Fantastic (weak) hyper filters in hyper BE-algebras

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Abstract. In this paper, fantastic (weak) hyper filters in hyper BE-algebras are introduced and investigated. The relationships between fantastic (weak) hyper filters and (weak) hyper filters are discussed and the related examples are delivered. Then, fantastic (weak) hyper filters are characterized respectively. Moreover, examples are given in which fantastic weak hyper filters and fantastic hyper filters may not be deduced from each other in hyper BE-algebras, meanwhile the conditions are found that fantastic weak hyper filters become fantastic weak hyper filters in hyper BE-algebras.

Keywords: hyper BE-algebra, $\circ$-reflexive subset, (weak) hyper filter, fantastic (weak) hyper filter.

1. Introduction

The hyper algebraic theory was introduced by Marty [15] at the 8th Congress of Scandinavian Mathematicians. Since then, hyper algebraic structure has been intensively researched such as hyper BCK-algebras [12, 13], hyper K-algebras [11, 18], hyper residuated lattices [2, 17], hyper EQ-algebras [4, 8] and hyper equality algebras [3, 7], etc. At present, hyper algebraic theory has been widely applied to many disciplines [9, 10]. Borzooei et al. investigated the filter theory of residuated lattices and hyper equality algebras in [2] and [3] respectively.

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Then, Borzooei and Aaly in [1] systematically summarized various of hyper algebraic structures and presented the relationships among these hyper algebraic structures. Radfar et al. [16] in 2014 introduced the notion of hyper BE-algebras as a generalization of BE-algebras [14]. Moreover, they proposed some special types of hyper BE-algebras and (weak) hyper filters in hyper BE-algebras. In fact, hyper BE-algebras are closely related to many hyper algebras and it is a generalization of dual hyper BCK-algebras, dual hyper K-algebras and hyper hoops [5]. Cheng and Xin in [6] focused on investigating (positive) implicative hyper filters in hyper BE-algebras and induced quotient hyper BE-algebras by use of implicative hyper filters. Based on the above, the present paper considers fantastic (weak) hyper filters in hyper BE-algebras so as to further explore the structure of hyper BE-algebras.

2. Preliminaries

In this section, we recollect some definitions and results about hyper BE-algebras which will be used in the following.

Definition 2.1 ([16]). *Let H be a nonempty set and $\circ : H \times H \to P^*(H)$ be a hyperoperation. Then, $(H, \circ, 1)$ is called a hyper BE-algebra provided it satisfies the following axioms:*

$(HBE1)$ $x \ll 1$ *and* $x \ll x$*;*

$(HBE2)$ $x \circ (y \circ z) = y \circ (x \circ z)$*;*

$(HBE3)$ $x \in 1 \circ x$*;*

$(HBE4)$ $1 \ll x$ *implies* $x = 1$*, for all* $x, y \in H$*, where the relation* $\ll$ *is defined by* $x \ll y \Leftrightarrow 1 \in x \circ y$*. For any two nonempty subsets* A *and* B *of* H*,* $A \ll B$ *means that there exist* $a \in A, b \in B$ *such that* $a \ll b$*.*

Notice that, in any hyper BE-algebra, $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$ and $A \leq B$ means for any $a \in A$, there exists $b \in B$ such that $a \ll b$.

In the following sequel, by H denote a hyper BE-algebra $(H, \circ, 1)$, unless otherwise specified.

Proposition 2.1 ([6, 16]). *In any hyper BE-algebra H, the following hold:*

(1) $A \circ (B \circ C) = B \circ (A \circ C)$*;*

(2) $A \subseteq 1 \circ A, 1 \in A \circ 1, 1 \in A \circ A$*;*

(3) $x \leq y \circ x, A \leq B \circ A$*;*

(4) $A \ll B$ *iff* $1 \in A \circ B$*;*

(5) $1 \in A$ *and* $A \leq B$ *imply* $1 \in B$*;*

(6) $1 \ll A$ *implies* $1 \in A$, *for all* $x, y \in H, A, B \subseteq H$.

Definition 2.2 ([16]). *We say that a hyper BE-algebra H is a*

(1) *C-hyper BE-algebra, if* $x \circ 1 = \{1\}$ *for all* $x \in H$;

(2) *R-hyper BE-algebra, if* $1 \circ x = \{x\}$ *for all* $x \in H$;

(3) *D-hyper BE-algebra, if* $x \circ x = \{1\}$ *for all* $x \in H$;

(4) *RD-hyper BE-algebra, if H is both a R-hyper BE-algebra and a D-hyper BE-algebra;*

(5) *RC-hyper BE-algebra, if H is both a R-hyper BE-algebra and a C-hyper BE-algebra.*

Definition 2.3 ([16]). *A nonempty subset F containing 1 of H is said to be a*

(1) *hyper filter if* $x \circ y \cap F \neq \emptyset$ *and* $x \in F$ *imply* $y \in F$, *for any* $x, y \in H$;

(2) *weak hyper filter if* $x \circ y \subseteq F$ *and* $x \in F$ *imply* $y \in F$, *for any* $x, y \in H$.

It is well known that every hyper filter is a weak hyper filter in a hyper BE-algebra, but the converse is not true. Moreover, every hyper filter satisfies the condition (F):

(F) $x \in F$ and $x \ll y$ imply $y \in F$ for all $x, y \in H$.

3. Fantastic (weak) hyper filters

In this section, we introduce fantastic (weak) hyper filters in hyper BE-algebras and deliver some related results of them.

Definition 3.1. *A nonempty subset F containing 1 of H is said to be a*

(1) *fantastic hyper filter, if* $z \circ (x \circ y) \cap F \neq \emptyset$ *and* $z \in F$ *imply* $((y \circ x) \circ x) \circ y \cap F \neq \emptyset$, *for any* $x, y, z \in H$;

(2) *fantastic weak hyper filter, if* $z \circ (x \circ y) \subseteq F$ *and* $z \in F$ *imply* $((y \circ x) \circ x) \circ y \subseteq F$, *for any* $x, y, z \in H$.

Example 3.2. Let $H = \{a, b, 1\}$. Define the operation $\circ$ on H as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a, b\}$	$\{b\}$
a	$\{1, b\}$	$\{1\}$	$\{1\}$
b	$\{1, b\}$	$\{1\}$	$\{1\}$

Then, $(H, \circ, 1)$ is a hyper BE-algebras [16]. It is easy to verify that $F = \{1\}$ is a fantastic weak hyper filter and $G = \{1, a\}$ is a fantastic hyper filter of H.

Proposition 3.1. *Let H be a RC-hyper BE-algebra. If F is a fantastic (weak) hyper filter of H, then F is a (weak) hyper filter of H.*

Proof. (1) Let $x \circ y \cap F \neq \emptyset$ and $x \in F$, for any $x, y \in F$. Then, by $x \circ y \subseteq x \circ (1 \circ y)$ we have $x \circ (1 \circ y) \cap F \neq \emptyset$. Again since $x \in F$ and F is a fantastic hyper filter of H, we can obtain that $\{y\} \cap F = 1 \circ y \cap F = ((y \circ 1) \circ 1) \circ y \cap F \neq \emptyset$ and thus $y \in F$. Therefore, F is a hyper filter of H.

(2) Let $x \circ y \subseteq F$ and $x \in F$, for any $x, y \in F$. Since H is a R-hyper BE-algebra, we have $x \circ (1 \circ y) = x \circ y \subseteq F$. Again since $x \in F$ and F is a fantastic weak hyper filter of H, then $\{y\} = 1 \circ y = ((y \circ 1) \circ 1) \circ y \subseteq F$ and thus $y \in F$. Therefore, F is a weak hyper filter of H. □

Example 3.3. Let $H = \{a, b, 1\}$. Define the operation $\circ$ on H as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1\}$	$\{1, a, b\}$	$\{b\}$
b	$\{1\}$	$\{a, b\}$	$\{1, b\}$

Then, $(H, \circ, 1)$ is a RC-hyper BE-algebras [16]. One can calculate that $F = \{1, a\}$ is both a (weak) hyper filter and a fantastic (weak) hyper filter of H.

Notice that the condition of the RC-hyper from Proposition 3.1 is not necessary in general. In fact, in Example 3.2 H is not a RC-hyper BE-algebra, but it is easy to see that $F = \{1\}$ is both a (weak) hyper filter and a fantastic (weak) hyper filter of H.

The converse of Proposition 3.1 may not be true and see the following example.

Example 3.4. (1) Let $H = \{a, b, 1\}$. Define the operation $\circ$ on H as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1\}$	$\{1\}$	$\{1, a\}$
b	$\{1\}$	$\{1\}$	$\{1, a\}$

Then, $(H, \circ, 1)$ is a RC-hyper BE-algebras [16]. It is not difficult to check that $F = \{1\}$ is a weak hyper filter of H, but it is not a fantastic weak hyper filter of H since $1 \in F$ and $1 \circ (b \circ a) \subseteq F$ while $((a \circ b) \circ b) \circ a = \{1, a\} \nsubseteq F$.
(2) Let $H = \{1, a, b, c\}$. Define the operation $\circ$ on H as follows:

$\circ$	1	a	b	c
1	$\{1\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
b	$\{1\}$	$\{a\}$	$\{1, b\}$	$\{c\}$
c	$\{1\}$	$\{a\}$	$\{1, b\}$	$\{1, b\}$

Then, $(H, \circ, 1)$ is a RC-hyper BE-algebra [11]. It is routine to verify that $F = \{1, a\}$ is a hyper filter of H, but it is not a fantastic hyper filter of H since $1 \in F$ and $1 \circ (a \circ b) \cap F \neq \emptyset$ while $((b \circ a) \circ a) \circ b = \{b\} \cap F = \emptyset$.

In what follows we deliver a characterization of the fantastic (weak) hyper filter of H, respectively.

Theorem 3.1. *Let F be a hyper filter of H. Then, the following are equivalent:*

(1) *F is a fantastic hyper filter of H;*

(2) *$x \circ y \cap F \neq \emptyset$ implies $((y \circ x) \circ x) \circ y \cap F \neq \emptyset$, for any $x, y \in H$.*

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds and $x \circ y \cap F \neq \emptyset$, for any $x, y \in H$. Since $x \circ y \subseteq 1 \circ (x \circ y)$ then $1 \circ (x \circ y) \cap F \neq \emptyset$. Since $1 \in F$ and F is a fantastic hyper filter of H, we have $((y \circ x) \circ x) \circ y \cap F \neq \emptyset$.

(2) $\Rightarrow$ (1) Assume that (2) holds. Let $z \circ (x \circ y) \cap F \neq \emptyset$ and $z \in F$, for any $x, y, z \in H$. Since F is a hyper filter of H, then $x \circ y \cap F \neq \emptyset$ and so by hypothesis we can obtain $((y \circ x) \circ x) \circ y \cap F \neq \emptyset$. It concludes that F is a fantastic hyper filter of H. □

Theorem 3.2. *Let F be a weak hyper filter of a R-hyper BE-algebra H. Then, the following are equivalent:*

(1) *F is a fantastic weak hyper filter of H;*

(2) *$x \circ y \subseteq F$ implies $((y \circ x) \circ x) \circ y \subseteq F$, for any $x, y \in H$.*

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds and $x \circ y \subseteq F$, for any $x, y \in H$. Since $1 \in F, 1 \circ (x \circ y) = x \circ y \subseteq F$ and F is a fantastic weak hyper filter of H, we have $((y \circ x) \circ x) \circ y \subseteq F$.

(2) $\Rightarrow$ (1) Assume that (2) holds. Let $z \circ (x \circ y) \subseteq F$ and $z \in F$, for any $x, y, z \in H$. Since F is a weak hyper filter of H, then $x \circ y \subseteq F$ and so by hypothesis we can obtain $((y \circ x) \circ x) \circ y \subseteq F$. It concludes that F is a fantastic weak hyper filter of H. □

In general, a fantastic hyper filter of H may not be a fantastic weak hyper filter and vice versa.

Example 3.5. (1) In Example 3.2 one can check that the set $M = \{1, b\}$ is a fantastic hyper filter of H, but it is not a fantastic weak hyper filter since $b \in M$ and $b \circ (1 \circ a) = \{1\} \subseteq M$ while $((a \circ 1) \circ 1) \circ a = \{1, a, b\} \nsubseteq M$.
(2) Let $H = \{a, b, 1\}$. Define the operation $\circ$ on H as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a, b\}$	$\{b\}$
a	$\{1\}$	$\{1, a\}$	$\{1, b\}$
b	$\{1\}$	$\{1, a, b\}$	$\{1\}$

Then, $(H, \circ, 1)$ is a hyper BE-algebras [16]. It can be calculated that $F = \{1, a\}$ is a fantastic weak hyper filter of H, but it is not a fantastic hyper filter since $a \in F$ and $a \circ (1 \circ b) = \{1, b\} \cap F \neq \emptyset$ while $((b \circ 1) \circ 1) \circ b = \{b\} \cap F = \emptyset$.

In what follows, we provide the conditions that fantastic weak hyper filters become fantastic hyper filters in hyper BE-algebras.

Definition 3.6 ([6])**.** *A nonempty subset S of H is said to be $\circ$-reflexive if $x \circ y \cap S \neq \emptyset$ implies $x \circ y \subseteq S$ for all $x, y \in H$.*

Proposition 3.2. *Let H be a RC-hyper BE-algebra. If F is a $\circ$-reflexive fantastic weak hyper filters of H, then it is a fantastic hyper filter of H.*

Proof. As F is a $\circ$-reflexive weak hyper filters of H, we have that F is a hyper filter of H. Now, set $x \circ y \cap F \neq \emptyset$, for any $x, y \in H$. It follows from the $\circ$-reflexivity of F that $x \circ y \subseteq F$. Since F is a fantastic weak hyper filter of H, then by Theorem 3.2 we obtain $((y \circ x) \circ x) \circ y \subseteq F$ and so $((y \circ x) \circ x) \circ y \cap F \neq \emptyset$. Therefore, by Theorem 3.1 F is a fantastic hyper filter of H. $\square$

Definition 3.7. *A hyper BE-algebra H is called right-ordered, if $x \ll y$ implies $y \circ z \ll x \circ z$ for all $x, y, z \in H$.*

Example 3.8. It is easy to verify that the hyper BE-algebra H from Example 3.5 (2) is right-ordered.

Theorem 3.3. *Let H be a right-ordered RD-hyper BE-algebra, and F, G be $\circ$-reflexive weak hyper filters of H. If $F \subseteq G$ and F is a fantastic weak hyper filter of H, then G is a fantastic hyper filter of H.*

Proof. Let $x \circ y \cap G \neq \emptyset$, for any $x, y \in H$. Denote $m = x \circ y$, since G is $\circ$-reflexive then $m \subseteq G$ Again since H is a D-hyper BE-algebra, we have $x \circ (m \circ y) = m \circ (x \circ y) = \{1\} \subseteq F$. Notice that H is a R-hyper BE-algebra and F is a fantastic weak hyper filter, it follows from Theorem 3.2 that $m \circ ((((m \circ y) \circ x) \circ x) \circ y) = (((m \circ y) \circ x) \circ x) \circ (m \circ y) \subseteq F$ and hence $m \circ ((((m \circ y) \circ x) \circ x) \circ y) \subseteq G$. Combing that $m \in G$ and G is a weak hyper filter, we can obtain $(((m \circ y) \circ x) \circ x) \circ y \subseteq G$. Again since $y \ll m \circ y$ and H is right-ordered, we get that $(((m \circ y) \circ x) \circ x) \circ y \ll ((y \circ x) \circ x) \circ y$. Considering $(((m \circ y) \circ x) \circ x) \circ y \subseteq G$ and the $\circ$-reflexivity of G, it can conclude that $((y \circ x) \circ x) \circ y \cap G \neq \emptyset$. Therefore, using Theorem 3.1 G is a fantastic hyper filter of H. $\square$

4. Conclusions

Filters are an important tool in the research of algebraic structures. In this paper, fantastic (weak) hyper filters are proposed in hyper BE-algebras and also the relation between them is delivered. What is more, the characterizations of fantastic (weak) hyper filters are showed. In the further work, we shall explore some applications of fantastic (weak) hyper filters such as in quotient hyper BE-algebras and in the state theory of hyper BE-algebras.

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