The separator of Green's classes of the full transformation semigroup

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Abstract. This paper investigates the separator of Green's classes of the full transformation semigroup. The separator of a subset A of a semigroup S is the set of all elements $x \in S$ satisfying the following conditions: $xA \subseteq A$, $Ax \subseteq A$, $x(S\backslash A) \subseteq S\backslash A$ and $(S\backslash A)x \subseteq S\backslash A$. We establish the relationship between the separator of Green's classes and the permutations preserving partition and/or permuting image.

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1. Introduction

The separator of a subset A of a semigroup S is the set of all elements $x \in S$ satisfying the following conditions: $xA \subseteq A$, $Ax \subseteq A$, $x(S\backslash A) \subseteq S\backslash A$ and $(S\backslash A)x \subseteq S\backslash A$. Let π be an equivalence relation on a set X. We say that $\alpha : X \to X$ preserves π if, for all $x, y \in X$, $(x, y) \in \pi$ implies $(x\alpha, y\alpha) \in \pi$. Let T_n and S_n denote the full transformation semigroup and symmetric group, respectively, on $\underline{n} = \{1, \ldots, n\}$. Denote by $S_n(\pi)$ the set of all permutations on $\underline{n}$ that preserve π. For a nonempty subset Y of $\underline{n}$, denote by $S_n(Y)$ the set of all permutations on $\underline{n}$ that permute Y. Moreover, let $S_n(\pi, Y) = S_n(\pi) \cap S_n(Y)$. The Green's relations on a semigroup were first studied by J.A. Green [7] in 1951. Let a and b be elements of a semigroup S. We define $a\mathscr{L}b$ ($a\mathscr{R}b$) if a and b generate the same principal left (right) ideal of S. The join of $\mathscr{L}$ and $\mathscr{R}$ is

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denoted by $\mathscr{D}$ and their intersection by $\mathscr{H}$ (see [3]). In 2011, A. Nagy proved that the separator of a proper ideal of T_n is the symmetric group S_n. Guided by the result put forth by C.G. Doss [5], we will describe the separator of the Green's classes of T_n. Following the convention used in [3], by a partition π of a set X we mean the partition X/π determined by an equivalence relation π on X. First, we show that the separator of a $\mathscr{D}$-class of T_n is the symmetric group S_n. Then, we prove that $S_n(Y)$ is the separator of the $\mathscr{L}$-class consisting of all elements of T_n whose image is Y. Next, we show that $S_n(\pi)$ is the separator of the $\mathscr{R}$-class consisting of all elements of T_n with partition π. Finally, we show that $S_n(\pi, Y)$ is the separator of the $\mathscr{H}$-class consisting of all elements of T_n with partition π and image Y.

2. Preliminaries

The following definitions are found in [3]. A *transformation* of a set X is a single-valued mapping of X into itself. The image of an element x of X under a transformation or mapping α is denoted by $x\alpha$ (rather than αx or $\alpha(x)$). The *product* (or *iterate* or *composition*) of two transformations α and β of X is the transformation $\alpha\beta$ defined by $x(\alpha\beta) = (x\alpha)\beta$, for all $x \in X$ (that is, α followed by β). The set T_X of all transformations of X is a semigroup with respect to iteration. We call T_X the *full transformation semigroup on* X. A one-to-one mapping of a set X onto itself will be called a *permutation of* X, even when X is infinite. The *symmetric group* S_X *on* X consists of all permutations of X under the operation of iteration.

Definition 2.1 ([3]). *With each element α of T_X we associate two things: (1) the image $X\alpha$ of α, also denoted by $Im(\alpha)$, which is defined by $X\alpha = \{x\alpha \mid x \in X\}$ and (2) the partition $\pi_\alpha = \alpha \circ \alpha^{-1}$ of X corresponding to α, i.e., the equivalence relation on X defined by $(x, y) \in \pi_\alpha$ if $x\alpha = y\alpha$, where $x, y \in X$. Let $\pi_\alpha^\natural$ be the natural mapping of X upon the set X/π_α of equivalence classes of X mod π_α. Then, $x\pi_\alpha^\natural \mapsto x\alpha$ is a one-to-one mapping of X/π_α upon $X\alpha$. It follows that $|X/\pi_\alpha| = |X\alpha|$, and this cardinal number is called the rank of α.*

The following theorem characterizes Green's classes in terms of rank, partition, and image.

Theorem 2.1 ([3]). *Let T_X be the full transformation semigroup on a set X.*

i. In the semigroup T_X, we have $\mathscr{D} = \mathscr{J}$.

ii. There is a one-to-one correspondence between the set of all principal ideals of T_X and the set of all cardinal numbers $r \leq |X|$ such that the principal ideal corresponding to r consists of all elements of T_X of rank $\leq r$.

iii. There is a one-to-one correspondence between the set of all $\mathscr{D}$-classes of T_X and the set of all cardinal numbers $r \leq |X|$ such that the $\mathscr{D}$-class D_r corresponding to r consists of all elements of T_X of rank r.

iv. *Let r be a cardinal number $\leq |X|$. There is a one-to-one correspondence between the set of all $\mathscr{L}$-classes in D_r and the set of all subsets Y of X of cardinal r such that the $\mathscr{L}$-class corresponding to Y consists of all elements of T_X having image Y.*

v. *Let r be a cardinal number $\leq |X|$. There is a one-to-one correspondence between the set of all $\mathscr{R}$-classes contained in D_r and the set of all partitions π of X for which $|X/\pi| = r$ such that the $\mathscr{R}$-class corresponding to π consists of all elements of T_X having partition π.*

vi. *Let r be a cardinal number $\leq |X|$. There is a one-to-one correspondence between the set of all $\mathscr{H}$-classes in D_r and the set of all pairs (π, Y) where π is a partition of X and Y is a subset of X such that $|X/\pi| = |Y| = r$, such that the $\mathscr{H}$-class corresponding to (π, Y) consists of all elements of T_X having partition π and image Y.*

Throughout this paper, we will only consider the finite full transformation semigroup. Let T_n and S_n denote the full transformation semigroup and symmetric group, respectively, on $\underline{n} = \{1, \ldots, n\}$.

Lemma 2.1 ([6]). *Let $\alpha \in T_n$. Then, the following conditions are equivalent:*

i) *α is surjective.*

ii) *α is injective.*

iii) *α is bijective.*

Lemma 2.2 ([4]). *Let $\alpha, \beta \in T_n$. Then, $rank(\alpha\beta) \leq \min\{rank(\alpha), rank(\beta)\}$.*

Lemma 2.3 ([2]). *If $\alpha \in S_n$ and $\beta \in T_n$, then $rank(\alpha\beta) = rank(\beta\alpha) = rank(\beta)$.*

Next, we introduce notations for the Green's classes of T_n. Let $k \leq n$. We denote by D_k the set of all $\alpha \in T_n$ whose rank is k. For a partition π of $\underline{n}$ and $Y \subseteq \underline{n}$ where $|\underline{n}/\pi| = |Y| = k$, let $L_k(Y)$ be the set of all $\alpha \in D_k$ with image Y. Moreover, let $R_k(\pi)$ be the set of all $\alpha \in D_k$ with $\pi_\alpha = \pi$. Finally, we denote by $H_k(\pi, Y)$ the set of all $\alpha \in D_k$ with $\pi_\alpha = \pi$ and $Im\alpha = Y$. Then, $H_k(\pi, Y) = R_k(\pi) \cap L_k(Y)$. By Theorem 2.1, D_k, $L_k(Y)$, $R_k(\pi)$, and $H_k(\pi, Y)$ are precisely the $\mathscr{D}$-, $\mathscr{L}$-, $\mathscr{R}$-, and $\mathscr{H}$-classes of T_n.

Definition 2.2 ([8]). *Let S be a semigroup and let $A \subseteq S$. The separator of A, denoted by $Sep(A)$, is the set of all elements $x \in S$ satisfying the following conditions: $xA \subseteq A$, $Ax \subseteq A$, $x(S \backslash A) \subseteq S \backslash A$ and $(S \backslash A)x \subseteq S \backslash A$.*

2.1 Transformations preserving a partition

Definition 2.3 ([1]). *Let $\mathcal{P}$ be a partition of a set X. We say that $\alpha \in T_X$ preserves $\mathcal{P}$ if, for all $P \in \mathcal{P}, \exists Q \in \mathcal{P}$ such that $P\alpha \subseteq Q$.*

Let $T(X, \mathcal{P})$ denote the semigroup of all full transformations of X that preserve the partition $\mathcal{P}$. We now define a transformation preserving an equivalence relation π. It is straightforward to show that this definition is equivalent to the definition of a transformation preserving X/π.

Definition 2.4. *Let π be an equivalence relation on a set X. We say that $\alpha \in T_X$ preserves π if, for all $x, y \in X$, $(x, y) \in \pi$ implies $(x\alpha, y\alpha) \in \pi$.*

Definition 2.5 ([10]). *Let E be an equivalence relation on a set X. A selfmap $\alpha : X \to X$ is said to be E^*-preserving if α satisfies the following: $(x, y) \in E$ if and only if $(x\alpha, y\alpha) \in E$.*

Remark 2.1. In view of Definition 2.4, an E^*-preserving map preserves E and satisfies the condition that $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$.

Definition 2.6 ([10]). *Let $\mathcal{P} = \{X_i \mid i \in I\}$ be a partition of an arbitrary set X, and let $\alpha \in T(X, \mathcal{P})$. The character of α is a selfmap $\chi^{(\alpha)} : I \to I$ defined by $i\chi^{(\alpha)} = j$ whenever $X_i\alpha \subseteq X_j$.*

Denote by $\Sigma(X, \mathcal{P})$ the set of all $\alpha \in T(X, \mathcal{P})$ whose image intersects every block of $\mathcal{P}$. Sarkar and Singh [10] gave a characterization of elements in $\Sigma(X, \mathcal{P})$. It is useful in proving our result on the separator of an $\mathscr{R}$-class.

Corollary 2.1 ([10]). *Let $\mathcal{P} = \{X_1, \ldots, X_m\}$ be an m-partition associated with an equivalence relation E on an arbitary set X, and let $\alpha \in T(X, \mathcal{P})$. Then, the following statements are equivalent:*

(i) $\alpha \in \Sigma(X, \mathcal{P})$.

(ii) $\chi^{(\alpha)}$ *is a bijective map on* $\{1, \ldots, m\}$.

(iii) α *is an* E^**-preserving map.*

3. Main Results

In view of the definition of the separator of a subset of a semigroup [8], we have the following remark.

Remark 3.1. Let S be a semigroup. Let $A \subseteq S$ and $x \in S$. Then, $x \in Sep(A)$ if and only if x satisfies the following four conditions:

i) $xa \in A$, for all $a \in A$.

ii) $ax \in A$, for all $a \in A$.

iii) $xb \in S \backslash A$, for all $b \in S \backslash A$.

iv) $bx \in S \backslash A$, for all $b \in S \backslash A$.

Remark 3.2 ([8]). Let S be a semigroup. Then, $Sep(\emptyset) = Sep(S) = S$.

Using Theorem 2.2 (ii), Nagy proved the following result.

Theorem 3.1 ([8]). *If I is a proper ideal of T_n, then $Sep(I) = S_n$.*

3.1 The separator of $\mathscr{D}$-classes

Lemma 3.1. *If $k \geq 2$ and $\beta \in T_n \backslash S_n$, then $\exists \alpha \in D_k$ such that $rank(\alpha\beta) \leq k-1$.*

Proof. Suppose $k \geq 2$ and $\beta \in T_n \backslash S_n$. Then, $\exists x \neq y$ such that $x\beta = y\beta$. Choose an element $\alpha \in D_k$ such that $x, y \in Im\alpha$. Then, $|\underline{n}/\pi_\alpha| = |Im\alpha| = k$ so we may choose distinct elements $p_1, p_2, \ldots, p_k \in \underline{n}$ such that the equivalence classes $[p_s]_{\pi_\alpha}$ and $[p_t]_{\pi_\alpha}$ are disjoint for $s \neq t$. Let $m_i = p_i\alpha$ for $i = 1, 2, \ldots, k$. Then, $Im\alpha = \{m_1, m_2, \ldots, m_k\}$. Since $x, y \in Im\alpha$, we have $x = m_{i_1}$ and $y = m_{i_2}$, for some $1 \leq i_1, i_2 \leq k$ with $i_1 \neq i_2$; hence, $m_{i_1}\beta = x\beta = y\beta = m_{i_2}\beta$. Note that, $(Im\alpha)\beta = \{m_i\beta \mid i = i_1, i_2\} \cup \{m_i\beta \mid i \in \{1, 2, \ldots, k\} \backslash \{i_1, i_2\}\}$. Therefore, $|Im(\alpha\beta)| = |(Im\alpha)\beta| \leq 1 + (k-2) = k-1$. □

Applying Lemma 2.3, we have the following results.

Lemma 3.2. *If $\alpha \in S_n, \beta \in D_k$, and $\gamma \in T_n \backslash D_k$, then $\alpha\beta, \beta\alpha \in D_k$ and $\alpha\gamma, \gamma\alpha \in T_n \backslash D_k$.*

Lemma 3.3. *If $\alpha \in S_n$ and $\beta \in \bigcup_{i=1}^{m} D_{k_i}$, then $\alpha\beta, \beta\alpha \in \bigcup_{i=1}^{m} D_{k_i}$.*

Lemma 3.4. *Let $\alpha, \gamma \in T_n$. If $\alpha \in S_n$ and $\gamma \notin \bigcup_{i=1}^{m} D_{k_i}$, where $m < n$, then $\alpha\gamma, \gamma\alpha \notin \bigcup_{i=1}^{m} D_{k_i}$.*

Theorem 3.2. *$Sep(D_k) = S_n$*

Proof. If $n = 1$, then $D_1 = S_1 = T_1$. By Remark 3.2, $Sep(D_1) = Sep(T_1) = T_1 = S_1$. Suppose $n \geq 2$ and $k = 1$. Note that, D_1 is a proper ideal of T_n. By Theorem 3.1, $Sep(D_1) = S_n$. Suppose $k \geq 2$. By Lemma 3.2, $S_n \subseteq Sep(D_k)$. Suppose $\beta \notin S_n$. By Lemma 3.1, $\exists \alpha \in D_k$ such that $rank(\alpha\beta) \leq k-1$. Hence, $\alpha\beta \notin D_k$. Therefore, $\beta \notin Sep(D_k)$. □

Next, we investigate the separator of union of $\mathscr{D}$-classes. The following result is a generalization of Theorem 3.1.

Theorem 3.3. *If $1 \leq k_1 < \ldots < k_m \leq n$ where $m < n$, then $Sep(\bigcup_{i=1}^{m} D_{k_i}) = S_n$.*

Proof. If $m = 1$, apply Theorem 3.2. Suppose $m \geq 2$. By Lemmas 3.3 and 3.4, $S_n \subseteq Sep(\bigcup_{i=1}^{m} D_{k_i})$. Suppose $\alpha \notin S_n$.

Case 1. $k_1 \geq 2$. By Lemma 3.1, $\exists\beta \in D_{k_1}$ such that $rank(\beta\alpha) \leq k_1 - 1$. Hence, $\beta\alpha \notin \bigcup_{i=1}^{m} D_{k_i}$. Therefore, $\alpha \notin Sep(\bigcup_{i=1}^{m} D_{k_i})$.

Case 2. $k_1 = 1$. Suppose $k_1, k_2, \ldots, k_m$ are consecutive positive integers. Then, $\bigcup_{i=1}^{m} D_{k_i}$ is a proper ideal of T_n. Since $\alpha \notin S_n$, by Theorem 3.1, $\alpha \notin Sep(\bigcup_{i=1}^{m} D_{k_i})$. Suppose $k_{i+1} - k_i > 1$, for some $1 \leq i \leq m - 1$. By the Well-ordering principle, $b = \min\{i \mid k_{i+1} - k_i > 1\}$ exists. Then, $k_1, \ldots, k_b$ are consecutive positive integers and $k_b < k_b + 1 < k_{b+1}$. But Lemma 3.1 tells us that $\exists\beta$ with $rank(\beta) = k_b + 1$ such that $rank(\beta\alpha) \leq k_b$. Note that, $\beta \notin \bigcup_{i=1}^{m} D_{k_i}$ but $\beta\alpha \in \bigcup_{i=1}^{m} D_{k_i}$. Therefore, $\alpha \notin Sep(\bigcup_{i=1}^{m} D_{k_i})$. □

3.2 The separator of $\mathscr{L}$-classes

Given a subset Y of $\underline{n}$ with $|Y| = k$, let $S_n(Y) = \{\alpha \in S_n \mid Y\alpha = Y\}$ and $L_k(Y) = \{\alpha \in D_k \mid Im\alpha = Y\}$.

Remark 3.3. If $n = k = 1$, then $|Y| = 1$ so that $L_1(Y) = T_1 = S_1 = S_1(Y)$. Then, $Sep(L_1(Y)) = Sep(T_1) = T_1 = S_1(Y)$.

We will show that $S_n(Y)$ is the separator of the $\mathscr{L}$-class consisting of all elements of T_n whose image is Y. The next two lemmas follow immediately from the properties of S_n and $L_k(Y)$.

Lemma 3.5. *If $\alpha \in S_n(Y)$ and $\beta \in L_k(Y)$, then $\alpha\beta, \beta\alpha \in L_k(Y)$.*

Lemma 3.6. *If $\alpha \in S_n(Y)$ and $\beta \in T_n \setminus L_k(Y)$, then $\alpha\beta, \beta\alpha \in T_n \setminus L_k(Y)$.*

For $m = 1, \ldots, n$, let c_m denote the constant transformation on $\underline{n}$ defined by $x \mapsto m$.

Theorem 3.4 ([2]). *Let $n \geq 2$. If $A = \{c_{k_1}, \ldots, c_{k_r}\}$, then $Sep(A) = S_n(K)$, where $K = \{k_1, \ldots, k_r\}$.*

Lemma 3.7. *If $k \geq 2$ and $\alpha \in T_n \setminus S_n$ with $Y\alpha = Y$, then $\exists\gamma \in L_k(Y)$ such that $\alpha\gamma \notin L_k(Y)$.*

Proof. Suppose $k \geq 2$ and $\alpha \in T_n \setminus S_n$ with $Y\alpha = Y$. Since $\alpha \notin S_n$, it is not surjective. Let $s \in \underline{n} \setminus Im\alpha$, $Y = \{y_1, \ldots, y_k\}$, and $Z = \underline{n} \setminus (Y \cup \{s\})$. Then, $s \notin Y$ since $Y = Y\alpha \subseteq Im\alpha$. For $i = 1, 2, \ldots, k$, let

$$P_i = \begin{cases} \{s\}, & \text{if } i = 1 \\ \{y_1, y_2\} \cup Z, & \text{if } i = 2 \\ \{y_i\}, & \text{if } i \notin \{1, 2\}. \end{cases}$$

Consider $\gamma : \underline{n} \to \underline{n}$ where $\underline{n}/\pi_\gamma = \{P_1, \ldots, P_k\}$ and $P_i\gamma = \{y_i\}, \forall i = 1, 2, \ldots, k$. Then, $\gamma \in L_k(Y)$ since $Im\gamma = Y$. Note that, $P_1\gamma = \{s\}\gamma = \{y_1\}$.

Claim. $y_1 \notin Im\alpha\gamma$. Suppose $y_1 \in Im\alpha\gamma$. Then, $\exists x \in Im\alpha$ such that $x\gamma = y_1 = s\gamma$. Hence, $(x, s) \in \pi_\gamma$ which implies that $x \in [s]_{\pi_\gamma} = P_1$. Then, $x = s$, a contradiction, since $s \notin Im\alpha$. Hence, $y_1 \notin Im\alpha\gamma$ which implies that $Y \neq Im\alpha\gamma$. Therefore, $\alpha\gamma \notin L_k(Y)$. □

Theorem 3.5. $Sep(L_k(Y)) = S_n(Y)$

Proof. If $n = k = 1$, by Remark 3.3, $Sep(L_1(Y)) = S_1(Y)$. Suppose $n \geq 2$ and $k = 1$. Then, $|Y| = 1$. Let $Y = \{m\}$. Then, $L_1(Y) = \{c_m\}$. By Theorem 3.4, $Sep(L_1(Y)) = S_n(Y)$. Now, suppose $k \geq 2$. By Lemmas 3.5 and 3.6, $S_n(Y) \subseteq Sep(L_k(Y))$. Suppose $\alpha \notin S_n(Y)$.

Case 1. $Y\alpha \neq Y$. Let $\beta \in L_k(Y)$. Then, $Im\beta\alpha = (Im\beta)\alpha = Y\alpha \neq Y$ which implies that $\beta\alpha \notin L_k(Y)$. Therefore, $\alpha \notin Sep(L_k(Y))$.

Case 2. $\alpha \notin S_n$ with $Y\alpha = Y$. By Lemma 3.7, $\alpha \notin Sep(L_k(Y))$. □

3.3 The separator of $\mathscr{R}$-classes

The next two lemmas are immediate from the definitions.

Lemma 3.8. *Let $\alpha, \beta \in T_X$ and $x, y \in X$. Then, $(x\alpha, y\alpha) \in \pi_\beta$ if and only if $(x, y) \in \pi_{\alpha\beta}$.*

Lemma 3.9. *If $\alpha, \beta \in T_X$, then $\pi_\alpha \subseteq \pi_{\alpha\beta}$.*

Lemma 3.10. *If $\alpha \in S_X$ and $\beta \in T_X$, then $\pi_{\beta\alpha} = \pi_\beta$.*

Proof. Let $x, y \in X$. Since α is injective,

$$x(\beta\alpha) = y(\beta\alpha) \iff (x\beta)\alpha = (y\beta)\alpha \iff x\beta = y\beta. \qquad \square$$

Let π be an equivalence relation on $\underline{n}$. Then, $\underline{n}/\pi$ is a partition of $\underline{n}$. Denote by $T_n(\pi)$ the semigroup $T(\underline{n}, \underline{n}/\pi)$. Moreover, let $\Sigma_n(\pi) = \Sigma(\underline{n}, \underline{n}/\pi)$ and $S_n(\pi) = S(\underline{n}, \underline{n}/\pi)$. Since $S_n(\pi) = T_n(\pi) \cap S_n$ and $S_n(\pi) \subseteq \Sigma_n(\pi) \subseteq T_n(\pi)$, we have $S_n(\pi) \subseteq S_n \cap \Sigma_n(\pi) \subseteq S_n \cap T_n(\pi) = S_n(\pi)$. Thus, we have the following remark.

Remark 3.4. $S_n(\pi) = S_n \cap \Sigma_n(\pi) = S_n \cap T_n(\pi)$.

Let $R_k(\pi)$ denote the $\mathscr{R}$-class consisting of all $\alpha \in D_k$ with partition π.

Lemma 3.11. *If $\alpha \in S_n$ and $\beta \in R_k(\pi)$, then $\beta\alpha \in R_k(\pi)$.*

Proof. By Lemma 3.10, $\pi_{\beta\alpha} = \pi_\beta = \pi$. Therefore, $\beta\alpha \in R_k(\pi)$. □

Lemma 3.12. *If $\alpha \in \Sigma_n(\pi)$ and $\beta \in R_k(\pi)$, then $\alpha\beta \in R_k(\pi)$.*

Proof. Let $x, y \in \underline{n}$. By Corollary 2.1, α is π^*-preserving. Then, by Lemma 3.8, $(x, y) \in \pi_{\alpha\beta} \iff (x\alpha, y\alpha) \in \pi_\beta = \pi \iff (x, y) \in \pi$. Thus, $\pi_{\alpha\beta} = \pi$. Therefore, $\alpha\beta \in R_k(\pi)$. □

Lemma 3.13. *If $\alpha \in S_n(\pi)$ and $\gamma \in T_n\backslash R_k(\pi)$, then $\alpha\gamma, \gamma\alpha \in T_n\backslash R_k(\pi)$.*

Proof. By Remark 3.4, $S_n(\pi) = S_n \cap \Sigma_n(\pi)$. Suppose $\alpha \in S_n(\pi)$ and $\gamma \in T_n\backslash R_k(\pi)$. Since $S_n(\pi)$ is a group, $\alpha^{-1} \in S_n(\pi)$. Suppose $\gamma \notin D_k$. By Lemma 3.2, $\alpha\gamma, \gamma\alpha \notin D_k$ which implies that $\alpha\gamma, \gamma\alpha \notin R_k(\pi)$. Suppose $\pi_\gamma \neq \pi$.

Case 1. $\pi \not\subseteq \pi_\gamma$. Then, $\exists(u, v) \in \pi$ such that $(u, v) \notin \pi_\gamma$. Then, $u\gamma \neq v\gamma$. Since α is injective, $u\gamma\alpha \neq v\gamma\alpha$. Then, $(u, v) \notin \pi_{\gamma\alpha}$. Thus, $\pi \neq \pi_{\gamma\alpha}$. Let $u' = u\alpha^{-1}$ and $v' = v\alpha^{-1}$. Then, $u'\alpha = u$ and $v'\alpha = v$. Since α^{-1} preserves π, we have that

$$(u, v) \in \pi \implies (u\alpha^{-1}, v\alpha^{-1}) \in \pi \implies (u', v') \in \pi.$$

However, since $u'\alpha\gamma = u\gamma \neq v\gamma = v'\alpha\gamma$, we have $(u', v') \notin \pi_{\alpha\gamma}$. Thus, $\pi \neq \pi_{\alpha\gamma}$. Therefore, $\alpha\gamma, \gamma\alpha \notin R_k(\pi)$.

Case 2. $\pi_\gamma \not\subseteq \pi$. Then, $\exists(x, y) \in \pi_\gamma$ such that $(x, y) \notin \pi$. Then, $x\gamma = y\gamma$ and

$$x\gamma = y\gamma \implies x\gamma\alpha = y\gamma\alpha \implies (x, y) \in \pi_{\gamma\alpha}.$$

Thus, $\pi_{\gamma\alpha} \neq \pi$. Let $x' = x\alpha^{-1}$ and $y' = y\alpha^{-1}$. Then, $x'\alpha = x$ and $y'\alpha = y$. By Corollary 2.1, α^{-1} is π^*-preserving. Then

$$(x, y) \notin \pi \implies (x\alpha^{-1}, y\alpha^{-1}) \notin \pi \implies (x', y') \notin \pi.$$

However, since $x'\alpha\gamma = x\gamma = y\gamma = y'\alpha\gamma$, we have $(x', y') \in \pi_{\alpha\gamma}$. Thus, $\pi_{\alpha\gamma} \neq \pi$. Therefore, $\alpha\gamma, \gamma\alpha \notin R_k(\pi)$. □

Note that, $|\underline{n}/\pi| = 1$ if and only if $\underline{n}/\pi = \{\underline{n}\}$. Clearly, $R_1(\pi) \subseteq D_1$. Let $\alpha \in D_1$. Then, α has rank 1 which means that it only has one equivalence class. Then, $\pi_\alpha = \pi$. Thus, we have the following remark

Remark 3.5. $R_1(\pi) = D_1$.

Theorem 3.6. *$Sep(R_k(\pi)) = S_n(\pi)$.*

Proof. Suppose $k = 1$. By Theorem 3.2, $Sep(R_1(\pi)) = Sep(D_1) = S_n = S_n(\pi)$. Suppose $k \geq 2$. Since $S_n(\pi) = S_n \cap \Sigma_n(\pi)$, by Lemmas 3.11, 3.12, and 3.13, $S_n(\pi) \subseteq Sep(R_k(\pi))$. Now, suppose $\alpha \notin S_n(\pi)$. Let $\beta \in R_k(\pi)$.

Case 1. $\alpha \notin T_n(\pi)$. Then, α does not preserve π; hence, $\exists(x, y) \in \pi$ such that $(x\alpha, y\alpha) \notin \pi = \pi_\beta$. By Lemma 3.8, $(x, y) \notin \pi_{\alpha\beta}$. Thus, $\pi \neq \pi_{\alpha\beta}$ which implies that $\alpha\beta \notin R_k(\pi)$. Therefore, $\alpha \notin Sep(R_k(\pi))$.

Case 2. $\alpha \notin S_n$. Then, $\exists x, y \in \underline{n}$ with $x \neq y$ such that $x\alpha = y\alpha$. Suppose $(x, y) \notin \pi$. Since $x\alpha\beta = y\alpha\beta$, we have $(x, y) \in \pi_{\alpha\beta}$. Thus, $\pi_{\alpha\beta} \neq \pi$ which implies that $\alpha\beta \notin R_k(\pi)$. Therefore, $\alpha \notin Sep(R_k(\pi))$.

Suppose $(x, y) \in \pi$. Since $k \geq 2$, we can choose $q \in \underline{n}$ such that $(x, q) \notin \pi$. Consider an element $\gamma \in R_k(\pi)$ such that $x\gamma = x$ and $q\gamma = y$. Then, $x\gamma\alpha = x\alpha = y\alpha = q\gamma\alpha$ which implies that $(x, q) \in \pi_{\gamma\alpha}$. Thus, $\pi_{\gamma\alpha} \neq \pi$. It follows that $\gamma\alpha \notin R_k(\pi)$. Therefore, $\alpha \notin Sep(R_k(\pi))$. □

3.4 The separator of $\mathscr{H}$-classes

For a partition π of $\underline{n}$ and $Y \subseteq \underline{n}$ with $|\underline{n}/\pi| = |Y|$, let $H_k(\pi, Y)$ denote the $\mathscr{H}$-class consisting of all $\alpha \in D_k$ with partition π and image Y. Clearly, $H_k(\pi, Y) = R_k(\pi) \cap L_k(Y)$. Moreover, denote by $S_n(\pi, Y)$ the intersection of $S_n(\pi)$ and $S_n(Y)$. We will show that $S_n(\pi, Y)$ is the separator of $H_k(\pi, Y)$.

Lemma 3.14. *$S_n(\pi, Y) \subseteq Sep(H_k(\pi, Y))$.*

Proof. Suppose $\alpha \in S_n(\pi, Y)$. Let $\beta \in H_k(\pi, Y)$. Applying Lemma 3.5, we have $\alpha\beta, \beta\alpha \in L_k(Y)$. Then, by Lemma 3.12, $\alpha\beta \in R_k(\pi)$. Moreover, by Lemma 3.10, $\pi_{\beta\alpha} = \pi_\beta = \pi$, which implies that $\beta\alpha \in R_k(\pi)$. Therefore, $\alpha\beta, \beta\alpha \in H_k(\pi, Y)$. Let $\gamma \in T_n \backslash H_k(\pi, Y)$. Suppose $\gamma \notin R_k(\pi)$. By Lemma 3.13, $\alpha\gamma, \gamma\alpha \notin R_k(\pi)$. Suppose $\gamma \notin L_k(Y)$. By Lemma 3.6, $\alpha\gamma, \gamma\alpha \notin L_k(Y)$. Then, $\alpha\gamma, \gamma\alpha \in T_n \backslash H_k(\pi, Y)$. Therefore, $\alpha \in Sep(H_k(\pi, Y))$. □

Lemma 3.15. *If $\alpha \in T_n \backslash S_n$ with $Y\alpha = Y$ such that $(\underline{n} \backslash Im\alpha)\alpha \cap Y \neq \emptyset$, then $\exists \beta \in T_n \backslash H_k(\pi, Y)$ such that $\beta\alpha \in H_k(\pi, Y)$.*

Proof. Let $\underline{n}/\pi = \{P_1, \ldots, P_k\}$ and $Y = \{y_1, \ldots, y_k\}$. Suppose $\alpha \in T_n \backslash S_n$ with $Y\alpha = Y$ such that $(\underline{n} \backslash Im\alpha)\alpha \cap Y \neq \emptyset$. Let $t \in (\underline{n} \backslash Im\alpha)\alpha \cap Y$. Then, $t = sa$, for some $s \in \underline{n} \backslash Im\alpha$. Since $Y = Y\alpha$, $\exists y_m \in Y$ such that $t = y_m\alpha$. Note that, $s \notin Y$ since $Y = Y\alpha \subseteq Im\alpha$. Let $Y' = Y \backslash \{y_m\} \cup \{s\}$ and consider $\beta \in T_n$ with $\pi_\beta = \pi$ and $Im\beta = Y'$, where $P_m\beta = \{s\}$ and $P_i\beta = \{y_i\}$, for all $i \neq m$. Since $Im\beta \neq Y$, we have $\beta \notin H_k(\pi, Y)$. By Lemma 3.9, $\pi_\beta \subseteq \pi_{\beta\alpha}$.

Claim. $\pi_{\beta\alpha} \subseteq \pi_\beta$. Suppose $(x, y) \notin \pi_\beta$. Then, $x\beta \neq y\beta$. Then, at least one of $x\beta$ or $y\beta$ must belong to Y; otherwise, $x\beta = s = y\beta$, a contradiction. Suppose both are elements of Y, that is, $x\beta, y\beta \in Y$. Since $Y\alpha = Y$, the map $\alpha|_Y : Y \to Y$ is surjective hence injective. Then, $x\beta\alpha \neq y\beta\alpha$ which implies that $(x, y) \notin \pi_{\beta\alpha}$. Suppose only one of them is an element of Y. Without loss of generality, assume $x\beta \in Y$ and $y\beta \notin Y$. Then, $x\beta = y_i$, for some $i \neq m$ and $y\beta = s$. Since $\alpha|_Y$ is injective, we have

$$x\beta\alpha = y_i\alpha \neq y_m\alpha = t = s\alpha = y\beta\alpha.$$

Hence, $(x, y) \notin \pi_{\beta\alpha}$. This proves our claim. We have shown that $\pi_{\beta\alpha} = \pi_\beta = \pi$. Moreover, $P_m\beta\alpha = \{s\}\alpha = \{t\} = \{y_m\alpha\}$ and $P_i\beta\alpha = \{y_i\}\alpha = \{y_i\alpha\}$, for all $i \neq m$. Hence, $Im\beta\alpha = Y\alpha = Y$. Therefore, $\beta\alpha \in H_k(\pi, Y)$. □

Lemma 3.16. *If $\alpha \in \Sigma_n(\pi) \backslash S_n$ with $Y\alpha = Y$, then $\alpha \notin Sep(H_k(\pi, Y))$.*

Proof. Let $\underline{n}/\pi = \{P_1, \ldots, P_k\}$ and $Y = \{y_1, \ldots, y_k\}$. Suppose $\alpha \in \Sigma_n(\pi) \backslash S_n$ with $Y\alpha = Y$. Since $\alpha \notin S_n$, α is not surjective; hence, $\underline{n} \backslash Im\alpha \neq \emptyset$. If $(\underline{n} \backslash Im\alpha)\alpha \cap Y \neq \emptyset$, by Lemma 3.15, $\alpha \notin Sep(H_k(\pi, Y))$. Suppose $(\underline{n} \backslash Im\alpha)\alpha \cap Y = \emptyset$. Let $s \in \underline{n} \backslash Im\alpha$. Then, $s\alpha \notin Y$. Since $\underline{n}/\pi$ is a partition of $\underline{n}$, $s \in P_j$, for some j with $1 \leq j \leq k$. Then, by Corollary 2.1, $\chi^{(\alpha)}$ is bijective; hence there exists m with $1 \leq m \leq k$ such that $m\chi^{(\alpha)} = j$, that is, $P_m\alpha \subseteq P_j$. Let $z \in P_m$.

Then, $z\alpha \in P_j$ and $z\alpha \neq s$, since $s \notin Im\alpha$. Thus, $P_j \backslash \{s\} \neq \emptyset$. Consider an element $\beta \in D_{k+1}$ with $\underline{n}/\pi_\beta = \{Q_1, \ldots, Q_{k+1}\}$, where

$$Q_i = \begin{cases} P_i, & \text{if } i \notin \{j, k+1\} \\ P_j \backslash \{s\}, & \text{if } i = j \\ \{s\}, & \text{if } i = k+1, \end{cases}$$

with $Q_i\beta = \{y_i\}$, for all $i = 1, \ldots, k$ and $Q_{k+1}\beta = \{s\alpha\}$.

Claim 1. $\pi_\beta \subseteq \pi$. Suppose $(x, y) \in \pi_\beta$. Then, $x, y \in Q_i$, for some $1 \leq i \leq k+1$. If $i \notin \{j, k+1\}$, then $x, y \in P_i$. If $i = j$, then $Q_i = P_j \backslash \{s\}$ so $x, y \in P_j$. If $i = k+1$, then $x = y = s$. Thus, $(x, y) \in \pi$. This proves Claim 1.

Claim 2. $(x\alpha, y\alpha) \in \pi$ implies $(x\alpha, y\alpha) \in \pi_\beta$. Suppose $(x\alpha, y\alpha) \in \pi$. Then $x\alpha, y\alpha \in P_i$, for some $1 \leq i \leq k$. If $i \neq j$, then $x\alpha, y\alpha \in Q_i$. We now consider the case where $i = j$. Then, $x\alpha, y\alpha \in P_j$. Note that, $x\alpha$ and $y\alpha$ are both not equal to s, since $s \notin Im\alpha$. Then

$$x\alpha, y\alpha \in P_j \implies x\alpha, y\alpha \in P_j \backslash \{s\} \implies x\alpha, y\alpha \in Q_j.$$

Thus, $(x\alpha, y\alpha) \in \pi_\beta$. This proves Claim 2. Note that, the converse of Claim 2 is true by Claim 1. By Corollary 2.1, α is π^*-preserving. By Lemma 3.8,

$$(x, y) \in \pi_{\alpha\beta} \iff (x\alpha, y\alpha) \in \pi_\beta \iff (x\alpha, y\alpha) \in \pi \iff (x, y) \in \pi.$$

Thus, $\pi_{\alpha\beta} = \pi$. Let $P_i \in \underline{n}/\pi$. Then, by Corollary 2.1, $\chi^{(\alpha)}$ is bijective; hence there exists i^* with $1 \leq i^* \leq k$ such that $i^*\chi^{(\alpha)} = i$, that is, $P_{i^*}\alpha \subseteq P_i$. For $i \neq j$, we have $P_i = Q_i$. Then $P_{i^*}\alpha\beta \subseteq P_i\beta = Q_i\beta = \{y_i\}$. Suppose $i = j$. Then, $Q_j = P_j \backslash \{s\}$. Since $s \notin Im\alpha$, we have $s \notin P_{j^*}\alpha$ which implies that $P_{j^*}\alpha \subseteq P_j \backslash \{s\}$. Then $P_{j^*}\alpha\beta \subseteq (P_j \backslash \{s\})\beta = Q_j\beta = \{y_j\}$. Hence, $Im\alpha\beta = Y$. Note that, $\beta \notin H_k(\pi, Y)$ but $\alpha\beta \in H_k(\pi, Y)$. Therefore, $\alpha \notin Sep(H_k(\pi, Y))$. □

Theorem 3.7. $Sep(H_k(\pi, Y)) = S_n(\pi, Y)$.

Proof. By Lemma 3.14, $S_n(\pi, Y) \subseteq Sep(H_k(\pi, Y))$. Suppose $\alpha \notin S_n(\pi, Y)$. Let $T_n(Y) = \{\alpha \in T_n \mid Y\alpha = Y\}$. Note that,

$$S_n(\pi, Y) = S_n(\pi) \cap S_n(Y) = S_n \cap \Sigma_n(\pi) \cap S_n \cap T_n(Y) = \Sigma_n(\pi) \cap S_n \cap T_n(Y).$$

Let $\beta \in H_k(\pi, Y)$. Then, $\pi_\beta = \pi$ and $Im\beta = Y$.

Case 1. $\alpha \notin T_n(Y)$. Then, $Y\alpha \neq Y$ which implies that $Im\beta\alpha = (Im\beta)\alpha = Y\alpha \neq Y$. Thus, $\beta\alpha \notin H_k(\pi, Y)$. Therefore, $\alpha \notin Sep(H_k(\pi, Y))$.

Case 2. $\alpha \notin \Sigma_n(\pi)$. Suppose $\alpha \in T_n \backslash T_n(\pi)$. Then, α does not preserve π so $\exists (x, y) \in \pi$ such that $(x\alpha, y\alpha) \notin \pi = \pi_\beta$. By Lemma 3.8, $(x, y) \notin \pi_{\alpha\beta}$. Thus, $\pi \neq \pi_{\alpha\beta}$ which implies that $\alpha\beta \notin H_k(\pi, Y)$.

Suppose $\alpha \in T_n(\pi) \backslash \Sigma_n(\pi)$. Then, α preserves π but is not π^*-preserving. By Remark 2.1, $\exists (u\alpha, v\alpha) \in \pi$ such that $(u, v) \notin \pi$. Since $\pi = \pi_\beta$, by Lemma

3.8, $(u, v) \in \pi_{\alpha\beta}$. Thus, $\pi_{\alpha\beta} \neq \pi$ which implies that $\alpha\beta \notin H_k(\pi, Y)$. Therefore, $\alpha \notin Sep(H_k(\pi, Y))$.

Case 3. $\alpha \notin S_n$ but $\alpha \in \Sigma_n(\pi)$ with $Y\alpha = Y$. Then, by Lemma 3.16, $\alpha \notin Sep(H_k(\pi, Y))$. □

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