Picture fuzzy multisets

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Abstract. In this paper, the notion of picture fuzzy multiset was introduced. Some standard operations on picture fuzzy multiset such as intersection, union, complement were defined and their properties were investigated. Also, cut set and Cartesian product of picture fuzzy multiset were defined and the connections of Cartesian product with intersection and union were obtained.

Keywords: multiset, fuzzy multiset, intuitionistic fuzzy multiset, picture fuzzy multiset.

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1. Introduction

In 1965, the theory of fuzzy sets was introduced by Zadeh [34] as a generalisation of classical set theory. The theory only takes into consideration membership degree of an element belonging to a particular set. Atanassov [1], extended the work of Zadeh by introducing the theory of intuitionistic fuzzy sets which deals

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with both the membership and non-membership degrees of an element belonging to a set.

Cuong and Kreinovich [8] generalised the works of Zadeh and Atanassov into the theory of *picture fuzzy sets* (PFSs). This theory is a new concept for computational intelligence which is a set of Nature-inspired computational methodologies and approaches based on mathematics, computer science, artificial intelligence, to address applications of the real world complex problems that can not be solved by traditional methodologies and approaches. Basically, picture fuzzy sets based models may be appropriate in situations involving more answers of type: yes, abstain, no, refusal. A good example of such a situation is voting system in which human voters may decide to: vote for, vote against, abstain and refusal to vote. Thus, according to Cuong and Kreinovich [8], a given set is represented by three membership degrees i.e; positive membership degree, neutral membership degree and negative membership degree.

Picture fuzzy set has been extensively studied such as; in 2014, Cuong [11] investigated some characteristics of PFSs, introduced distance measure and defined convex combination between two PFSs. Cuong and Hai [13] investigated main fuzzy logic operators: negations, conjunctions, disjunctions and implications on picture fuzzy sets. Son [31], introduced a generalised picture distance measure and applied it to establish an Hierarchical Picture Clustering.

The theory of picture fuzzy set has been widely applied in decision making problems in the area of medical diagnosis, building material and minerals field recognitions, Covid-19 medicine selection among others (see [19, 26, 27, 32] for more details).

Yagar in 1986 [33], put forward the notion of *fuzzy multiset* (FM). In 2012, Shinoj and John [29] initiated *intuitionistic fuzzy multiset* (IFMS) from the combination of the concepts of fuzzy multisets and intuitionistic fuzzy sets and this was applied in medicine to diagnosis diseases. In 2013, Shinoj and John [30] defined some operations on intuitionistic fuzzy multisets and established some of its properties. Some researchers have also studied this notion of intuitionistic fuzzy multisets and applied it to medical diagnosis, binomial distribution (see [16, 17, 21] for more details). Due to the fact that the idea of intuitionistic fuzzy multisets also lacks accuracy in handling imprecision and uncertainties because of not taking into account neutrality degree, it is important to study the concept of picture fuzzy multiset as a generalisation of intuitionistic fuzzy multiset.

In this paper, we introduce the concept of *picture fuzzy multisets* (PFMSs), standard operations such as intersection, union, complement are defined and their properties are obtained. Cartesian product of picture fuzzy multiset are also defined and the connections of Cartesian product with intersection and union are established. The paper is organised as follows. Section 2 defines basic terms. Section 3 introduces the notion of PFMS, and some of its properties are obtained.

2. Preliminaries

In this section, some basic definitions are stated. Throughout this paper, E denotes a nonempty set.

Definition 2.1 ([21]). *A fuzzy set (FS) P drawn from E is defined as*

$$P = \{\langle y, \sigma_P(y)\rangle | y \in E\},$$

where $\sigma_P : E \longrightarrow [0,1]$ is the membership function of the fuzzy set P.

Definition 2.2 ([20]). *A fuzzy multiset (FMS) P drawn from E is characterised by a count membership function CM_P such that $CM_P : E \to N$, where N is the set of all crisp multisets drawn from $[0,1]$. Then, for any $y \in E$, the value $CM_P(y)$ is a crisp multiset drawn from $[0,1]$. For any $y \in E$, the membership sequence is defined as the decreasingly ordered sequence of elements in $CM_P(y)$. It is denoted by $(\sigma_P^1(y), \sigma_P^2(y), \cdots, \sigma_P^d(y))$ where $\sigma_P^1(y) \geq \sigma_P^2(y) \geq \cdots \geq \sigma_P^d(y)$.*

Definition 2.3 ([1]). *An intuitionistic fuzzy set (IFS) P of E is defined as*

$$P = \{\langle y, \sigma_P(y), \tau_P(y)\rangle | y \in E\},$$

where the functions $\sigma_P : E \to [0,1]$ and $\tau_P : E \to [0,1]$ are called the membership and non-membership degrees of $y \in P$, respectively, and for every $y \in E$,

$$0 \leq \sigma_P(y) + \tau_P(y) \leq 1.$$

Definition 2.4 ([18]). *An intuitionistic fuzzy multiset (IFMS) P drawn from E is characterised by count membership function CM_P and count nonmembership function CN_P such that $CM_P : E \to N$ and $CN_P : E \to N$, respectively, where N is the set of all crisp multisets drawn from $[0,1]$, such that for any $y \in E$, the membership sequence is defined as the decreasingly ordered sequence of elements in $CM_P(y)$, denoted by $(\sigma_P^1(y), \sigma_P^2(y), \cdots, \sigma_P^d(y))$ where $\sigma_P^1(y) \geq \sigma_P^2(y) \geq \cdots \geq \sigma_P^d(y)$ and the nonmembership sequence is given as $(\tau_P^1(y), \tau_P^2(y) \cdots, \tau_P^d(y))$ such that $0 \leq \sigma_P^i(y) + \tau_P^i(y) \leq 1$ for any $y \in E$, $i = 1, 2, \cdots, d$.*
Thus, an IFMS is given as

$$P = \{\langle y, (\sigma_P^1(y), \sigma_P^2(y) \cdots, \sigma_P^d(y)), (\tau_P^1(y), \tau_P^2(y), \cdots, \tau_P^d(y))\rangle | y \in E\}.$$

Definition 2.5 ([8]). *A picture fuzzy set P of E is defined as*

$$P = \{\langle y, \sigma_P(y), \tau_P(y), \gamma_P(y)\rangle | y \in E\},$$

where the functions

$$\sigma_P : E \to [0,1],\ \tau_P : E \to [0,1] \text{ and } \gamma_P : E \to [0,1]$$

are called the positive, neutral and negative membership degrees of $y \in P$, respectively, and $\sigma_P, \tau_P, \gamma_P$ satisfy

$$0 \leq \sigma_P(y) + \tau_P(y) + \gamma_P(y) \leq 1,\ \forall y \in E.$$

For each $y \in E$, $S_P(y) = 1 - (\sigma_P(y) + \tau_P(y) + \gamma_P(y))$ is called the refusal membership degree of $y \in P$.

Definition 2.6 ([16])**.** *The Cut set of PFS P, denoted by $C_{r,s,t}(P)$ is defined by*

$$C_{r,s,t}(P) = \{y \in E | \sigma_P(y) \geq r,\ \tau_P(y) \geq s, \gamma_P(y) \leq t\},$$

where $r, s, t \in [0,1]$ with the condition $0 \leq r+s+t \leq 1$.

Definition 2.7 ([8])**.** *Let P and Q be two PFSs. Then, the inclusion, equality, union, intersection and complement are defined as follow:*

- *$P \subseteq Q$ if and only if for all $y \in E$, $\sigma_P(y) \leq \sigma_Q(y)$, $\tau_P(y) \leq \tau_Q(y)$ and $\eta_P(y) \geq \eta_Q(y)$.*
- *$P = Q$ if and only if $P \subseteq Q$ and $Q \subseteq P$.*
- *$P \cup Q = \{(y, \sigma_p(y) \vee \sigma_Q(y), \tau_P(y) \wedge \tau_Q(y)), \eta_P(y) \wedge \eta_Q(y))|\ y \in\ E\}$.*
- *$P \cap Q = \{(y, \sigma_P(y) \wedge \sigma_Q(y), \tau_P(y) \wedge \tau_Q(y)), \eta_P(y) \vee \eta_Q(y))|\ y \in\ E\}$.*
- *$\overline{P} = \{(y, \eta_P(y), \tau_P(y), \sigma_P(y))|\ y \in\ E\}$.*

3. Picture fuzzy multisets

Here, we define picture fuzzy multiset, some basic operations and investigate some properties related to the operations.

Definition 3.1. *A picture fuzzy multiset (PFMS) P drawn from E is characterised by count positive membership function C_pM_P, count neutral membership function $C_{ne}M_P$ and count negative membership function C_nM_P such that $C_pM_P : E \to N$, $C_{ne}M_P : E \to N$ and $C_nM_P : E \to N$, respectively, where N is the set of all crisp multisets drawn from $[0,1]$, such that for any $y \in E$, the positive membership sequence is defined as the decreasingly ordered sequence of elements in $C_pM_P(y)$, denoted by $(\sigma_P^1(y), \sigma_P^2(y), \cdots, \sigma_P^d(y))$ where $\sigma_P^1(y) \geq \sigma_P^2(y) \geq \cdots \geq \sigma_P^d(y)$, the neutral membership sequence and negative membership sequence is $(\tau_P^1(y), \tau_P^2(y), \cdots, \tau_P^d(y))$ and $(\eta_P^1(y), \eta_P^2(y), \cdots, \eta_P^d(y))$, respectively such that $0 \leq \sigma_P^i(y) + \tau_P^i(y) + \eta_P^i(y) \leq 1$ for any $y \in E$, $\ i = 1, 2, \cdots, d$.*

So, a PFMS is denoted by

$$\begin{aligned} P \;=\; & \{\langle y, (\sigma_P^1(y), \sigma_P^2(y), \cdots, \sigma_P^d(y)), (\tau_P^1(y), \tau_P^2(y), \cdots, \tau_P^d(y)), \\ & (\eta_P^1(y), \eta_P^2(y), \cdots, \eta_P^d(y))\rangle | y \in E\}. \end{aligned}$$

Remark 3.1. Notice that since the positive membership sequence is arranged in decreasing order, neutral or negative membership sequence may not be decreasing or increasing order.

Definition 3.2. *Let $P = \{\langle y, \sigma_P^i(y), \tau_P^i(y), \eta_P^i(y)\rangle|\ y \in\ E\}$ be a PFMS. Then, the (r,s,t)-cut of P denoted by $[P]_{r,s,t}$ is defined by*

$$[P]_{(r,s,t)} = \{a \in P|\ \sigma_P^i(a) \geq r,\ \tau_P^i(a) \leq s,\ \eta_P^i(a) \leq t\},\ i = 1, 2, \cdots, d,$$

where $r, s, t \in [0,1]$ such that $0 \leq r+s+t \leq 1$.

Definition 3.3. *Let $P = \{\langle y, (\sigma_P^i(y)), (\tau_P^i(y)), (\eta_P^i(y))\rangle | y \in E\},\ i = 1, 2, \cdots, d$ be a PFMS and $y \in P$. Then, the size of $y \in P$, denoted by $S(y; P)$ is defined as the cardinality of $C_pM_P(y)$ or $C_{ne}M_P(y)$ or $C_nM_P(y)$, for which $0 \leq \sigma_P^1(y) + \tau_P^1(y) + \eta_P^1(y) \leq 1$. That is*

$$S(y;\ P) = |C_pM_P(y)| = |C_{ne}M_P(y)| = |C_nM_P(y)|.$$

Definition 3.4. *Given two PFMSs P and Q drawn from E. Then, the size of P and Q is defined as*

$$S(y;\ P,\ Q) = S(y;\ P) \vee S(y;\ Q).$$

Example 3.1. Let $E = \{a, b, c\}$

$$\begin{aligned} P &= \{\langle a; (0.5,\ 0.2), (0.3,\ 0.1), (0.2, 0.4)\rangle, \\ &\quad \langle c; (0.0,\ 0.4,\ 0.1),\ (0.2,\ 0.1,\ 0.3), (0.5, 0.2, 0.6)\rangle\} \end{aligned}$$

and

$$\begin{aligned} Q &= \{\langle a; (0.1,\ 0.5), (0.2,\ 0.4), (0.0, 0.0)\rangle, \\ &\quad \langle b; (0.2,\ 0,\ 0.3), (0.1,\ 1.0,\ 0.2), (0.2, 0.0, 0.4)\rangle, \\ &\quad \langle c; (0.8,\ 0.1),\ (0.1,\ 0.3),\ (0.1, 0.5)\rangle\}. \end{aligned}$$

Then

$$\begin{aligned} &S(a;\ P) = 2,\ S(b;\ P) = 0,\ S(c;\ P) = 3, \\ &S(a;\ Q) = 2,\ S(b;\ Q) = 3,\ S(c;\ Q) = 2, \\ &S(a; P,\ Q) = 2,\ S(b;\ P,\ Q) = 3,\ S(c;\ P,\ Q) = 3. \end{aligned}$$

3.1 Standard operations on picture fuzzy multisets

Definition 3.5. *Let*

$$P = \{\langle y, \sigma_P^i(y), \tau_P^i(y), \eta_P^i(y)\rangle |\ y \in\ E\}$$

and

$$Q = \{\langle y, \sigma_Q^i(y), \tau_Q^i(y), \eta_Q^i(y)\rangle |\ y \in\ E\},$$

where $i = 1, 2, \cdots, d$, be two PFMSs drawn from E. Then, the inclusion, equality, union, intersection and complement are defined as follow:

- *$P \subseteq Q$ if and only if, $\sigma_P^i(y) \leq \sigma_Q^i(y)$, $\tau_P^i(y) \leq \tau_Q^i(y)$ and $\eta_P^i(y) \geq \eta_Q^i(y)$; $i = 1, 2, \cdots, S(y;\ P,\ Q),\ y \in E$.*
- *$P = Q$ if and only if $P \subseteq Q$ and $Q \subseteq P$.*
- *$P \cup Q = \{(y, \max(\sigma_P^i(y), \sigma_Q^i(y)), \min(\tau_P^i(y), \tau_Q^i(y)), \min(\eta_P^i(y), \eta_Q^i(y)))| y \in E\}$, where $i = 1, 2, \cdots, S(y; P, Q)$.*

- $P \cap Q = \{(y, \min(\sigma^i_P(y), \sigma^i_Q(y)), \min(\tau^i_P(y), \tau^i_Q(y)), \max(\eta^i_P(y), \eta^i_Q(y)))|y \in E\}$, *where* $i = 1, 2, \cdots, S(y;\ P,\ Q)$.
- $\overline{P} = \{(y, \eta^i_P(y), \tau^i_P(y), \sigma^i_P(y))|\ y \in\ E\},\ i = 1, 2, \cdots, S(y;\ P,\ Q)$.

Example 3.2. Let $E = \{a, b, c\}$

$$\begin{aligned} P &= \{\langle a; (0.1,\ 0.5), (0.2,\ 0.4), (0.0, 0.0)\rangle, \\ &\quad \langle b; (0.1,\ 0.4,\ 0.7), (0.1,\ 0.6,\ 0.0), (0.5, 0.0, 0.3)\rangle, \\ &\quad \langle c; \langle c; (0.4,\ 0.1), (0.1,\ 0.7), (0.0, 0.0)\rangle\} \end{aligned}$$

and

$$\begin{aligned} Q &= \{\langle a; (0.5,\ 0.2), (0.3,\ 0.1), (0.2, 0.4)\rangle, \\ &\quad \langle b; (0.2,\ 0.0,\ 0.3), (0.1,\ 0.6,\ 0.4), (0.2, 0.3, 0.1)\rangle, \\ &\quad \langle c; \langle c; (0.8,\ 0.1),\ (0.1,\ 0.3),\ (0.1, 0.5)\rangle\}. \end{aligned}$$

Then

$$\begin{aligned} P \cup Q &= \{\langle a; (0.5,\ 0.5), (0.2,\ 0.1), (0.0, 0.0)\rangle, \\ &\quad \langle b; (0.2,\ 0.4,\ 0.7),\ (0.1,\ 0.6,\ 0.0), (0.2, 0.0, 0.1)\rangle, \\ &\quad \langle c; \langle c; (0.8,\ 0.1),\ (0.1,\ 0.3), (0.0, 0.0)\rangle\}, \\ P \cap Q &= \{\langle a; (0.1,\ 0.2),\ (0.2,\ 0.1), (0.2, 0.4)\rangle, \\ &\quad \langle b; (0.1,\ 0.0,\ 0.3),\ (0.1,\ 0.6,\ 0.0), (0.5, 0.3, 0.3)\rangle, \\ &\quad \langle c; \langle c; (0.4,\ 0.1),\ (0.1,\ 0.3), (0.1, 0.5)\rangle\}, \\ \overline{P} &= \{\langle c; (0.4,\ 0.1), (0.1,\ 0.7), (0.0, 0.0)\rangle, \\ &\quad \langle b; (0.1,\ 0.4,\ 0.7), (0.1,\ 0.6,\ 0.0), (0.5, 0.0, 0.3)\rangle, \\ &\quad \langle c; \langle a; (0.1,\ 0.5), (0.2,\ 0.4), (0.0, 0.0)\rangle\}. \end{aligned}$$

3.2 Basic properties

Proposition 3.1. *For every PFMS* P, Q, R.

1. **Involution**

   $\overline{\overline{P}} = P.$

2. **Commutative rule**

   $P \cap Q = Q \cap P,\ \ P \cup Q = Q \cup P.$

3. **Associative rule**

   $P \cap (Q \cap R) = (P \cap Q) \cap R,\ \ P \cup (Q \cup R) = (P \cup Q) \cup R.$

4. **Distributive rule**

$P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R), \;\; P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R).$

5. **Idempotent rule**

$P \cap P = P, \;\; P \cup P = P.$

6. **De Morgan's rule**

$\overline{P \cap Q} = \overline{P} \cup \overline{Q}, \;\; \overline{P \cup Q} = \overline{P} \cap \overline{Q}.$

Proof. Let

$$\begin{aligned} P &= \{\langle y, (\sigma_Q^1(y), \sigma_Q^2(y), \cdots, \sigma_Q^d(y)), (\tau_Q^1(y), \tau_Q^2(y), \cdots, \tau_Q^d(y)), \\ &\quad (\eta_P^1(y), \eta_P^2(y), \cdots, \eta_P^d(y))\rangle \mid y \in \; E\}, \\ Q &= \{\langle y, (\sigma_Q^1(y), \sigma_Q^2(y), \cdots, \sigma_Q^d(y)), (\tau_Q^1(y), \tau_Q^2(y), \cdots, \tau_Q^d(y)), \\ &\quad (\eta_Q^1(y), \eta_Q^2(y), \cdots, \eta_Q^d(y))\rangle \mid y \in \; E\} \end{aligned}$$

and

$$\begin{aligned} R &= \{\langle y, (\sigma_R^1(y), \sigma_R^2(y), \cdots, \sigma_R^d(y)), (\tau_R^1(y), \tau_r^2(y), \cdots, \tau_R^d(y)), \\ &\quad (\eta_R^1(y), \eta_R^2(y), \cdots, \eta_R^d(y))\rangle \mid y \in \; E\}. \end{aligned}$$

Then

1.

$$\begin{aligned} \overline{P} &= \{\langle y, (\eta_Q^1(y), \eta_Q^2(y), \cdots, \eta_Q^d(y)), (\tau_P^1(y), \tau_P^2(y), \cdots, \tau_P^d(y)), \\ &\quad (\sigma_P^1(y), \sigma_P^2(y), \cdots, \sigma_P^d(y))\rangle \mid y \in \; E\}, \\ \overline{\overline{P}} &= \{\langle y, (\sigma_P^1(y), \sigma_P^2(y), \cdots, \sigma_P^d(y)), (\tau_P^1(y), \tau_P^2(y), \cdots, \tau_P^d(y)), \\ &\quad (\eta_Q^1(y), \eta_Q^2(y), \cdots, \eta_Q^d(y))\rangle \mid y \in \; E\} \\ &= P. \end{aligned}$$

2.

$$\begin{aligned} & P \cap Q \\ & = \{\langle y, (\sigma_P^1(y), \sigma_P^2(y), \cdots, \sigma_P^d(y)), (\tau_P^1(y), \tau_P^2(y), \cdots, \tau_P^d(y)), \\ & (\eta_P^1(y), \eta_P^2(y), \cdots, \eta_P^d(y)) \\ & \cap (\sigma_Q^1(y), \sigma_Q^2(y), \cdots, \sigma_Q^d(y)), (\tau_Q^1(y), \tau_Q^2(y), \cdots, \tau_Q^d(y)), \end{aligned}$$

$$(\eta_Q^1(y), \eta_Q^2(y), \cdots, \eta_Q^d(y)))\rangle |\ y \in\ E\rangle\}$$
$$= \{\langle y, (\sigma_P^1(y) \wedge \sigma_Q^1(y), \sigma_P^2(y) \wedge \sigma_Q^2(y)), \cdots, (\sigma_P^d(y) \wedge \sigma_Q^d(y)),$$
$$(\tau_P^1(y) \wedge \tau_Q^1(y), \tau_P^2(y) \wedge \tau_Q^2(y)), \cdots (\tau_P^d(y) \wedge \tau_Q^d(y)),$$
$$(\eta_P^1(y) \vee \eta_Q^1(y), \eta_P^2(y) \vee \eta_Q^2(y)), \cdots, (\eta_P^d(y) \vee \eta_Q^d(y))|\ y \in\ E\rangle\}$$
$$= \{\langle y, (\sigma_Q^1(y) \wedge \sigma_P^1(y)), \cdots, (\sigma_Q^d(y) \wedge \sigma_P^d(y)), (\tau_Q^1(y) \wedge \tau_P^1(y)),$$
$$\cdots, (\tau_Q^d(y) \wedge \tau_P^d(y)), (\eta_Q^1(y) \vee \eta_P^1(y)), \cdots, (\eta_Q^d(y) \vee \eta_P^d(y))|\ y \in\ E\rangle\}$$
$$= Q \cap P.$$

Similarly, we can prove $P \cup Q = Q \cup P$.

3.

$$P \cap (Q \cap R)$$
$$= \{\langle y, (\sigma_P^1(y), \cdots, \sigma_P^d(y)), (\tau_P^1(y), \cdots, \tau_P^d(y)), (\eta_P^1(y), \cdots, \eta_P^d(y))\rangle |\ y \in\ E\}$$
$$\cap \{\langle y, ((\sigma_Q^1(y) \wedge \sigma_R^1(y)), \cdots, (\sigma_Q^d(y) \wedge \sigma_R^d(y)), (\tau_Q^1(y) \wedge \tau_R^1(y)),$$
$$\cdots, (\tau_Q^d(y) \wedge \tau_R^d(y)), (\eta_Q^1(y) \wedge \eta_R^1(y)), \cdots, (\eta_Q^d(y) \wedge \eta_R^d(y))))\rangle |\ y \in\ E\}$$
$$= \{\langle y, (\sigma_P^1(y) \wedge \sigma_Q^1(y)) \wedge \sigma_R^1(y), \cdots, (\sigma_P^d(y) \wedge \sigma_Q^d(y)) \wedge \sigma_R^d(y),$$
$$(\tau_P^1(y)\tau_Q^1(y)) \wedge \tau_R^1(y), \cdots, (\tau_P^d(y) \wedge \tau_Q^d(y)) \wedge \tau_R^d(y),$$
$$(\eta_P^1(y) \vee \eta_Q^1(y)) \vee \eta_R^1(y), \cdots, (\eta_P^d(y) \vee \eta_Q^d(y)) \vee \eta_R^d(y)\rangle |\ y \in\ E\}$$
$$= (P \cap Q) \cap R.$$

Similarly, we can prove $P \cup (Q \cup R) = (P \cup Q) \cup R$.

4.

$$P \cap (Q \cup R)$$
$$= \{\langle y, (\sigma_P^1(y) \wedge (\sigma_Q^1(y)) \vee \sigma_R^1(y)), \cdots, \sigma_P^d(y) \wedge (\sigma_Q^d(y)) \vee \sigma_R^d(y)),$$
$$\tau_P^1(y) \wedge (\tau_Q^1(y)) \vee \tau_R^1(y)), \cdots, \tau_P^d(y) \wedge (\tau_Q^d(y)) \vee \tau_R^d(y)),$$
$$\eta_P^1(y) \vee (\eta_Q^1(y)) \wedge \eta_R^1(y)), \cdots, \eta_P^d(y) \vee (\eta_Q^d(y)) \wedge \eta_R^d(y))))\rangle |\ y \in\ E\}$$
$$= \{\langle y, ((\sigma_P^1(y) \wedge \sigma_Q^1(y)) \vee (\sigma_P^1(y) \wedge \sigma_R^1(y)), \cdots, (\sigma_P^d(y) \wedge \sigma_Q^d(y))$$
$$\vee (\sigma_P^d(y) \wedge \sigma_R^d(y))), ((\tau_P^1(y) \wedge \tau_Q^1(y)) \vee (\tau_P^1(y) \wedge \tau_R^1(y)),$$
$$\cdots, (\tau_P^d(y) \wedge \tau_Q^d(y)) \vee (\tau_P^d(y) \wedge \tau_R^d(y))), ((\eta_P^1(y) \vee \eta_Q^1(y))$$
$$\wedge (\eta_P^1(y) \vee \eta_R^1(y)), \cdots, (\eta_P^d(y) \vee \eta_Q^d(y)) \wedge (\eta_P^d(y) \vee \eta_R^d(y))))\rangle |\ y \in\ E\}$$
$$= (P \cap Q) \cup (P \cap R).$$

Similarly, we can prove $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$.

5.

$$\begin{aligned}
&P \cap P \\
&= \{\langle y, (\sigma_P^1(y) \wedge \sigma_P^1(y)), \cdots, (\sigma_P^d(y) \wedge \sigma_P^d(y)), (\tau_P^1(y) \wedge \tau_P^1(y)), \\
&\cdots, (\tau_P^d(y) \wedge \tau_P^d(y)), (\eta_P^1(y) \vee \eta_P^1(y)), (\eta_P^d(y) \vee \eta_P^d(y))|\ y \in\ E\rangle\} \\
&= \{\langle y, (\sigma_P^1(y), \cdots, \sigma_P^d(y)), (\tau_P^1(y), \cdots, \tau_P^d(y)), (\eta_P^1(y), \cdots, \eta_P^d(y))|\ y \in\ E\rangle\} \\
&= P.
\end{aligned}$$

Similarly, we can prove $P \cup P = P$.

6.

$$\begin{aligned}
&\overline{P \cap Q} \\
&= \{\langle y, ((\eta_P^1(y) \vee \eta_Q^1(y)), \cdots, (\eta_P^d(y) \vee \eta_Q^d(y), ((\tau_P^1(y) \wedge \tau_Q^1(y)), \\
&\cdots, (\tau_P^d(y) \wedge \tau_Q^d(y)), ((\sigma_P^1(y) \wedge \sigma_Q^1(y)), \cdots, (\sigma_P^d(y) \wedge \sigma_Q^d(y))\rangle|\ y \in\ E\} \\
&= \{\langle y, ((\sigma_P^1(y) \vee \sigma_Q^1(y)), \cdots, (\sigma_P^d(y) \vee \sigma_Q^d(y)), ((\tau_P^1(y) \wedge \tau_Q^1(y)), \\
&\cdots, (\tau_P^d(y) \wedge \tau_Q^d(y)), ((\eta_P^1(y) \wedge \eta_Q^1(y)), \cdots, (\eta_P^d(y) \wedge \eta_Q^d(y)\rangle|\ y \in\ E\} \\
&= \{\langle y, (\eta_P^1(y), \cdots, \eta_P^d(y)), (\tau_P^1(y), \cdots, \tau_P^d(y)), (\sigma_P^1(y), \cdots, \sigma_P^d(y))\rangle|\ y \in\ E\} \\
&\cup \{\langle y, (\eta_Q^1(y), \cdots, \eta_Q^d(y)), (\tau_Q^1(y), \cdots, \tau_Q^i(y)), (\sigma_Q^1(y), \cdots, \sigma_Q^d(y))\rangle|\ y \in\ E\} \\
&= \overline{P} \cup \overline{Q}.
\end{aligned}$$

Similarly, we can prove $\overline{P \cup Q} = \overline{P} \cap \overline{Q}$. □

Definition 3.6. *Let*

$$\begin{aligned}
P &= \{\langle y, (\sigma_Q^1(y), \sigma_Q^2(y), \cdots, \sigma_Q^d(y)), (\tau_Q^1(y), \tau_Q^2(y), \cdots, \tau_Q^d(y)), \\
&\quad (\eta_P^1(y), \eta_P^2(y), \cdots, \eta_P^d(y))\rangle|\ y \in\ E\}
\end{aligned}$$

and

$$\begin{aligned}
Q &= \{\langle y, (\sigma_Q^1(y), \sigma_Q^2(y), \cdots, \sigma_Q^d(y)), (\tau_Q^1(y), \tau_Q^2(y), \cdots, \tau_Q^d(y)), \\
&\quad (\eta_Q^1(y), \eta_Q^2(y), \cdots, \eta_Q^d(y))\rangle|\ y \in\ E\}
\end{aligned}$$

be two PFMSs on E.

Then, the Cartesian product of P and Q, $P \times Q$ is defined as

$$\begin{aligned}
P \times Q &= \{\langle (x, y), (\sigma_{P\times Q}^1(x, y), \sigma_{P\times Q}^2(x, y), \cdots, \sigma_{P\times Q}^d(x, y)), \\
&\quad (\tau_{P\times Q}^1(x, y), \tau_{P\times Q}^2(x, y), \cdots, \tau_{P\times Q}^d(x, y)), \\
&\quad (\eta_{P\times Q}^1(x, y), \eta_{P\times Q}^2(x, y), \cdots, \eta_{P\times Q}^d(x, y))\rangle|\ x, y \in\ E\}
\end{aligned}$$

where

$$\begin{aligned}
\sigma_{P\times Q}^i(x, y) &= \sigma_P^i(x) \wedge \sigma_Q^i(y), \\
\tau_{P\times Q}^i(x, y) &= \tau_P^i(x) \wedge \tau_Q^i(y)
\end{aligned}$$

and

$$\eta^i_{P\times Q}(x,y) = \eta^i_P(x) \vee \eta^i_Q(y)$$

with $i = 1, 2, \cdots, d$.

3.3 Basic properties

Proposition 3.2. *Let* P, Q, R *be PFMSs. Then*

1. $P \times Q = Q \times P$.
2. $(P \times Q) \times R = P \times (Q \times R)$.
3. $P \times (Q \cup R) = (P \times Q) \cup (P \times R)$.
4. $P \times (Q \cap R) = (P \times Q) \cap (P \times R)$.

Proof. Let PFMSs P, Q, R be defined as

$$\begin{aligned} P &= \{\langle y, (\sigma^1_Q(y), \sigma^2_Q(y), \cdots, \sigma^d_Q(y)), (\tau^1_Q(y), \tau^2_Q(y), \cdots, \tau^d_Q(y)), \\ &\quad (\eta^1_P(y), \eta^2_P(y), \cdots, \eta^d_P(y))\rangle |\ y \in\ E\}, \end{aligned}$$

$$\begin{aligned} Q &= \{\langle y, (\sigma^1_Q(y), \sigma^2_Q(y), \cdots, \sigma^d_Q(y)), (\tau^1_Q(y), \tau^2_Q(y), \cdots, \tau^d_Q(y)), \\ &\quad (\eta^1_Q(y), \eta^2_Q(y), \cdots, \eta^d_Q(y))\rangle |\ y \in\ E\} \end{aligned}$$

and

$$\begin{aligned} R &= \{\langle y, (\sigma^1_R(y), \sigma^2_R(y), \cdots, \sigma^d_R(y)), (\tau^1_R(y), \tau^2_r(y), \cdots, \tau^d_R(y)), \\ &\quad (\eta^1_R(y), \eta^2_R(y), \cdots, \eta^d_R(y))\rangle |\ y \in\ E\}. \end{aligned}$$

(1) and (2) are obvious from the definition

3.
$$\begin{aligned} & P \times (Q \cup R) \\ = \ & \{\langle (x,y), (\sigma^1_P(y) \wedge (\sigma^1_Q(y) \vee \sigma^1_R(y)), \cdots, \sigma^d_P(y) \wedge (\sigma^d_Q(y) \vee \sigma^d_R(y))), \\ & (\tau^1_P(y) \wedge (\tau^1_Q(y) \wedge \tau^1_R(y)), \cdots, \tau^d_P(y) \wedge (\tau^d_Q(y) \wedge \tau^d_R(y))), \\ & (\eta^1_P(y) \vee (\eta^1_Q(y) \wedge \eta^1_R(y)), \cdots, \eta^d_P(y) \vee (\eta^d_Q(y) \wedge \eta^d_R(y)))\rangle |\ x, y \in\ E\} \\ = \ & \{\langle (x,y), (\sigma^1_P(y) \wedge \sigma^1_Q(y), \cdots, \sigma^d_P(y) \wedge \sigma^d_Q(y)), (\tau^1_P(y) \wedge \tau^1_Q(y), \\ & \cdots, \tau^d_P(y) \wedge \tau^d_Q(y)), (\eta^1_P(y) \vee \eta^1_Q(y), \cdots, \eta^d_P(y) \vee \eta^d_Q(y))\rangle |\ x, y \in\ E\} \\ & \cup \{\langle (x,y), (\sigma^1_P(y) \wedge \sigma^1_R(y), \cdots, \sigma^d_P(y) \wedge \sigma^d_R(y)), (\tau^1_P(y) \wedge \tau^1_R(y), \\ & \cdots, \tau^d_P(y) \wedge \tau^d_R(y)), (\eta^1_P(y) \vee \eta^1_R(y), \cdots, \eta^d_P(y) \vee \eta^d_R(y))\rangle |\ x, y \in\ E\} \\ = \ & (P \times Q) \cup (P \times R). \quad \square \end{aligned}$$

Property 4 can also be proved in the same way as property 3.

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