The matrix inverse based on the EP-nilpotent decomposition of a complex matrix

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Abstract. A generalized inverse for matrices is introduced, which is called the MPEPN-inverse. Let A be a complex matrix, the MPEPN-inverse can be described

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by using the part A_1 in the EP-nilpotent decomposition of A and the Moore-Penrose inverse of A. Let $A = A_1 + A_2$ be the EP-nilpotent decomposition of A, $A^{E,\ddagger}$ be the MPEPN-inverse of A and $A^\dagger$ be the Moore-Penrose inverse of A, one can show that $A^{E,\ddagger}AA^{E,\ddagger} = A^{E,\ddagger}$ does not hold in general, moreover, necessary and sufficient conditions to make the MPEPN-inverse to be an outer inverse of A are given, that is $A^{E,\ddagger}AA^{E,\ddagger} = A^{E,\ddagger}$ hold if and only if one of the conditions $(A_1A^\dagger)^2 = A_1A^\dagger$ and $P_{\mathcal{R}(A_2)}A^{\textcircled{\dagger}} = 0$ holds, where $A^{\textcircled{\dagger}}$ is the Core-EP inverse of A and $P_{\mathcal{R}(A_2)}$ is the projection on $\mathcal{R}(A_2)$. If $A_1A^\dagger$ is an idempotent, then the MPEPN-inverse of A coincides with the $(A^\dagger A_1 P_{\mathcal{R}(A^*)}, P_{\mathcal{R}(A)}A_1A^\dagger)$-inverse of A , i.e. coincides the inverse along $A^\dagger A_1 P_{\mathcal{R}(A^*)}$ and $P_{\mathcal{R}(A)}A_1A^\dagger$.
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1. Introduction

Let $\mathbb{C}$ be the complex field. The set $\mathbb{C}^{m\times n}$ denotes the set of all $m \times n$ complex matrices over the complex field $\mathbb{C}$. Let $A \in \mathbb{C}^{m\times n}$. The symbol A^* denotes the conjugate transpose of A. Notations $\mathcal{R}(A) = \{y \in \mathbb{C}^m : y = Ax, x \in \mathbb{C}^n\}$ and $\mathcal{N}(A) = \{x \in \mathbb{C}^n : Ax = 0, x \in \mathbb{C}^n\}$ will be used in the sequel. An integer k is called the index of $A \in \mathbb{C}^{n\times n}$ if k is the smallest positive integer such that $\text{rank}\,(A^k) = \text{rank}(A^{k+1})$ holds and is denoted by $\text{ind}(A)$.

Let $A \in \mathbb{C}^{m\times n}$. A matrix $X = A^\dagger \in \mathbb{C}^{n\times m}$ is called the Moore-Penrose inverse of A [8, 12] if $AXA = A,\ XAX = X,\ (AX)^* = AX$ and $(XA)^* = XA$ hold. Let $A, X \in \mathbb{C}^{n\times n}$ with $\text{ind}\,(A) = k$. Then algebraic definition of the Drazin inverse as follows: if

$$AXA = A,\ XA^{k+1} = A^k \text{ and } AX = XA,$$

then X is called a Drazin inverse of A. If such X exists, then it is unique and denoted by A^D [4]. More generalized inverses can be seen as follows:core inverse [2] by using $\Sigma - K - L$ decomposition [7], core-EP inverse [9] and DMP inverse [11].

Let $A, B, C \in \mathbb{C}^{n\times n}$. The (B, C)-inverse of A is unique (see [1, 5, 13]). Several kinds of generalized inverses are all special cases of the (B, C)-inverse of the matrix A: Moore-Penrose inverse [8, 12], Drazin inverse [4], core inverse [2], DMP-inverse [11] and core-EP inverse [9].

For a complex matrix with a given index, there are three important matrix decompositions: core-nilpotent decomposition [10], Core-EP decomposition [14] and EP-nilpotent decomposition [15]. The CMP inverse can be introduced by the core-nilpotent decomposition and the MPCEP-inverse can be introduced by the Core-EP decomposition. Motivated by the idea of the CMP inverse and the MPCEP-inverse of a complex matrix, in this paper, the MPEPN-inverse was introduced. Specifically, the CMP inverse of $A \in \mathbb{C}^{n\times n}$ was introduced by Mehdipour and Salemi in [10], this inverse using the core part in core-nilpotent decomposition of A and the Moore-Penrose inverse of A. The MPCEP-inverse can be described by using the core part in Core-EP decomposition of A and the Moore-Penrose inverse of A [3]. Motivated by the above method, we have a natural question as follows: Using the core part A_1 in EP-nilpotent decomposition of A and the Moore-Penrose inverse of A to introduce a matrix $X = A^\dagger A_1 A^\dagger$. Thus, the MPEPN-inverse can be described by using the core part in EP-nilpotent decomposition of A and the Moore-Penrose inverse of A [15].

2. Existence criteria and expressions of the MPEPN-inverse

The EP-nilpotent decomposition of A was introduced by Wang and Liu in [15]. That is A can be written as $A = A_1 + A_2$, where k is the index of A, A_1 is an EP matrix (i.e. $A_1 A_1^\dagger = A_1^\dagger A_1$), $A_2^{k+1} = 0$ and $A_2 A_1 = 0$. The following lemma holds by [15, Theorem 2.2].

Lemma 2.1 ([15, Theorem 2.1]). *Let $A \in \mathbb{C}^{n\times n}$ and $A = A_1 + A_2$ be the EP-nilpotent decomposition of A. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$A_1 = U \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} U^* \text{ and } A_2 = U \begin{bmatrix} 0 & S \\ 0 & N \end{bmatrix} U^*, \tag{1}$$

where $\operatorname{ind}(A) = k$*, T is nonsingular, S and N are matrices with some suitable sizes.*

The Core-EP decomposition in the following lemma is useful in the study of the Core-EP inverse. Ferreyra et al.[6] given the explicit expressions of the Moore-Penrose inverse by using the Core-EP decomposition, which can be seen in Lemma 2.3.

Lemma 2.2 ([14, Theorem 2.1]). *Let $A \in \mathbb{C}^{n\times n}$ and $A = A_1' + A_2'$ be the Core-EP decomposition of A. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$A_1' = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* \text{ and } A_2' = U \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} U^*, \tag{2}$$

where $\operatorname{ind}(A) = k$*, T is nonsingular, S and N are matrices with some suitable sizes.*

Lemma 2.3 ([6, Theorem 3.9]). *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$*. If A has the Core-EP decomposition of A as* (2.2) *in Lemma* 2.2*, then*

$$A^\dagger = U \begin{bmatrix} T^*\Delta & -T^*\Delta S N^\dagger \\ (E_{n-t} - N^\dagger N) S^\dagger \Delta & N^\dagger - (E_{n-t} - N^\dagger N) S^* \Delta S N^\dagger \end{bmatrix} U^*,$$

where $t = \operatorname{rank}(A^k)$, $\Delta = [TT^* + S(E_{n-t} - N^\dagger N)S^*]^{-1}$ and E_{n-t} is the identity of size $n - t$.

Lemma 2.4 ([8])**.** *Let* $A \in \mathbb{C}^{n\times n}$*.Then*

(1) $A^*B = A^*C$ *if and only if* $A^\dagger B = A^\dagger C$ *for any* $B, C \in \mathbb{C}^{n\times n}$*;*

(2) $BA^* = CA^*$ *if and only if* $BA^\dagger = CA^\dagger$ *for any* $B, C \in \mathbb{C}^{n\times n}$*.*

The core part of the EP-nilpotent decomposition can be expressed by the Moore-Penrose inverse of A^k, where $ind(A) = k$. The core part of the EP-nilpotent decomposition is useful in our paper.

Lemma 2.5 ([15, Theorem 2.2])**.** *Let* $A \in \mathbb{C}^{n\times n}$ *with the index of* A *is* k *and* $A = A_1 + A_2$ *be the EP-nilpotent decomposition of* A *as* (2.1)*. Then* $A_1 = AA^k(A^k)^\dagger$*.*

Lemma 2.6 ([5, Theorem 2.1 and Proposition 6.1])**.** *Let* $A \in \mathbb{C}^{n\times n}$*. Then* $Y \in \mathbb{C}^{n\times n}$ *is a (B,C)-inverse of* A *if and only if* $YAY = Y$*,* $\mathcal{R}(Y) = \mathcal{R}(B)$ *and* $\mathcal{N}(X) = \mathcal{N}(C)$*.*

Motivated by the definition of the CMP inverse in [10], in the following definition we will introduced the MPEPN-inverse of a complex matrix by using the Moore-Penrose inverse of such matrix and the core part of the EP-nilpotent decomposition of this matrix, then one can prove that this inverse is unique.

Definition 2.1. *Let* $A \in \mathbb{C}^{n\times n}$ *with the index of* A *is* k *and* $A = A_1 + A_2$ *be the EP-nilpotent decomposition of* A *as* (1)*. Then* $X = A^\dagger A_1 A^\dagger$ *is called the MPEPN-inverse of* A*.*

Example 2.1. The MPEPN-inverse $A^\dagger A_1 A^\dagger$ is different to $A^\dagger A^D A^\dagger$. Since by Lemma *2.5*, we have $A_1 = AA^k(A^k)^\dagger$ and by [5], we have $A^D = A^k(A^{2k+1})^\dagger A^k$, thus $A^\dagger A_1 A^\dagger = A^\dagger A^{k+1}(A^k)^\dagger A^\dagger$ and $A^\dagger A^D A^\dagger = A^\dagger A^k(A^{2k+1})^\dagger A^k A^\dagger$. Let $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{C}^{4\times 4}$, one check that $A^\dagger A_1 A^\dagger = \begin{bmatrix} \frac{5}{9} & -\frac{4}{9} & \frac{1}{9} & 0 \\ -\frac{4}{9} & \frac{5}{9} & \frac{1}{9} & 0 \\ \frac{1}{9} & \frac{1}{9} & \frac{2}{9} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $A^\dagger A^D A^\dagger = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k. The equality $AA^k(A^k)^\dagger = A^k(A^k)^\dagger A$ does not hold in general, a counterexample will be given in the following example.

Example 2.2. Let $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{C}^{4\times 4}$. Then it is easy to check that the index of A is $k = 2$, but $AA^k(A^k)^\dagger = AA^2(A^2)^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^k(A^k)^\dagger A = A^2(A^2)^\dagger A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, that is, $AA^k(A^k)^\dagger \neq A^k(A^k)^\dagger A$. Moreover, we have $A^D = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^\dagger = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and $A^{E,\ddagger} = \begin{bmatrix} \frac{5}{9} & -\frac{4}{9} & \frac{1}{9} & 0 \\ -\frac{4}{9} & \frac{5}{9} & \frac{1}{9} & 0 \\ \frac{1}{9} & \frac{1}{9} & \frac{2}{9} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k. The following Example 2.3 shows that the equality $AA^k(A^k)^\dagger = A^k(A^k)^\dagger A$ can hold for some matrices.

Example 2.3. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{5} & \frac{4}{5} \end{bmatrix} \in \mathbb{C}^{4\times 4}$. Then it is easy to check that $\mathrm{ind}(A) = k = 1$ and $AA^k(A^k)^\dagger = AAA^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{5} & \frac{4}{5} \end{bmatrix}$, $A^k(A^k)^\dagger A = AA^\dagger A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{5} & \frac{4}{5} \end{bmatrix}$, that is, $AA^k(A^k)^\dagger = A^k(A^k)^\dagger A$. Moreover, we have $A^{E,\ddagger} = A^D = A^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{5} & \frac{4}{5} \end{bmatrix}$.

Example 2.2 and Example 2.3 show that the equality $AA^k(A^k)^\dagger = A^k(A^k)^\dagger A$ does not hold in general. One sufficient condition such that the equality holds can be seen in the following proposition.

Theorem 2.1. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = k$. *If* $PA^*A^k = 0$, *then* $AA^k(A^k)^\dagger = A^k(A^k)^\dagger A$, *where* $P = E_n - A^k(A^k)^\dagger$ *and* E_n *is the identity of size* n.

Proof. Since $P = E_n - A^k(A^k)^\dagger$, then $PA^*A^\dagger = 0$ is equivalent to $[E_n - A^k(A^k)^\dagger]A^*A^k = 0$, which is equivalent to

$$A^*A^k = A^k(A^k)^\dagger A^*A^k. \tag{3}$$

Taking $*$ on (3) gives $(A^*A^k)^* = [A^k(A^k)^\dagger A^*A^k]^*$, then

$$(A^k)^*A = (A^k)^*A[A^k(A^k)^\dagger]^* = (A^k)^*AA^k(A^k)^\dagger. \tag{4}$$

By (4) and Lemma 2.4, we have

$$(A^k)^\dagger A = (A^k)^\dagger AA^k(A^k)^\dagger. \tag{5}$$

Pre-multiplying by A^k on (5) gives

$$A^k(A^k)^\dagger A = A^k(A^k)^\dagger AA^k(A^k)^\dagger = A^k(A^k)^\dagger A^kA(A^k)^\dagger = A^kA(A^k)^\dagger = AA^k(A^k)^\dagger,$$

that is, $AA^k(A^k)^\dagger = A^k(A^k)^\dagger A$. □

By using the Moore-Penrose inverse of A and the core part in the EP-nilpotent decomposition of A, the formula of the MPEPN-inverse of A was given. Moreover, we can get the formula $A^\dagger A^{k+1}(A^k)^\dagger A^\dagger$ is the MPEPN-inverse of A.

Theorem 2.2. *Let* $A \in \mathbb{C}^{n\times n}$ *with the index of* A *is* k *and* A_1 *be the core part in the EP-nilpotent decomposition of* A, *then* $A^\dagger A^{k+1}(A^k)^\dagger A^\dagger$ *is the MPEPN-inverse of* A.

Proof. Let X be the MPEPN-inverse of A, we have $A_1 = AA^k(A^k)^\dagger$ by Lemma 2.5. By Definition 2.1, we have $X = A^\dagger A_1A^\dagger$. Thus, the conditions $A_1 = AA^k(A^k)^\dagger$ and $X = A^\dagger A_1A^\dagger$ give

$$X = A^\dagger A_1A^\dagger = A^\dagger AA^k(A^k)^\dagger A^\dagger = A^\dagger A^{k+1}(A^k)^\dagger A^\dagger.$$ □

3. When the MPEPN-inverse of complex matrix is an outer inverse of this matrix

Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k and $X \in \mathbb{C}^{n\times n}$ be the MPEPN-inverse of A. In general, the MPEPN-inverse is an outer inverse of A? The answer is no, $X = XAX$ does not hold, a counterexample will be given in the following example.

Example 3.1. Let $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{C}^{4\times 4}$. Then $\operatorname{ind}(A) = 2$, but $A^{E,\ddagger} = \begin{bmatrix} \frac{5}{9} & -\frac{4}{9} & \frac{1}{9} & 0 \\ -\frac{4}{9} & \frac{5}{9} & \frac{1}{9} & 0 \\ \frac{1}{9} & \frac{1}{9} & \frac{2}{9} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^{E,\ddagger}AA^{E,\ddagger} = \begin{bmatrix} \frac{14}{27} & -\frac{13}{27} & \frac{1}{27} & 0 \\ -\frac{13}{27} & \frac{14}{27} & \frac{1}{27} & 0 \\ \frac{1}{27} & \frac{1}{27} & \frac{2}{27} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, that is, $A^{E,\ddagger} \neq A^{E,\ddagger}AA^{E,\ddagger}$. Moreover, $A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^\dagger = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, then

$$A_1A^\dagger = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (A_1A^\dagger)^2 = \begin{bmatrix} \frac{5}{9} & -\frac{4}{9} & \frac{1}{9} & 0 \\ -\frac{4}{9} & \frac{5}{9} & \frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Obviously, $A_1A^\dagger$ is not an idempotent.

The above counterexample shows that $X \neq XAX$, where X is the MPEPN-inverse of A. A natural question is: when $A^{E,\ddagger}$ is an outer inverse of A. One can show that if the condition $(A_1A^\dagger)^2 = A_1A^\dagger$ holds, then the MPEPN-inverse of A is an outer inverse of A.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k and A_1 be the core part in the EP-nilpotent decomposition of A. Then $XAX = X$ if and only if $(A_1A^\dagger)^2 = A_1A^\dagger$, where X is the MPEPN-inverse of A.*

Proof. Let X be the MPEPN-inverse of A, then by Definition 2.1 we have $X = A^\dagger A_1 A^\dagger$. We have the following conditions of equation $XAX = X$.

$$XAX = X \Longleftrightarrow A^\dagger A_1 A^\dagger = A^\dagger A_1 A^\dagger A A^\dagger A_1 A^\dagger = A^\dagger A_1 A^\dagger A_1 A^\dagger,$$

that is,

$$XAX = X \Longleftrightarrow A^\dagger A_1 A^\dagger = A^\dagger A_1 A^\dagger A_1 A^\dagger. \tag{6}$$

By Lemma 2.5, we know $A_1 = AA^k(A^k)^\dagger$, thus (6) gives

$$XAX = X \Longleftrightarrow A^\dagger AA^k(A^k)^\dagger A^\dagger = A^\dagger AA^k(A^k)^\dagger A^\dagger AA^k(A^k)^\dagger A^\dagger. \tag{7}$$

Pre-multiplying by A on the right of (7) implies

$$AA^\dagger AA^k(A^k)^\dagger A^\dagger = AA^\dagger AA^k(A^k)^\dagger A^\dagger AA^k(A^k)^\dagger A^\dagger.$$

Then,

$$AA^k(A^k)^\dagger A^\dagger = AA^k(A^k)^\dagger A^\dagger AA^k(A^k)^\dagger A^\dagger. \tag{8}$$

Thus, we have the equality in (8) is equivalent to $A_1A^\dagger = A_1A^\dagger A_1A^\dagger$ by $A_1 = AA^k(A^k)^\dagger$, that is, $(A_1A^\dagger)^2 = A_1A^\dagger$. □

In the following, we show that the MPEPN-inverse of A is an outer inverse under the condition $S(E_{n-t} - N^\dagger N)S^* = 0$, where E_{n-t} is the identity of size $n-t$ and reciprocally.

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k and $A = A_1 + A_2$ be the EP-nilpotent decomposition of A as (1). Then $XAX = X$ if and only if $S(E_{n-t} - N^\dagger N)S^* = 0$, where $t = rank(A^k)$ and X is the MPEPN-inverse of A.*

Proof. By Lemma 1, we have $A = A_1 + A_2$, where $A_1 = U \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} U^*$ and $A_2 = U \begin{bmatrix} 0 & S \\ 0 & N \end{bmatrix} U^*$, where t is the rank of A^k, the size of T and N are t and $n-t$, respectively. Then by Lemma 1 and Lemma 2.3, we have

$$\begin{aligned} A_1 A^\dagger &= U \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^*\Delta & -T^*\Delta S N^\dagger \\ (E_{n-t} - N^\dagger N)S^\dagger \Delta & N^\dagger - (E_{n-t} - N^\dagger N)S^*\Delta S N^\dagger \end{bmatrix} U^* \\ &= U \begin{bmatrix} TT^*\Delta & -TT^*\Delta S N^\dagger \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

By $(A_1A^\dagger)^2 = A_1A^\dagger$, we have

$$\left(U \begin{bmatrix} TT^*\Delta & -TT^*\Delta S N^\dagger \\ 0 & 0 \end{bmatrix} U^* \right)^2 = U \begin{bmatrix} TT^*\Delta & -TT^*\Delta S N^\dagger \\ 0 & 0 \end{bmatrix} U^*,$$

which is equivalent to

$$\begin{bmatrix} (TT^*\Delta)^2 & -TT^* \triangle TT^*\Delta S N^\dagger \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} TT^*\triangle & -TT^* \triangle SN^\dagger \\ 0 & 0 \end{bmatrix} \tag{9}$$

since U is nonsingular because U is unitary. The equality in (9) gives

$$\begin{cases} (TT^*\Delta)^2 = TT^*\Delta \\ TT^*\Delta TT^*\Delta S N^\dagger = TT^*\Delta S N^\dagger \end{cases} \tag{10}$$

By Lemma 2.4, we know that (10) is equivalent to

$$\begin{cases} (TT^*\Delta)^2 = TT^*\Delta \\ TT^*\Delta TT^*\Delta S N^* = TT^*\Delta S N^* \end{cases} \tag{11}$$

Since T is nonsingular, then TT^* is nonsingular, then (11) is equivalent to

$$\begin{cases} TT^*\Delta = E_t \\ TT^*\Delta S N^* = SN^* \end{cases} \tag{12}$$

which is equivalent to

$$TT^*\Delta = E_t. \tag{13}$$

Since Δ is invertible, (13) is equivalent to

$$TT^* = \Delta^{-1}. \tag{14}$$

By Lemma 2.3,

$$\Delta^{-1} = TT^* + S(E_{n-t} - N^\dagger N)S^*. \tag{15}$$

By (14) and (15), we have $TT^* = TT^* + S(E_{n-t} - N^\dagger N)S^*$, that is, $S(E_{n-t} - N^\dagger N)S^* = 0$. □

Remark 3.1. By the proof of Theorem *3.2*, we have $X = XAX$ if and only if $TT^* = \Delta^{-1}$, where X is the MPEPN-inverse of A and $\Delta = [TT^* + S(E_{n-t} - N^\dagger N)S^*]^{-1}$.

In the following, we show that the MPEPN-inverse of A is an outer inverse of A if and only if $A_2A_2^\dagger A^{\textcircled{\dagger}} = 0$.

Theorem 3.3. *Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k and $A = A_1 + A_2$ be the EP-nilpotent decomposition of A as* (1) *and $A = A_1' + A_2'$ be the Core-EP decomposition of A as* (2.2)*. Then $XAX = X$ if and only if $A^{\textcircled{\dagger}}A_1A_2A_2^*A_1A^{\textcircled{\dagger}} = A_1'(A_2')^\dagger A_2'(A_1')^*$, where X is the MPEPN-inverse of A.*

Proof. Let X be the MPEPN-inverse of A. By Theorem 3.2, we have $XAX = X$ if and only if $S(E_{n-t} - N^\dagger N)S^* = 0$, that is,

$$SS^* = SN^\dagger NS^*. \tag{16}$$

We have

$$A_2A_2^* = U\begin{bmatrix}0 & S\\ 0 & N\end{bmatrix}U^*U\begin{bmatrix}0 & 0\\ S^* & N^*\end{bmatrix}U^* = U\begin{bmatrix}SS^* & SN^*\\ NS^* & NN^*\end{bmatrix}U^* \tag{17}$$

by $A_2 = U\begin{bmatrix}0 & S\\ 0 & N\end{bmatrix}U^*$ and $A_2^* = U^*\begin{bmatrix}0 & 0\\ S^* & N^*\end{bmatrix}U$. Moreover, by Lemma 1 we have

$$A_1 = U\begin{bmatrix}T & 0\\ 0 & 0\end{bmatrix}U^*. \tag{18}$$

By (17) and (18), we have

$$A_2A_2^*A_1 = U\begin{bmatrix}SS^*T & 0\\ NS^*T & 0\end{bmatrix}U^*. \tag{19}$$

By (19), we have

$$A_1A_2A_2^*A_1 = U\begin{bmatrix}T & 0\\ 0 & 0\end{bmatrix}U^*U\begin{bmatrix}SS^*T & 0\\ NS^*T & 0\end{bmatrix}U^* = U\begin{bmatrix}TSS^*T & 0\\ 0 & 0\end{bmatrix}U^*. \tag{20}$$

By [14, Theorem 3.2], we have

$$A^{\oplus} = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{21}$$

By (20) and (21), we have

$$A^{\oplus} A_1 A_2 A_2^* A_1 A^{\oplus} = U \begin{bmatrix} SS^* & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{22}$$

By Lemma 2.2, we have $(A_2')^\dagger = U \begin{bmatrix} 0 & 0 \\ 0 & N^\dagger \end{bmatrix} U^*$, then

$$(A_2')^\dagger A_2' = U \begin{bmatrix} 0 & 0 \\ 0 & N^\dagger N \end{bmatrix} U^*. \tag{23}$$

Since $(A_1')^* = U \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} U^*$. Thus by (23), we have

$$\begin{aligned} A_1'(A_2')^\dagger A_2'(A_1')^* &= U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & N^\dagger N \end{bmatrix} \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} 0 & SN^\dagger N \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} SN^\dagger NS^* & 0 \\ 0 & 0 \end{bmatrix} U^*. \end{aligned} \tag{24}$$

By (22) and (24), the equality in (16) can be written as

$$A^{\oplus} A_1 A_2 A_2^* A_1 A^{\oplus} = A_1'(A_2')^\dagger A_2'(A_1')^*.$$

□

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k and $A = A_1 + A_2$ be the EP-nilpotent decomposition of A as (1). Then $XAX = X$ if and only if $A_2 A_2^\dagger A^{\oplus} = 0$, where X is the MPEPN-inverse of A.*

Proof. By Lemma 2.3, we have

$$A_2^\dagger = U \begin{bmatrix} 0 & 0 \\ (E_{n-t} - N^\dagger N)S^*\Delta & N^\dagger - (E_{n-t} - N^\dagger N)S^*\Delta S N^\dagger \end{bmatrix} U^*, \tag{25}$$

where $\Delta = [TT^* + S(E_{n-t} - N^\dagger N)S^*]^{-1}$. Then

$$\begin{aligned} A_2^\dagger A^{\oplus} &= U \begin{bmatrix} 0 & 0 \\ (E_{n-t} - N^\dagger N)S^*\Delta & N^\dagger - (E_{n-t} - N^\dagger N)S^*\Delta S N^\dagger \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} 0 & 0 \\ (E_{n-t} - N^\dagger N)S^*\Delta T^{-1} & 0 \end{bmatrix} U^*. \end{aligned} \tag{26}$$

By (26), we have

$$A_2A_2^{\dagger}A^{\oplus} = U\begin{bmatrix}0 & S\\ 0 & N\end{bmatrix}U^*U\begin{bmatrix}0 & 0\\ (E_{n-t}-N^{\dagger}N)S^*\Delta T^{-1} & 0\end{bmatrix}U^*$$
$$= U\begin{bmatrix}S(E_{n-t}-N^{\dagger}N)S^*\Delta T^{-1} & 0\\ N(E_{n-t}-N^{\dagger}N) & 0\end{bmatrix}U^*.$$

Thus,

$$S(E_{n-t}-N^{\dagger}N)S^* = 0 \Longleftrightarrow S(E_{n-t}-N^{\dagger}N)S^*\Delta T^{-1} = 0 \Longleftrightarrow A_2A_2^{\dagger}A^{\oplus} = 0. \qquad \square$$

Note that, the condition $A_2A_2^{\dagger}A^{\oplus} = 0$ in Theorem 3.4 can be written as $P_{\mathcal{R}(A_2)}A^{\oplus} = 0$, where $P_{\mathcal{R}(A_2)}$ is the orthogonal projectors onto $\mathcal{R}(A_2)$.

4. The "distance" between the MPEPN-inverse and the inverse along two matrices

In 2012, Drazin [5] introduced a new kind of generalized inverse based on two elements. In 2017, Benítez et al. [1] investigated the (B,C)-inverse of a rectangle complex matrix. The "distance" between the MPEPN-inverse and the inverse along two matrices can be stated by $A^{E,\ddagger}$ is the $(A^{\dagger}A_1P_{\mathcal{R}(A^*)}, P_{\mathcal{R}(A)}A_1A^{\dagger})$-inverse of A under the condition $(A_1A^{\dagger})^2 = A_1A^{\dagger}$.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k and A_1 be the core part in the EP-nilpotent decomposition of A. If $A_1A^{\dagger}$ is an idempotent, then X is the $(A^{\dagger}A_1P_{\mathcal{R}(A^*)}, P_{\mathcal{R}(A)}A_1A^{\dagger})$-inverse of A, where X is the MPEPN-inverse of A.*

Proof. By Theorem 3.1, when $A_1A^{\dagger}$ is an idempotent, we have $XAX = X$, where $X = A^{\dagger}A_1A^{\dagger} = A^{\dagger}A^{k+1}(A^k)^{\dagger}A^{\dagger}$. Let $B = A^{\dagger}A_1P_{\mathcal{R}(A^*)}$ and $C = P_{\mathcal{R}(A)}A_1A^{\dagger}$, then $X = XAX = A^{\dagger}A_1A^{\dagger}AX = A^{\dagger}A_1P_{\mathcal{R}(A^*)}X = BX$, which gives

$$\mathcal{R}(X) \subseteq \mathcal{R}(B). \tag{27}$$

Moreover, the condition $B = A^{\dagger}A_1P_{\mathcal{R}(A^*)} = A^{\dagger}A_1A^{\dagger}A = XA$ implies

$$\mathcal{R}(B) \subseteq \mathcal{R}(X). \tag{28}$$

By (27) and (28), we can get $\mathcal{R}(B) = \mathcal{R}(X)$. For any $u \in \mathcal{N}(P_{\mathcal{R}(A)}A_1A^{\dagger})$, that is, $P_{\mathcal{R}(A)}A_1A^{\dagger}u = 0$, then $Xu = XAXu = XAA^{\dagger}A_1A^{\dagger}u = XP_{\mathcal{R}(A)}A_1A^{\dagger}u = 0$, which gives

$$\mathcal{N}(P_{\mathcal{R}(A)}A_1A^{\dagger}) \subseteq \mathcal{N}(X). \tag{29}$$

For any $v \in \mathcal{N}(X)$, that is, $Xv = 0$, then the condition $P_{\mathcal{R}(A)}A_1A^{\dagger}v = AA^{\dagger}A_1A^{\dagger}v = AXv = 0$ implies

$$\mathcal{N}(X) \subseteq \mathcal{N}(P_{\mathcal{R}(A)}A_1A^{\dagger}). \tag{30}$$

By (29) and (30), we have $\mathcal{N}(C) = \mathcal{N}(X)$. Thus, by Lemma 2.6, we have X is the $(A^{\dagger}A_1P_{\mathcal{R}(A^*)}, P_{\mathcal{R}(A)}A_1A^{\dagger})$-inverse of A. $\square$

The MPEPN-inverse of A is different from the Moore-Penrose inverse, the DMP inverse $A^{D,\dagger}$ of A ([11]), the Core-EP inverse $A^{\oplus}$ of A ([9]) and the MPCEP-inverse $A^{\dagger,\oplus}$ of A ([3]). The example can been seen in the following example.

Example 4.1. Let $A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{C}^{4\times 4}$. Then it is easy to check that

$$A^{E,\ddagger} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^{\dagger} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^{D,\dagger} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^{\oplus} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^{\dagger,\oplus} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the MPEPN-inverse is different from the above generalized inverses.

5. Conclusions

Let A be a given complex matrix with a given index, then one can get that the computation of the MPEPN inverse of A by using the EP-nilpotent decomposition of this matrix. There is a interesting fact about the EP-nilpotent decomposition of A, that is one can using the Core-EP decomposition of A to get the the EP-nilpotent decomposition of A. The future perspectives for research are proposed:

Part 1. The MPEPN inverse is one of the useful tools to investigate the matrix partial orders.

Part 2. The rank properties of a given matrix, such as rank $(AA^{E,\ddagger} - A^{E,\ddagger}A)$.

Part 3. The weighted generalized inverse of matrices related given range space and null space.

Author contributions

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