \quad U \in \mathfrak{X}(\gamma). \tag{1.10}$$

Proof. By comparing $\mathcal{E}$ and $\mathcal{E}_f$ it follows:

$$\frac{k'(t)}{k(t)(k(t)-1)}(\nabla_{\gamma'}\gamma')(t) + (2k(t)-1)\gamma'(t) = 0. \tag{1.11}$$

Due to the unit speed parametrization of γ the vector fields $\nabla_{\gamma'}\gamma'$ and γ' are orthogonal and then $2k - 1 = 0$. □

Remarks 1.3. 1) Let (Γ^k_{ij}) denote the Christoffel symbols of the metric g in a local chart of M in which $\gamma(t) = (x^i(t))$, $1 \le i \le 2$. Then the differential equation (1.10) becomes a scalar third-order one for a fixed $k \in \{1, 2\}$:

$$\begin{aligned} &\frac{d}{dt}\left[\ddot{x}^k(t) + \Gamma^k_{ij}(\gamma(t))\dot{x}^i(t)\dot{x}^j(t)\right] \\ &+ \Gamma^k_{ij}(\gamma(t))\dot{x}^i(t)\left[\ddot{x}^j(t) + \Gamma^j_{ab}(\gamma(t))\dot{x}^a(t)\dot{x}^b(t)\right] + \frac{\dot{x}^k(t)}{4} = 0. \end{aligned} \tag{1.12}$$

2) In the same paper [3] the flow-frame is generalized with an arbitrary (smooth) angle function $\Omega = \Omega(t)$ obtaining the Ω-*curvature*:

$$k^\Omega(t) := k(t) - \frac{\Omega'(t)}{\|\gamma'(t)\|}. \tag{1.13}$$

Hence, an arc-length parametrized curve with $k^\Omega \neq 0$ will be called Ω-*flow-selfdual* if the differential equation (1.9) holds with k^f replaced by Ω. The characterization of the proposition 1.2 reads now:

$$k(t) = \frac{\Omega'(t)}{2} = -k^\Omega(t). \tag{1.14}$$

A second suitable generalization of our notion works at the level of vector fields $\xi \in \mathfrak{X}(M)$=the Lie algebra of vector fields on M. Fix a unit ξ; then we call

ξ as being *a tangential vector field* if there exists a strictly positive $k \in C^{\infty}(M)$ (which we call *the curvature* of ξ) such that:

$$\nabla_{\xi}\nabla_{\xi}\xi - \xi(\ln k)\nabla_{\xi}\xi + k^2\xi = 0. \tag{1.15}$$

Making the g-product of the left-hand-side term above with ξ gives, as is expected, that:

$$\|\nabla_{\xi}\xi\| = k > 0. \tag{1.16}$$

We remark that is not necessary to work in the initial dimension two. An example of tangential vector field is provided within the theory of *torse-forming vector fields*. Recall, after [2], that a fixed $V \in \mathfrak{X}(M)$ is called torse-forming if for all $X \in \mathfrak{X}(M)$ we have:

$$\nabla_X V = fX + \omega(X)V, \tag{1.17}$$

for a smooth function $f \in C^{\infty}(M)$ and a 1-form $\omega \in \Omega^1(M)$. Now, suppose that $\nabla_{\xi}\xi$ is a torse-forming vector field with the data:

$$f = -k^2, \quad \omega = d(\ln k), \tag{1.18}$$

for a given strictly positive function k. It follows:

$$\nabla_X\nabla_{\xi}\xi - X(\ln k)\nabla_{\xi}\xi + k^2X = 0 \tag{1.19}$$

and then for $X = \xi$ it results the definition (1.15).

Recall also that an important tool in dynamics on curves is the Fermi-Walker derivative, which is the map ([4]) $\nabla_{\gamma}^{FW} : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$:

$$\begin{aligned} \nabla_{\gamma}^{FW}(X) := \frac{d}{dt}X + \|r'(\cdot)\|k[\langle X, N\rangle T \\ - \langle X, T\rangle N] \to \nabla_{\gamma}^{FW}(T) = \nabla_{\gamma}^{FW}(N) = 0. \end{aligned} \tag{1.20}$$

Hence, we generalize this derivative as follows: the Fermi-Walker derivative generated by a tangential vector field ξ is the map $\nabla^{\xi} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by:

$$\nabla^{\xi}(X) := \nabla_{\xi}X + g(X, \nabla_{\xi}\xi)\xi - g(X, \xi)\nabla_{\xi}\xi. \tag{1.21}$$

From the equation (1.14) it results that ξ and $\nabla_{\xi}\xi$ are eigenvector fields of ∇^{ξ}:

$$\nabla^{\xi}(\xi) = 0, \quad \nabla^{\xi}(\nabla_{\xi}\xi) = \xi(\ln k)\nabla_{\xi}\xi. \tag{1.22}$$

2. Examples of flow-selfdual curves

Example 2.1. As is expected the plane Euclidean geometry $\mathbb{E}^2 := (\mathbb{R}^2, \langle \cdot, \cdot \rangle_{can})$ is the simplest case. The circle $\mathcal{C}(O, R = 2)$ is the "generic" Euclidean flow-orthogonal curve; it has the arc-length parametrization and Frenet data:

$$\gamma(t) = 2\left(\cos\frac{t}{2}, \sin\frac{t}{2}\right), \gamma'(t) = \left(-\sin\frac{t}{2}, \cos\frac{t}{2}\right),$$

$$N(t) = -\exp\left(i\frac{t}{2}\right) = -\frac{1}{2}\gamma(t) \tag{2.1}$$

and then the flow-frame is:

$$E_1^f(t) = \left(\sin\frac{t}{2}, \cos\frac{t}{2}\right), \quad E_2^f(t) = \left(-\cos\frac{t}{2}, \sin\frac{t}{2}\right) = \overline{N(t)}, \tag{2.2}$$

where we use the complex conjugate; the ordinary differential equation (1.10) is: $u'' + \frac{1}{4}u = 0$. An associated interesting problem is if there exists a Riemannian metric on (an open subset of) $\mathbb{R}^2$ having as geodesics the Euclidean circles; for the case of Finslerian metric this problem is already solved in [1].

Example 2.2. Fix (M, g) a rotationally symmetric surface i.e., conform [6], M is the product $\mathbb{S}^1 \times I$ with $\mathbb{S}^1$ the Euclidean unit circle and $I \subseteq \mathbb{R}$, endowed with the warped product metric:

$$g = dr^2 + f(r)^2 d\varphi^2, \quad r \in I, \quad \varphi \in \mathbb{S}^1. \tag{2.3}$$

This surface is oriented by the 2-form $dr \wedge d\varphi$ and then:

$$J\left(\frac{\partial}{\partial r}\right) = \frac{1}{f(r)}\frac{\partial}{\partial \varphi}, \quad J\left(\frac{\partial}{\partial \varphi}\right) = -f(r)\frac{\partial}{\partial r}. \tag{2.4}$$

Fix now the curve $\gamma(t) := (r(t), \varphi(t))$ parameterized by the arc-length t. Let $\sigma = \sigma(t)$ be the *structural angle of* γ i.e. the oriented angle between $\frac{\partial}{\partial r}$ and T. It follows the Frenet frame:

(2.5)

$$T(t) = \cos\sigma(t)\frac{\partial}{\partial r}|_t + \frac{\sin\sigma(t)}{f(r(t))}\frac{\partial}{\partial \varphi}|_t, \quad N(t) = -\sin\sigma(t)\frac{\partial}{\partial r}|_t + \frac{\cos\sigma(t)}{f(r(t))}\frac{\partial}{\partial \varphi}|_t.$$

The first derivative of T is then:

$$(\nabla_{\gamma'}\gamma')(t) = \left(\sigma'(t) + \frac{f'(r)}{f(r)}(t)\sin\sigma(t)\right)N(t) \tag{2.6}$$

which provides the expression of the geodesic curvature for γ:

$$k(t) = \sigma'(t) + \frac{f'(r)}{f(r)}(t)\sin\sigma(t). \tag{2.7}$$

The Proposition 1.1 of the cited paper [6] (or [7, p. 89]) offers a conservation law along γ, which for our constant $k = \frac{1}{2}$ reads as follows:

Proposition 2.3. *The smooth function:*

$$t \in [0, L(\gamma)) \to \mathcal{F}(t) := f(r(t)) \sin \sigma(t) - \frac{1}{2} \int_{r(0)}^{r(t)} f(\xi) d\xi \tag{2.8}$$

is constant along a given flow-selfdual curve γ.

The Euclidean plane geometry means $f(r) = r$ and the circle $\mathcal{C}(O, R > 0)$ gives $r = constant = R$, $\varphi(t) = \frac{t}{R}, t \in [0, 2\pi R]$ and $\sigma = constant = \frac{\pi}{2}$.

Example 2.4. For the hyperbolic plane geometry we use the Poincaré model of [5, p. 103]: $\mathbb{H}^2 := (\mathbb{R}^2_{y>0}; g = \frac{1}{y^2}(dx^2 + dy^2))$. Fix a curve $\gamma : t \in [0, +\infty) \to (x(t), y(t)) \in \mathbb{H}^2$ parametrized by arc-length. With the computation of the geodesic curvature from the cited book it follows that γ is a flow-selfdual curve if and only if the following differential system is satisfied:

$$[x'(t)]^2 + [y'(t)]^2 = [y(t)]^2, \quad x'(t) \left(\frac{1}{y(t)} + \frac{y''(t)}{y(t)^2} \right) - x''(t) \frac{y'(t)}{y(t)^2} = \frac{1}{2}. \tag{2.9}$$

A straightforward computation gives a single second order differential equation, which is written in a more simple form as:

$$\ddot{y} - \frac{2}{y} \dot{y} + y = \frac{\sqrt{y^2 - \dot{y}^2}}{2} \tag{2.10}$$

Unfortunately, being a nonlinear differential equation we cannot solve explicitly. In fact, we know that the types of hyperbolic curves with constant geodesic curvature k are as follows: a) circles, for $k > 1$; b) horocycles, with $k = 1$; c) equidistant curves (i.e. curves of finite distance from a hyperbolic geodesic), for $k \in (0, 1)$. Hence, the hyperbolic flow-selfdual curves are a family of equidistant curves.

Example 2.5. Let γ be a arc-length parametrized curve in the unit sphere $\mathbb{S}^2 := (S^2 \subset \mathbb{R}^3, g = (\langle \cdot, \cdot \rangle_{can})|_{S^2})$. Its usual Frenet curvature and torsion as space curve are $k^F > 0$ and τ^F. In fact, from the relationship:

$$k^F = \sqrt{k^2 + 1} \geq 1 \tag{2.11}$$

it follows that a flow-selfdual curve on $\mathbb{S}^2$ have a constant Frenet curvature $k^F = \frac{\sqrt{5}}{2}$. As concrete example we have the horizontal circle:

$$\gamma(t) = \frac{2}{\sqrt{5}} \left(\cos \left(\frac{\sqrt{5}}{2} t \right), \sin \left(\frac{\sqrt{5}}{2} t \right), \frac{1}{2} \right), t \in \mathbb{R}, \quad \tau^F \equiv 0. \tag{2.12}$$

More generally, recall that for the given arc-length parametrized curve γ on the regular surface $S \subset \mathbb{R}^3$ its geodesic curvature satisfies:

$$k = k^F \sin\theta \tag{2.13}$$

with θ the oriented angle between the normal N_γ of the curve and the normal N_S of S. For a flow-selfdual curve on $S = S^2$ it results the angle θ provided by:

$$\sin\theta = \frac{1}{\sqrt{5}}, \quad \cos\theta = \pm\frac{2}{\sqrt{5}}. \tag{2.14}$$

3. Conclusions

This note concerns with a particular class of curves in an orientable geometric surface (M^2, g). The curves in this class have a constant geodesic curvature, and hence they are remarkable objects for the differential geometry of the given pair (M^2, g). Four examples illustrate the significance of these curves in some important geometries. From a dynamical point of view we generalise the Fermi-Walker derivative and we hope this operator to become more suitable in some future works.

References

[1] M. Crampin, T. Mestdag, *A class of Finsler surfaces whose geodesics are circles*, Publ. Math. Debr., 84 (2014), 3-16.

[2] M. Crasmareanu, *Scalar curvature for middle planes in odd-dimensional torse-forming almost Ricci solitons*, Kragujevac J. Math., 43 (2019), 275-279.

[3] M. Crasmareanu, *The flow-curvature of curves in a geometric surface*, Commun. Korean Math. Soc., 38 (2023), 1261-1269.

[4] M. Crasmareanu, C. Frigioiu, *Unitary vector fields are Fermi-Walker transported along Rytov-Legendre curves*, Int. J. Geom. Methods Mod. Phys., 12 (2015), Article ID 1550111.

[5] G. R. Jensen, E. Musso, L. Nicolodi, *Surfaces in classical geometries. A treatment by moving frames*, Universitext, Springer, 2016.

[6] M. Ritoré, *Constant geodesic curvature curves and isoperimetric domains in rotationally symmetric surfaces*, Commun. Anal. Geom., 9 (2001), 1093-1138.

[7] M. Ritoré, *Isoperimetric inequalities in Riemannian manifolds*, Progress in Mathematics, 348, Cham: Birkhäuser, 2023.

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