Some even-odd mean graphs in the context of arbitrary super subdivision

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Abstract. In this paper, we establish some new results on even vertex odd mean labeling of graph. We prove that the graphs obtained by arbitrary super subdivision of cycle, comb, crown, slanting ladder and planar grid are even-odd mean graphs.

Keywords: labeling, even-odd mean labeling, biclique, arbitrary super subdivision.
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1. Introduction

Unless mentioned or otherwise, all graphs in this paper are simple, finite, connected and undirected. For all other standard terminology and notations in graph theory we follow Harary [7]. A (p, q)-graph G is a graph such that $\mid V(G) \mid = p$ and $\mid E(G) \mid = q$. A labeling (or valuation) of a graph is a function that carries graphs elements to numbers usually to non negative integers or positive. If the domain is the vertex set or edge set the labeling called vertex labeling or edge labeling respectively. If the domain is both vertices and edges then the labeling is called total labeling. According to Beineke and Hegde [1] graph labeling serves as frontier between number theory and structure of graphs. Labeled graph have variety of applications in coding theory, mathematical modeling, x-ray, crystallography and to determine optimal circuit layouts. For a dynamic survey of various graph labeling problems along with extensive bibliography we refer to Gallian [6]. The concept of even-odd mean labeling of the graph was introduced by Vasuki, Nagarajan and Arockiaraj [11]. They studied even-odd mean labeling of some standard graphs. The subject of even-odd mean labeling has been further studied in [2], [3], [4], [8], [9], [12]. The concept of super subdivision of graphs was introduced by Sethuraman and Selvaraju [10]. They proved that the arbitrary super subdivision of graphs admit graceful labeling. In [5] Basher et.al proved that the super subdivision of some families of graphs admit an even-odd mean labeling. Motivated by the work in [5], in this paper, we study the even-odd mean labeling of cycle, comb, crown, slanting ladder and planar grid. We will give a brief summary of definitions and terminology which are useful for our study.

Definition 1.1 ([11]). *A vertex labeling of G is an injective function $f : V(G) \to \{0, 2, 4, ..., 2q\}$. For a vertex labeling f, the induced edge labeling f^* is defined by $f^*(uv) = \frac{f(u)+f(v)}{2}$ for any edge uv in G, then the vertex labeling f is called even-odd mean labeling of graph of G if its induced edge labeling $f^* : E(G) \to \{1, 3, 5, ..., 2q-1\}$ is a bijection, that is $f^*(E) = \{1, 3, 5, ..., 2q-1\}$.*

If a graph G has even-odd mean labeling , then we say that G is an even-odd mean graph.

Definition 1.2 ([10]). *Let G be a graph with p vertices and q edges. A graph G' is said to be an arbitrary super subdivision of G if G' is obtained from G by replacing each edge e_i by a complete bipartite graph (biclique) K_{2,t_i} where t_i is any positive integer and may vary for each edge arbitrary by identifying the ends of each edge e_i with the two vertices of 2-vertices part of K_{2,t_i} after removing the edge from G.*

In this work a cycle on n vertices denoted by C_n, a slanting ladder $SL_n, n \geq 2$ is a graph obtained from two parallel paths with vertices $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_n$ respectively by joining each u_i with $v_{i+1}, 1 \leq i \leq n-1$. The corona $G \odot K_1$ of a graph G on p vertices $u_1, u_2, ..., u_p$ is the graph obtained from G by adding p new vertices $v_1, v_2, ..., v_p$ and joining each vertex u_i to a vertex v_i,$1 \leq i \leq n$. The graph $P_n \odot K_1$ is called a comb and the graph $C_n \odot K_1$ is called a crown. Let G_1 and G_2 be any two graphs with p_1 and p_2 vertices, respectively. The Cartesian product $G_1 \times G_2$ is the graph such that $V = p_1p_2$ with vertices set $\{(u, v) : u \in G_1, v \in G_2\}$ and the two vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1 = u_2$ and v_1, v_2 are adjacent in G_2 or $v_1 = v_2$ and u_1, u_2 are adjacent in G_1. The product $P_m \times P_n$ is called a planar grid and $P_2 \times P_n$ is called ladder, denoted by L_n. Let a and b be two positive numbers, we refer to $[a, b]$ an interval of numbers s, where $a \leq s \leq b$.

Notation. We denote the arbitrary super subdivision of G by $ASS(G)$.

2. Main results

Theorem 2.1. *$ASS(C_n)$ is an even-odd mean graph where the edges $u_iu_{i+1}(i \in [1, n-1])$, u_nu_1 of the cycle C_n are replaced by K_{2,t_i}, K_{2,t_n} respectively, such that $n \equiv 0 \pmod 4$, $t_{\frac{n}{2}} = t_n$ and $\sum_{i=1}^{\frac{n}{2}} t_i = \sum_{i=\frac{n}{2}+1}^{n} t_i$.*

Proof. Let C_n be a cycle graph of length n, where $n \equiv 0 \pmod 4$ whose vertex set is $V = \{u_1, u_2, ..., u_n\}$ and the edge set is $E = \{e_i = u_iu_{i+1}, e_n = u_nu_1 : i \in [1, n-1]\}$. Let $ASS(C_n)$ be an arbitrary super subdivision of a cycle graph C_n. That is, for $i \in [1, n]$ each edge e_i of the cycle C_n replaced by a biclique K_{t_i}, where t_i is any positive integer, $t_{\frac{n}{2}} = t_n$ and $\sum_{i=1}^{\frac{n}{2}} t_i = \sum_{i=\frac{n}{2}+1}^{n} t_i$. Let $u_{ij}(i \in [1, n], j \in [1, t_i])$ be the vertices which are used for arbitrary super subdivision. Therefore, the edge set is $E(ASS(C_n)) = \{u_iu_{ij}, u_{ij}u_{i+1}, u_{nj}u_1 : i \in [1, n], j \in$

$[1, t_i]\}$. Then, it is clear that $ASS(C_n)$ has $n + \sum_{i=1}^{n} t_i$ vertices and $2\sum_{i=1}^{n} t_i$ edges. Define labeling $f : V(ASS(C_n)) \to \{0, 2, 4, ..., 2q-2, 2q = 4\sum_{s=1}^{n} t_s\}$ as follows:

$$f(u_i) = \begin{cases} 0, & i = 1 \\ 4\sum_{s=1}^{i-1} t_s, & i \in [2, \frac{n}{2}] \\ 4\sum_{s=1}^{i-1} t_s + 4t_{\frac{n}{2}}, & i \in [\frac{n}{2}+1, n]. \end{cases}$$

For $j \in [1, t_i]$.

$$f(u_{ij}) = \begin{cases} 4j-2, & i = 1, \\ 4\sum_{s=1}^{i-1} t_s + 4j - 2, & i \in [2, n]. \end{cases}$$

Then, the induced edge labeling f^* is obtained as follows:

$$f^*(u_i u_{ij}) = \begin{cases} 2j-1, & i = 1 \\ 4\sum_{s=1}^{i-1} t_s + 2j - 1, & i \in [2, \frac{n}{2}] \\ 4\sum_{s=1}^{i-1} t_s + 2t_{\frac{n}{2}} + 2j - 1, & i \in [\frac{n}{2}+1, n]. \end{cases}$$

$$f^*(u_{ij} u_{(i+1)}) = \begin{cases} 2t_1 + 2j - 1, & i = 1 \\ 4\sum_{s=1}^{i-1} t_s + 2t_i + 2j - 1, & i \in [2, \frac{n}{2}-1] \\ 4\sum_{s=1}^{\frac{n}{2}} t_s + 2j - 1, & i = \frac{n}{2} \\ 4\sum_{s=1}^{i-1} t_s + 2(t_{\frac{n}{2}} + t_i) + 2j - 1, & i \in [\frac{n}{2}+1, n-1]. \end{cases}$$

$$f^*(u_{nj} u_1) = 2\sum_{s=1}^{n-1} t_s + 2j - 1.$$

Hence, f is an even-odd mean labeling of $ASS(C_n)$. Thus, $ASS(C_n)$ is an even-odd mean graph. □

Illustration 2.1. Consider $ASS(C_8)$ where the edges $u_i u_{i+1}, i \in [1, 7]$, $u_8 u_1$ are replaced by $K_{2,3}, K_{2,2}, K_{2,5}, K_{2,4}, K_{2,3}, K_{2,4}, K_{2,3}$ and $K_{2,4}$ respectively. An even-odd mean labeling of $ASS(C_8)$ is shown in Figure 1.

Theorem 2.2. *$ASS(P_n \odot K_1)$ is an even-odd mean graph where the edges $u_i u_{i+1}, u_i v_i$ and $u_n v_n$ are replaced by $K_{2,t_i}, K_{2,t_i'}$ and $K_{2,t_n'}$ respectively, such that $t_i' = t_{i+1}'$ when i is even, $i \in [1, n-1]$.*

Proof. Let $P_n \odot K_1$ be a comb graph. Let $u_i (i \in [1, n])$ be the vertices of the path P_n and v_i be the pendant vertex adjacent to $u_i (i \in [1, n-1])$. Then, the vertex set is $V = \{u_i, v_i : i \in [1, n]\}$ and the edge set is $E = \{e_i = u_i u_{i+1}, e_i' = u_i v_i, e_n' = u_n v_n : i \in [1, n-1]\}$. Let $ASS(P_n \odot K_1)$ be an arbitrary super subdivision of a comb graph $P_n \odot K_1$. The edges e_i, e_i' and

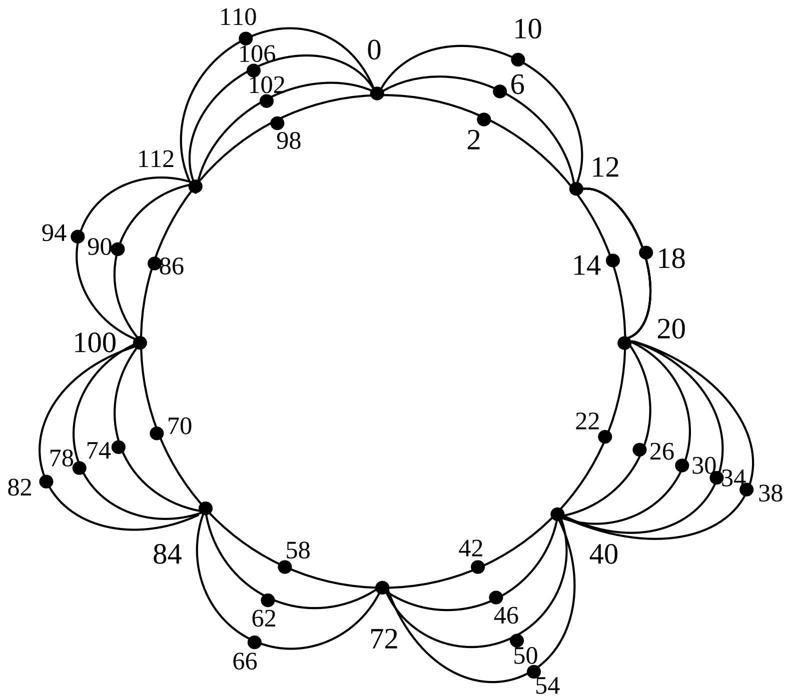

Figure 1: An even-odd mean graph of $ASS(C_8)$

$e_n^{'}(i \in [1, n-1])$ are replaced by bicliques K_{t_i},$K_{t_i^{'}}$ and $K_{t_n^{'}}(i \in [1, n-1])$ respectively, where $t_i, t_i^{'}$ are positive integer numbers, $t_i^{'} = t_{i+1}^{'}$ when i is even. Let $u_{ij}(i \in [1, n-1], j \in [1, t_i])$, $w_{ij}(i \in [1, n], j \in [1, t_i^{'}])$ be the vertices which are used for arbitrary super subdivision of $P_n \odot K_1$. Thus, the edge set is $E(ASS(P_n \odot K_1)) = \{\{u_i u_{ij}, u_{ij}u_{i+1} : i \in [1, n-1], j \in [1, t_i^{'}]\} \cup \{u_i w_{ij}, w_{ij} v_i : i \in [1, n], j \in [1, t_i^{'}]\}\}$. Here, we note that $ASS(P_n \odot K_1)$ has $2n + \sum_{i=1}^{n-1} t_i + \sum_{i=1}^{n} t_i^{'}$ vertices and $2(\sum_{i=1}^{n-1} t_i + \sum_{i=1}^{n} t_i^{'})$ edges. Define labeling $f : V(ASS(P_n \odot K_1)) \to \{0, 2, 4, ..., 2q-2, 2q = 4(\sum_{i=1}^{n-1} t_i + \sum_{i=1}^{n} t_i^{'})\}$ as follows:

$$f(u_i) = \begin{cases} 4t_1^{'}, & i = 1 \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}), & i \in [2, n], \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}), & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

$$f(v_i) = \begin{cases} 0, & i = 1 \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}), & i \in [2, n], \text{ and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}), & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i]$.

$$f(u_{ij}) = \begin{cases} 4t_1^{'} + 4j - 2, & i = 1 \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 4j - 2, & i \in [2, n], \text{ and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i+1} t_s^{'}) + 4j - 2, & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i^{'}]$.

$$f(w_{ij}) = \begin{cases} 4j - 2, & i = 1 \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 4j - 2, & i \in [2, n], \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 4j - 2, & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

Thus, the induced edge labeling f^* is obtained as follows:
For $j \in [1, t_i]$.

$$f^*(u_i u_{ij}) =$$
$$\begin{cases} 4t_1^{'} + 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 2j - 1, & i \in [2, n], \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 2(t_{(i+1)}^{'} + t_i^{'}) + 2j - 1, & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

$$f^*(u_{ij} u_{(i+1)}) =$$
$$\begin{cases} 4t_1^{'} + 2t_1 + 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 2t_i + 2j - 1, & i \in [2, n], \text{ and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i+1} t_s^{'}) + 2t_i + 2j - 1, & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i^{'}]$.

$$f^*(u_i w_{ij}) =$$
$$\begin{cases} 2t_1^{'} + 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 2t_{(i-1)} + 2j - 1, & i \in [2, n], \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 2j - 1, & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

$$f^*(w_{ij} v_i) =$$
$$\begin{cases} 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 2(t_{i-1} + t_i^{'}) + 2j - 1, & i \in [2, n], \text{ and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + t_s^{'}) + 2t_i^{'} + 2j - 1, & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

Hence, f is an even-odd mean labeling of $ASS(P_n \odot K_1)$. Then, $ASS(P_n \odot K_1)$ is an even-odd mean graph. □

Illustration 2.2. Consider $ASS(P_7 \odot K_1)$ where the edges $u_iu_{i+1}, i \in [1,6]$ are replaced by $K_{2,3}, K_{2,5}, K_{2,4}, K_{2,3}, K_{2,4}$ and $K_{2,2}$ respectively and the edges $u_iv_i, i \in [1,7]$ are replaced by $K_{2,5}, K_{2,4}, K_{2,4}, K_{2,2}, K_{2,2}, K_{2,3}$ and $K_{2,3}$ respectively . An even-odd mean labeling of $ASS(P_7 \odot K_1)$ is shown in Figure 2.

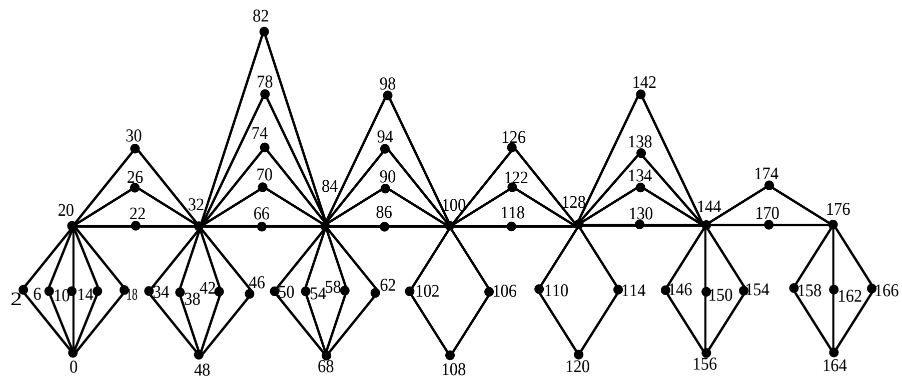

Figure 2: An even-odd mean graph of $ASS(P_7 \odot K_1)$

Theorem 2.3. *$ASS(C_n \odot K_1)$ is an even-odd mean graph where the edges $u_iu_{i+1}, u_iv_i, u_nu_1$ and u_nv_n of $C_n \odot K_1$ are replaced by $K_{2,t_i}, K_{2,t'_i}, K_{2,t_n}$ and K_{2,t'_n} respectively, such that $n \equiv 0$ (mod 4), $\sum_{i=1}^{\frac{n}{2}}(t_i + t'_i) = \sum_{i=\frac{n}{2}+1}^{n}(t_i + t'_i)$, $t_n = t_{\frac{n}{2}}$, $t'_n = t'_1$ and $t'_i = t'_{i+1}$ when i is even, $i \in [1, n-1]$.*

Proof. Let $C_n \odot K_1$ be a crown graph. Let $u_i(i \in [1,n])$ be the vertices of the cycle C_n, $n \equiv 0$ (mod 4). Let v_i be the pendant vertices adjacent to $u_i(i \in [1,n])$. Then, the vertex set of the crown $C_n \odot K_1$ is $V = \{u_i, v_i : i \in [1,n]\}$ and the edge set is $E = \{\{e_i = u_iu_{i+1}, e_n = u_nu_1 : i \in [1,n-1]\} \cup \{e'_i = u_iv_i : i \in [1,n]\}\}$. Let $ASS(C_n \odot K_1)$ be an arbitrary super subdivision of the crown $C_n \odot K_1$. The edges $e_i, e'_i(i \in [1,n])$ are replaced by the bicliques K_{2,t_i}, K_{2,t'_i} respectively where t_i, t'_i are positive integer numbers, $\sum_{i=1}^{\frac{n}{2}}(t_i + t'_i) = \sum_{i=\frac{n}{2}+1}^{n}(t_i + t'_i)$, $t_n = t_{\frac{n}{2}}$, $t'_n = t'_1$ and $t'_i = t'_{i+1}$ when i is even. Let $u_{ij}(i \in [1,n], j \in [1,t_i])$, $w_{ij}(i \in [1,n], j \in [1,t'_i])$ be the vertices which are used for arbitrary super subdivision of $C_n \odot K_1$. Therefore, the edge set of $ASS(C_n \odot K_1)$ is $E(ASS(C_n \odot K_1)) = \{\{u_iu_{ij}, u_{ij}u_{i+1}, u_{nt_n}u_1 : i \in [1,n], j \in [1,t_i]\} \cup \{u_iw_{ij}, w_{ij}v_i : i \in [1,n], j \in [1,t'_i]\}\}$. We observe that $ASS(C_n \odot K_1)$ has $2n + \sum_{i=1}^{n}(t_i + t'_i)$ vertices and $2\sum_{i=1}^{n}(t_i + t'_i)$ edges. Define labeling $f : V(ASS(C_n \odot K_1)) \to \{0, 2, 4, ..., 2q-2, 2q = 4\sum_{i=1}^{n}(t_i + t'_i)\}$ as follows:

$$f(u_i) = \begin{cases} 4t_1^{'}, & i = 1 \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}), & i \in [2, \frac{n}{2}],\ \text{and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}), & i \in [2, \frac{n}{2}],\ \text{and } i \text{ is even} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 4t_{\frac{n}{2}}, & i \in [\frac{n}{2}+1, n],\ \text{and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 4t_{\frac{n}{2}}, & i \in [\frac{n}{2}+1, n],\ \text{and } i \text{ is even.} \end{cases}$$

$$f(v_i) = \begin{cases} 0, & i = 1 \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}), & i \in [2, n],\ \text{and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}), & i \in [2, n],\ \text{and } i \text{ is even} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 4t_{\frac{n}{2}}, & i \in [\frac{n}{2}+1, n],\ \text{and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 4t_{\frac{n}{2}}, & i \in [\frac{n}{2}+1, n],\ \text{and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i]$.

$$f(u_{ij}) = \begin{cases} 4t_1^{'} + 4j - 2 & i = 1 \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 4j - 2, & i \in [2, n-1],\ \text{and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i+1} t_s^{'}) + 4j - 2, & i \in [2, n-1],\ \text{and } i \text{ is even} \\ 4\sum_{s=1}^{n-1}(t_s + t_s^{'}) + 4j - 2, & i = n. \end{cases}$$

For $j \in [1, t_i^{'}]$.

$$f(w_{ij}) = \begin{cases} 4j - 2, & i = 1 \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 4j - 2, & i \in [2, n],\ \text{and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 4j - 2, & i \in [2, n],\ \text{and } i \text{ is even} \\ 4(\sum_{s=1}^{n} t_s + \sum_{s=1}^{n-1} t_s^{'}) + 4j - 2, & i = n. \end{cases}$$

Then, the induced edge labeling f^* is obtained as follows:
For $j \in [1, t_i]$.

$$f^*(u_i u_{ij}) =$$
$$\begin{cases} 4t_1^{'} + 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 2j - 1, & i \in [2, \frac{n}{2}] \quad \text{and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 2j - 1, & i \in [2, \frac{n}{2}] \quad \text{and } i \text{ is even} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 2t_{\frac{n}{2}} + 2j - 1, & i \in [\frac{n}{2}+1, n-1] \quad \text{and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 2t_{\frac{n}{2}} + 2j - 1, & i \in [\frac{n}{2}+1, n-1] \quad \text{and } i \text{ is even} \\ 4\sum_{s=1}^{n-1}(t_s + t_s^{'}) + 2t_{\frac{n}{2}} + 2j - 1, & i = n. \end{cases}$$

$$f^*(u_{ij}u_{(i+1)}) = \begin{cases} 4t_1^{'} + 2t_1 + 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + & \\ 2t_i + 2j - 1, & i \in [2, \frac{n}{2} - 1] \text{ and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i+1} t_s^{'}) + & \\ 2t_i + 2j - 1, & i \in [2, \frac{n}{2} - 1] \text{ and } i \text{ is even} \\ 4(\sum_{s=1}^{\frac{n}{2}} t_s + \sum_{s=1}^{\frac{n}{2}+1} t_s^{'}) + & \\ 2j - 1, & i = \frac{n}{2} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 2(t_{\frac{n}{2}} + t_i) + & \\ 2j - 1, & i \in [\frac{n}{2} + 1, n - 1] \text{ and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i+1} t_s^{'}) + 2(t_{\frac{n}{2}} + t_i) + & \\ 2j - 1, & i \in [\frac{n}{2} + 1, n - 1] \text{ and } i \text{ is even .} \end{cases}$$

$f^*(u_{nj}u_1) = 2\sum_{s=1}^{n-1}(t_s + t_s^{'}) + 2t_1^{'} + 2j - 1.$
For $j \in [1, t_i^{'}]$.

$$f^*(u_iw_{ij}) = \begin{cases} 2t_1^{'} + 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 2(t_{(i-1)} + t_i^{'}) + & \\ 2j - 1, & i \in [2, \frac{n}{2}] \quad \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 2j - 1, & i \in [2, \frac{n}{2}] \text{ and } i \text{ is even} \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 2(t_{\frac{n}{2}} + t_{(i-1)} + t_i^{'}) + & \\ 2j - 1, & i \in [\frac{n}{2} + 1, n - 1] \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 2t_{\frac{n}{2}} + 2j - 1, & i \in [\frac{n}{2} + 1, n - 1] \text{ and } i \text{ is even} \\ 4(\sum_{s=1}^{n} t_s + \sum_{s=1}^{n-1} t_s^{'}) + 2j - 1, & i = n. \end{cases}$$

$$f^*(v_iw_{ij}) = \begin{cases} 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 2t_{(i-1)} + 2j - 1, & i \in [2, \frac{n}{2}] \quad \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 2t_i^{'} + 2j - 1, & i \in [2, \frac{n}{2}] \text{ and } i \text{ is even} \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 2(t_{(i-1)} + t_{\frac{n}{2}}) + 2j - 1, & i \in [\frac{n}{2} + 1, n - 1] \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 2(t_i^{'} + t_{\frac{n}{2}}) + 2j - 1, & i \in [\frac{n}{2} + 1, n - 1] \text{ and } i \text{ is even.} \\ 4(\sum_{s=1}^{n} t_s + \sum_{s=1}^{n-1} t_s^{'}) + 2t_n^{'} + 2j - 1, & i = n \end{cases}$$

Thus, f is an even-odd mean labeling of $ASS(C_n \odot K_1)$. Hence $ASS(C_n \odot K_1)$ is an even-odd mean graph. $\square$

Illustration 2.3. Consider $ASS(C_8 \odot K_1)$ where the edges $u_iu_{i+1}, i \in [1,7]$ and u_8u_1 are replaced by $K_{2,2}, K_{2,3}, K_{2,5}, K_{2,4}, K_{2,3}, K_{2,6}, K_{2,3}$ and $K_{2,4}$ respectively and the edges $u_iv_i, i \in [1,8]$ are replaced by $K_{2,2}, K_{2,4}, K_{2,4}, K_{2,5}, K_{2,5}, K_{2,3}, K_{2,3}$ and $K_{2,2}$ respectively . An even-odd mean labeling of $ASS(C_8 \odot K_1)$ is shown in Figure 3.

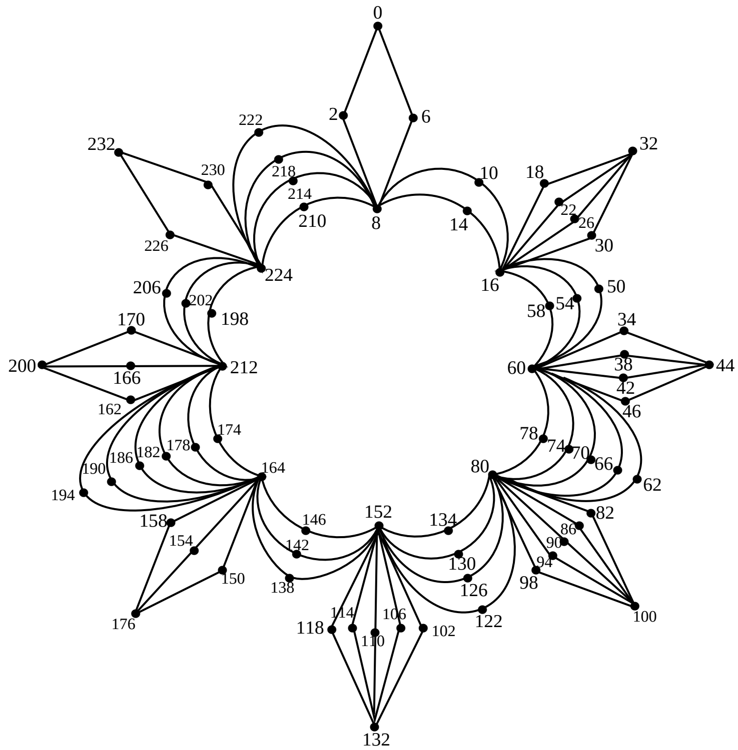

Figure 3: An even-odd mean graph of $ASS(C_8 \odot K_1)$

Theorem 2.4. *$ASS(SL_n)$ is an even-odd mean graph where the edges u_iu_{i+1}, v_iv_{i+1} and u_iv_{i+1} of SL_n are replaced by $K_{2,t_i}, K_{2,t_i^{'}}$ and $K_{2,t_i^{''}}(i \in [1, n-1])$ respectively, such that $t_i^{''} = t$ for all $(i \in [1, n-1])$, $t_{i+1}^{'} = t_i$, $t_{i+1} \geq t_i$ for all $(i \in [1, n-2])$ and $t_1^{'} = t_1 = t$ when n is odd.*

Proof. Let (SL_n) be a slanting ladder graph whose vertex set is $V = \{u_i, v_i : i \in [1, n]\}$ and edge set is $E = \{e_i = u_iu_{i+1}, e_i^{'} = v_iv_{i+1}, e^{''} = u_iv_{i+1} : i \in [1, n-1]\}$. Let $ASS(SL_n)$ be an arbitrary super subdivision of SL_n. the edges $e_i, e_i^{'}$ and $e_i^{''}(i \in [1, n-1])$ are replaced by the bicliques K_{2,t_i}, $K_{2,t_i^{'}}$ and $K_{2,t_i^{''}}$ respectively where $t_i, t_i^{'}$, and $t_i^{''}$ are positive integer numbers, $t_i^{''} = t$ for some fixed $t \in N$, $t_{i+1}^{'} = t_i$, $t_{i+1} \geq t_i$ for all $i \in [1, n-2]$ and $t_1^{'} = t_1 = t$ when n is odd. Let $u_{ij}(i \in [1, n-1], i \in [1, t_i]), v_{ij}(i \in [1, n-1], j \in [1, t_i^{'}])$, and $w_{ij}(i \in [1, n], j \in [1, t])$ be the vertices which are used for arbitrary super subdivision. Therefore, the edge set of $ASS(SL_n)$ is $E(ASS(SL_n) = \{\{u_iu_{ij}, u_{ij}u_{i+1} : i \in [1, n-1], j \in [1, t_i]\} \cup \{v_iv_{ij}, v_{ij}v_{i+1} : i \in [1, n-1], j \in [1, t_i^{'}]\} \cup \{u_iw_{ij}, w_{ij}v_{i+1} : i \in [1, n-1], j \in [1, t]\}\}$. Then, it obvious that $ASS(SL_n)$ has $2n+2\sum_{i=1}^{n-2} t_i + t_{(n-1)} + (n-1)t + t_1$ vertices and $4\sum_{i=1}^{n-2} t_i + 2((n-1)t + t_{(n-1)} + t_1)$ edges. Define labeling $f : V(SS(C_n \odot K_1)) \to \{0, 2, 4, ..., 2q-2, 2q = 8\sum_{i=1}^{n-2} t_i + 4((n-1)t + t_{(n-1)} + t_1)\}$ as follows:

Case (i). n is odd.

$$f(u_i) = \begin{cases} 0, & i = 1 \\ 8\sum_{s=1}^{i-1} t_s + 4t(i+1), & i \in [1, n-1] \\ 8\sum_{s=1}^{n-2} t_s + 4(nt + t_{(n-1)}), & i = n. \end{cases}$$

$$f(v_i) = \begin{cases} 4t, & i = 1 \\ 8t, & i = 2 \\ 8\sum_{s=1}^{i-2} t_s + 4t(i-1), & i \in [3, n] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s + 4ti, & i \in [3, n] \quad \text{and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i]$.

$$f(u_{ij}) = \begin{cases} 4t + 4j - 2, & i = 1 \\ 8\sum_{s=1}^{i-1} t_s + 4ti + 4j - 2, & i \in [2, n-2] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s + 4(t(i+1) + t_i) + 4j - 2, & i \in [2, n-2] \quad \text{and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s + 4nt + 4j - 2, & i = n-1. \end{cases}$$

For $i \in [1, t_i^{'}]$.

$$f(v_{ij}) = \begin{cases} 8t + 4j - 2, & i = 1 \\ 12t + 4j - 2, & i = 2 \\ 8\sum_{s=1}^{i-2} t_s + 4t(i-1) + 4j - 2, & i \in [3, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s + 4(ti + t_{(i-1)}) + 4j - 2, & i \in [3, n-1] \quad \text{and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i^{'}]$.

$$f(w_{ij}) = \begin{cases} 4j - 2, & i = 1 \\ 8\sum_{s=1}^{i-1} t_s + 4(ti + t_i) + 4j - 2, & i \in [2, n-2] \\ 8\sum_{s=1}^{n-2} t_s + 4t(n-1) + 4j - 2, & i = n-1. \end{cases}$$

Hence the induced edge labeling f^* is obtained as follows:
For $i \in [1, t_i]$.

$$f^*(u_iu_{ij}) = \begin{cases} 2t + 2j - 1, & i = 1 \\ 8\sum_{s=1}^{i-1} t_s + 4ti + 2j - 1, & i \in [2, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s + 4t(i+1) + 2t_i + 2j - 1, & i \in [1, n-2] \quad \text{and } i \text{ is even} \\ 8\sum_{s=1}^{n-1} t_s + 4nt + 2j - 1, & i = n-1. \end{cases}$$

$$f^*(u_{ij}u_{(i+1)}) = \begin{cases} 12t+2j-1, & i=1 \\ 8\sum_{s=1}^{i-1} t_s + 4t(i+1) + 4t_i + 2j - 1, & i \in [2, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s + 4t(i+1) + 6t_i + 2j - 1, & i \in [1, n-2] \quad \text{and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s + 4nt + 2t_{n-1} + 2j - 1, & i = n-1. \end{cases}$$

For $i \in [1, t_i^{'}]$.

$$f^*(v_iv_{ij}) = \begin{cases} 6t+2j-1, & i=1 \\ 10t+2j-1, & i=2 \\ 8\sum_{s=1}^{i-2} t_s + 4t(i-1) + 2j - 1, & i \in [3, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s + 4ti + 2t_{(i-1)} + 2j - 1, & i \in [3, n-1] \quad \text{and } i \text{ is even.} \end{cases}$$

$$f^*(v_{ij}v_{(i+1)}) = \begin{cases} 8t+2j-1, & i=1 \\ 14t+2j-1, & i=2 \\ 8\sum_{s=1}^{i-2} t_s + 4ti + 4t_{(i-1)} + 2j - 1, & i \in [3, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s + 4ti + 6t_{(i-1)} + 2j - 1, & i \in [3, n-1] \quad \text{and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i]$.

$$f^*(u_iw_{ij}) = \begin{cases} 2j-1, & i=1 \\ 8\sum_{s=1}^{i-2} t_s + 4ti + 2t_i + 2j - 1, & i \in [2, n-2] \\ 8\sum_{s=1}^{n-2} t_s + 4nt - 2t + 2j - 1, & i = n-1. \end{cases}$$

$$f^*(w_{ij}v_{(i+1)}) = \begin{cases} 4t+2j-1, & i=1 \\ 8\sum_{s=1}^{i-1} t_s + 4ti + 2(t+t_i) + 2j - 1, & i \in [2, n-2] \text{ and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s + 4ti + 2t_i + 2j - 1, & i \in [2, n-2] \text{ and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s + 4(n-1)t + 2j - 1, & i = n-1. \end{cases}$$

Then, f is an even-odd mean labeling of $ASS(SL_n)$. Thus, $ASS(SL_n)$ is an even-odd mean graph.

Case (ii). n is even.

$$f(u_i) = \begin{cases} 4(t_1^{'}+t), & i=1 \\ 8\sum_{s=1}^{i-1} t_s + 4(t_1^{'} + ti), & i \in [2, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s + 4(t_1^{'} + t(i-1)), & i \in [2, n-1] \quad \text{and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s + 4(t_1^{'} + t(n-1) + t_{(n-1)}), & i = n. \end{cases}$$

$$f(v_i) = \begin{cases} 0, & i = 1 \\ 4t_1^{'}, & i = 2 \\ 8\sum_{s=1}^{i-2} t_s + 4(t_1^{'} + t(i-1)), & i \in [1, n] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s + 4(t_1^{'} + t(i-2)), & i \in [1, n] \quad \text{and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i]$.

$$f(u_{ij}) = \begin{cases} 4(t_1^{'} + t_1 + t) + 4j - 2, & i = 1 \\ 8\sum_{s=1}^{i-1} t_s + 4(t_1^{'} + ti + t_i) + 4j - 2, & i \in [2, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s + 4(t_1^{'} + t(i-1)) + 4j - 2, & i \in [2, n-1] \quad \text{and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s + 4(t_1^{'} + t(n-1)) + 4j - 2, & i = n-1. \end{cases}$$

For $i \in [1, t_i^{'}]$.

$$f(v_{ij}) = \begin{cases} 4j - 2, & i = 1 \\ 4t_1^{'} + 4j - 2, & i = 2 \\ 8\sum_{s=1}^{i-2} t_s + 4(t_1^{'} + t(i-1) + & \\ t_{(i-1)}) + 4j - 2, & i \in [2, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s + 4(t_1^{'} + t(i-2)) + & \\ 4j - 2, & i \in [2, n-1] \quad \text{and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i]$.

$$f(w_{ij}) = \begin{cases} 4(t_1^{'} + t_1) + 4j - 2, & i = 1 \\ 8\sum_{s=1}^{i-1} t_s + 4(t_1^{'} + t(i-1) + t_i) + 4j - 2, & i \in [2, n-2] \\ 8\sum_{s=1}^{n-2} t_s + 4(t_1^{'} + t(n-2)) + 4j - 2, & i = n-1. \end{cases}$$

Thus, the induced edge labeling f^* is obtained as follows:
For $i \in [1, t_i]$.

$$f^*(u_i u_{ij}) = \begin{cases} 4(t_1^{'} + t) + 2t_1 + 2j - 1, & i = 1 \\ 8\sum_{s=1}^{i-1} t_s + 4(t_1^{'} + ti) + 2t_i + 2j - 1, & i \in [2, n-1], \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s + 4(t_1^{'} + t(i-1)) + 2j - 1, & i \in [2, n-1], \quad \text{and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s + 4(t_1^{'} + t(n-1)) + 2j - 1, & i = n-1. \end{cases}$$

$$f^*(u_{ij}u_{(i+1)}) = \begin{cases} 4(t_1^{'}+t)+6t_1+2j-1, & i=1 \\ 8\sum_{s=1}^{i-1} t_s+4(t_1^{'}+ti)+6t_i+2j-1, & i\in[2,n-2] \text{ and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s+4(t_1^{'}+ti+t_i)+2j-1, & i\in[2,n-2] \text{ and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s+4(t_1^{'}+t(n-1))+ & \\ 2t_{(n-1)}+2j-1 & i=n-1. \end{cases}$$

For $i \in [1, t_i^{'}]$.

$$f^*(v_iv_{ij}) = \begin{cases} 2j-1, & i=1 \\ 4t_1^{'}+2j-1, & i=2 \\ 8\sum_{s=1}^{i-2} t_s+4(t_1^{'}+t(i-1))+ & \\ 2t_{(i-1)}+2j-1, & i\in[2,n-1], \text{ and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s+4(t_1^{'}+t(i-2))+2j-1, & i\in[2,n-1], \text{ and } i \text{ is even.} \end{cases}$$

$$f^*(v_{ij}v_{(i+1)}) = \begin{cases} 2t_1^{'}+2j-1, & i=1 \\ 4(t_1^{'}+t+t_1)+2j-1, & i=2 \\ 8\sum_{s=1}^{i-2} t_s+4(t_1^{'}+t(i-1))+ & \\ 6t_{(i-1)}+2j-1, & i\in[3,n-1], \text{ and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s+4(t_1^{'}+t(i-1)+ & \\ t_{(i-1)})+2j-1, & i\in[3,n-1], \text{ and } i \text{ is even.} \end{cases}$$

For $i \in [1, t]$.

$$f^*(u_iw_{ij}) = \begin{cases} 4t_1^{'}+2(t+t_1)+2j-1, & i=1 \\ 8\sum_{s=1}^{i-1} t_s+4(t_1^{'}+ti)+2(t_i-t)+2j-1, & i\in[2,n-2] \text{ and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s+4(t_1^{'}+t(i-1))+2t_i+2j-1, & i\in[2,n-2] \text{ and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s+4(t_1^{'}+tn)-6t+2j-1, & i=n-1. \end{cases}$$

$$f^*(w_{ij}v_{(i+1)}) = \begin{cases} 4t_1^{'}+2t_1+2j-1, & i=1 \\ 8\sum_{s=1}^{i-1} t_s+4(t_1^{'}+t(i-1))+2t_i+2j-1, & i\in[2,n-2] \text{ and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s+4(t_1^{'}+ti)+2(t_i-t)+2j-1, & i\in[2,n-2] \text{ and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s+4(t_1^{'}+t(n-2))+2j-1, & i=n-1. \end{cases}$$

Then, f is an even-odd mean labeling of $ASS(SL_n)$. Thus, $ASS(SL_n)$ is an even-odd mean graph. □

Illustration 2.4. Consider $ASS(SL_5)$ where the edges $u_iu_{i+1}, i \in [1,4]$, are replaced by $K_{2,2}, K_{2,3}, K_{2,4}$ and $K_{2,3}$ respectively and the edges $v_iv_{i+1}, i \in [1,4]$ are replaced by $K_{2,2}, K_{2,2}, K_{2,3}, K_{2,4}$ respectively and all the edges $u_ivi+1, i \in [1,4]$, are replaced $K_{2,2}$. An even-odd mean labeling of $ASS(SL_5)$ is shown in Figure 4.

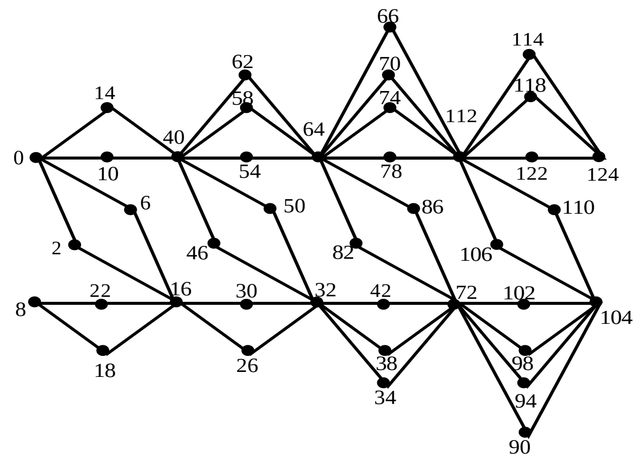

Figure 4: An even-odd mean graph of $ASS(SL_5)$

Illustration 2.5. Consider $ASS(SL_6)$ where the edges $u_iu_{i+1}, i \in [1,5]$, are replaced by $K_{2,2}, K_{2,3}, K_{2,4}, K_{2,5}, K_{2,3}$ respectively and the edges $v_iv_{i+1}, i \in [1,5]$ are replaced by $K_{2,4}, K_{2,2}, K_{2,3}, K_{2,4}, K_{2,5}$ respectively and all the edges $u_ivi+1, i \in [1,5]$, are replaced $K_{2,3}$. An even-odd mean labeling of $ASS(SL_6)$ is shown in Figure 5.

Theorem 2.5. *$ASS(P_m \times P_n)$ is an even-odd mean graph where the edges $u_{ij}u_{i(j+1)}$, $(i \in [1,m], j \in [1,n-1])$, $u_{ij}u_{(i+1)j}(i \in [1,m-1], j \in [1,n])$ of $P_m \times P_n$ are replaced by $K_{2,t_{ij}}$, $K_{2,t'_{ij}}$ respectively such that t_{ij} are equals for all j and t'_{ij} are equals for all i.*

Proof. Let the vertex set of planar grid $P_m \times P_n$ be $V = \{u_{ij} : i \in [1,m], j \in [1,n]\}$ and the edge set be $E = \{\{e_{ij} = u_{ij}u_{i(j+1)} : i \in [1,m], j \in [1,n-1]\} \cup \{e'_{ij} = u_{ij}u_{(i+1)j} : i \in [1,m-1], j \in [1,n]\}$. Let $ASS(P_m \times P_n)$ be an arbitrary super subdivision of the planar grid $P_m \times P_n$. The horizontal and vertical edges e_{ij}, e'_{ij} are replaced by the bicliques $K_{2,t_{ij}}, K_{2,t'_{ij}}$ respectively where t_{ij}, t'_{ij} are positive integer numbers, t_{ij} are equals for all j and t'_{ij} are equal for all i. Let $v_{ij,k}(i \in [1,m], j \in [1,n-1], k \in [1,t_{ij}])$ and $w_{ij,k}(i \in [1,m-1], j \in [1,n], k \in [1,t'_{ij}])$ be the vertices which are used for arbitrary super subdivision of the edges e_{ij} and e'_{ij} respectively. Thus, the edge set of $ASS(P_m \times P_n)$ is $E(ASS(P_m \times P_n)) = \{\{u_{ij}v_{ij,k}, v_{ij,k}u_{i(j+1)} : i \in [1,m], j \in [1,n-1], k \in$

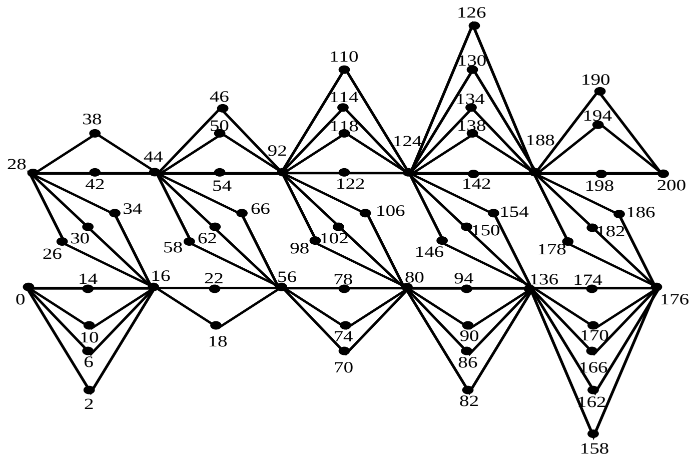

Figure 5: An even-odd mean graph of $ASS(SL_6)$

$[1, t_{ij}]\} \cup \{u_{ij}w_{ij,k}, w_{ij,k}u_{(i+1)j} : i \in [1, m-1], j \in [1, n], k \in [1, t'_{ij}]\}$. Therefore, it is clear that $ASS(P_m \times P_n)$ has $mn + m\sum_{s=1}^{n-1} t_{is} + n\sum_{s=1}^{m-1} t'_{sj}$ vertices and $2(m\sum_{s=1}^{n-1} t_{is} + n\sum_{s=1}^{m-1} t'_{sj})$ edges. Define labeling $f : V(ASS(P_m \times P_n)) \to \{0, 2, 4, ..., 2q-2, 2q = 4(m\sum_{s=1}^{n-1} t_{is} + n\sum_{s=1}^{m-1} t'_{sj})\}$ as follows:

$$f(u_{1j}) = \begin{cases} 0, & j = 1 \\ 4\sum_{s=1}^{j-1} t_{1s}, & j \in [2, n]. \end{cases}$$

$$f(u_{ij}) =$$
$$\begin{cases} 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1}, & j = 1, i \in [2, m] \text{ and } i \text{ is odd} \\ 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} + 4\sum_{s=1}^{j-1} t_{is}, & j \in [2, n], i \in [2, m] \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1}, & j = 1, i \in [2, m] \text{ and } i \text{ is even} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} - 4\sum_{s=1}^{j-1} t_{is}, & j \in [2, n], i \in [2, m] \text{ and } i \text{ is even.} \end{cases}$$

For $k \in [1, t_{ij}]$.

$$f(v_{1j,k}) = \begin{cases} 4t'_{11} + 4k - 2, & j = 1 \\ 4\sum_{s=1}^{j-1} t_{1s} + 4jt'_{1j} + 4k - 2, & j \in [2, n-1]. \end{cases}$$

$$f(v_{2j,k}) = \begin{cases} 8\sum_{s=1}^{n-1} t_{2s} + 4nt'_{11} - 4t_{21} + 4k - 2, & j = 1 \\ 8\sum_{s=1}^{n-1} t_{2s} + 4nt'_{1j} - 4\sum_{s=1}^{j} t_{2s} + 4k - 2, & j \in [2, n-1]. \end{cases}$$

$f(v_{ij,k}) =$

$$\begin{cases} 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-2} t'_{s1} & \\ +4t'_{(i-1)1} + 4k - 2, & j = 1, i \in [3,m] \text{ and } i \text{ is odd} \\ 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-2} t'_{sj} & \\ +4jt'_{(i-1)j} + 4\sum_{s=1}^{j-1} t_{is} + 4k - 2, & j \in [2,n-1], i \in [3,m] \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1} & \\ -4t'_{(i-1)1} - 4t_{i1} + 4k - 2, & j = 1, i \in [3,m] \text{ and } i \text{ is even} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} - 4\sum_{s=1}^{j} t_{is} & \\ -4jt'_{(i-1)j} + 4k - 2, & j \in [2,n-1], i \in [3,m] \text{ and } i \text{ is even.} \end{cases}$$

For $k \in [1, t'_{ij}]$.

$$f(w_{1j,k}) = \begin{cases} 4k-2, & j = 1 \\ 4\sum_{s=1}^{j-1} t_{1s} + 4(j-1)t'_{1j} + 4k - 2, & j \in [2,n-1]. \end{cases}$$

$f(w_{ij,k}) =$

$$\begin{cases} 4(i+1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i} t'_{s1} & \\ -4(t'_{i1}) + 4k - 2, & j = 1, i \in [2,m-1] \text{ and } i \text{ is odd} \\ 4(i+1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i} t'_{sj} & \\ -4\sum_{s=1}^{j-1} t_{is} - 4j(t'_{ij}) + 4k - 2, & j \in [2,n-1], i \in [2,m-1] \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1} & \\ +4k - 2, & j = 1, i \in [2,m-1] \text{ and } i \text{ is even} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} & \\ +4(j-1)t'_{ij} + 4\sum_{s=1}^{j-1} t_{is} + 4k - 2, & j \in [2,n-1], i \in [2,m-1] \text{ and } i \text{ is even.} \end{cases}$$

Then, the induced edge labeling f^* is obtained as follows:
For $k \in [1, t_{ij}]$.

$$f^*(u_{1j}\ v_{1j,k}) = \begin{cases} 2t'_{11} + 2k - 1, & j = 1 \\ 4\sum_{s=1}^{j-1} t_{1s} + 2jt'_{1j} + 2k - 1, & j \in [2,n-1]. \end{cases}$$

$f^*(u_{2j}\ v_{2j,k}) =$

$$\begin{cases} 8\sum_{s=1}^{n-1} t_{2s} + 4nt'_{11} - 2t_{21} + 2k - 1, & j = 1 \\ 8\sum_{s=1}^{n-1} t_{2s} + 4nt'_{1j} - 4\sum_{s=1}^{j-1} t_{2s} - 2t_{2j} + 2k - 1, & j \in [2,n-1]. \end{cases}$$

$$f^*(u_{ij}\ v_{ij,k}) = \begin{cases} 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-2} t'_{s1} \\ +2(n+1)t'_{(i-1)1} + 2k-1, & j=1, i\in[3,m] \text{ and } i \text{ is odd} \\ 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-2} t'_{sj} \\ +2(n+j)t'_{(i-1)j} + 4\sum_{s=1}^{j-1} t_{is} + 2k-1, & j\in[2,n-1], i\in[3,m] \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1} - 2t'_{(i-1)1} \\ -2t_{i1} + 2k-1, & j=1, j\in[3,m] \text{ and } i \text{ is even} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} - 4\sum_{s=1}^{j-1} t_{is} \\ -2jt'_{(i-1)j} - 2t_{ij} + 2k-1, & j\in[2,n-1], j\in[3,m] \text{ and } i \text{ is even.} \end{cases}$$

$$f^*(v_{1j,k}\ u_{1(j+1)}) = \begin{cases} 2t_{11} + 2t'_{11} + 2k-1, & j=1 \\ 4\sum_{s=1}^{j-1} t_{1s} + 2t_{ij} + 2jt'_{1j} + 2k-1, & j\in[2,n-1]. \end{cases}$$

$$f^*(v_{2j,k}\ u_{2(j+1)}) = \begin{cases} 8\sum_{s=1}^{n-1} t_{2s} + 4nt'_{11} - 4t_{21} + 2k-1, & j=1 \\ 8\sum_{s=1}^{n-1} t_{2s} + 4nt'_{1j} - 4\sum_{s=1}^{j} t_{2s} + 2k-1, & j\in[2,n-1]. \end{cases}$$

$$f^*(v_{ij,k}\ u_{i(j+1)}) = \begin{cases} 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-2} t'_{s1} \\ +2(n+1)t'_{(i-1)1} + 2t_{i1} + 2k-1, & j=1, i\in[3,m] \text{ and } i \text{ is odd} \\ 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-2} t'_{sj} \\ +4\sum_{s=1}^{j-1} t_{is} + 2(n+j)t'_{(i-1)j} \\ +2t_{ij} + 2k-1, & j\in[2,n-1], i\in[3,m] \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1} - 2t'_{(i-1)1} \\ -4t_{i1} + 2k-1, & j=1, j\in[3,m] \text{ and } i \text{ is even} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} - 4\sum_{s=1}^{j} t_{is} \\ -2jt'_{(i-1)j} + 2k-1, & j\in[2,n-1], i\in[3,m] \text{ and } i \text{ is even.} \end{cases}$$

For $k \in [1, t'_{ij}]$.

$$f^*(u_{1j}\ w_{1j,k}) = \begin{cases} 2k-1, & j=1 \\ 4\sum_{s=1}^{j-1} t_{1s} + 2(j-1)t'_{1j} + 2k-1, & j\in[2,n] \end{cases}$$

$$f^*(u_{ij}\ w_{ij,k}) = \begin{cases} 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1} + \\ 2(n-1)t'_{i1} + 2k - 1, & j = 1, i \in [2, m-1] \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} + \\ 2nt'_{ij} - 2t'_{ij} + 2k - 1, & j \in [2, n], i \in [2, m-1] \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1} + \\ 2k - 1, & j = 1, i \in [2, m-1] \text{ and } i \text{ is even} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} + \\ 2(j-1)t'_{ij} + 2k - 1, & i \in [2, n], i \in [2, m-1] \text{ and } i \text{ is even.} \end{cases}$$

$$f^*(w_{1j,k}\ u_{2j}) = \begin{cases} 4\sum_{s=1}^{n-1} t_{1s} + 2nt'_{11} + 2k - 1, & j = 1 \\ 4\sum_{s=1}^{n-1} t_{1s} + 2(n+j-1)t'_{1j} + 2k - 1, & j \in [2, n]. \end{cases}$$

For $i \in [2, m-1]$.

$$f^*(w_{ij,k}\ u_{(i+1)j}) = \begin{cases} 4(i+1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i} t'_{s1} \\ -2t'_{i1} + 2k - 1, & j = 1, \text{ and } i \text{ is odd} \\ 4(i+1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i} t'_{sj} \\ -4\sum_{s=1}^{j-1} t_{is} - 2jt'_{ij} + 2k - 1, & j \in [2, n], \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1} \\ +2nt'_{i1} + 2k - 1, & j = 1, \text{ and } i \text{ is even} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} + 4\sum_{s=1}^{j-1} t_{is} \\ +2(n+j-1)t'_{ij} + 2k - 1, & j \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

Hence, f is an even-odd mean labeling of $ASS(P_m \times P_n)$. Thus, $ASS(P_m \times P_n)$ is an even-odd mean graph. □

Illustration 2.6. Consider $ASS(P_6 \times P_5)$ where the edges $u_{i1}u_{i2}, u_{i2}u_{i3}, u_{i3}u_{i4}$ and $u_{i4}u_{i5}$ are replaced by $K_{2,5}, K_{2,2}, K_{2,4}$ and $K_{2,5}$ respectively for all $i \in [1, 6]$ and the edges $u_{1j}u_{2j}, u_{2j}u_{3j}, u_{3j}u_{4j}, u_{4j}u_{5j}$ and $u_{5j}u_{6j}$ are replaced by $K_{2,2}, K_{2,4}$, $K_{2,3}, K_{2,5}$ and $K_{2,2}$ respectively for all $j \in [1, 5]$. An even-odd mean labeling of $ASS(P_6 \times P_5)$ is shown in Figure 6.

Corollary 2.1. *$ASS(L_n)$ is an even-odd mean graph for all n.*

Proof. From the definition of Ladder L_n and by Theorem 2.5, the arbitrary super subdivision of L_n is also an even-odd mean graph. □

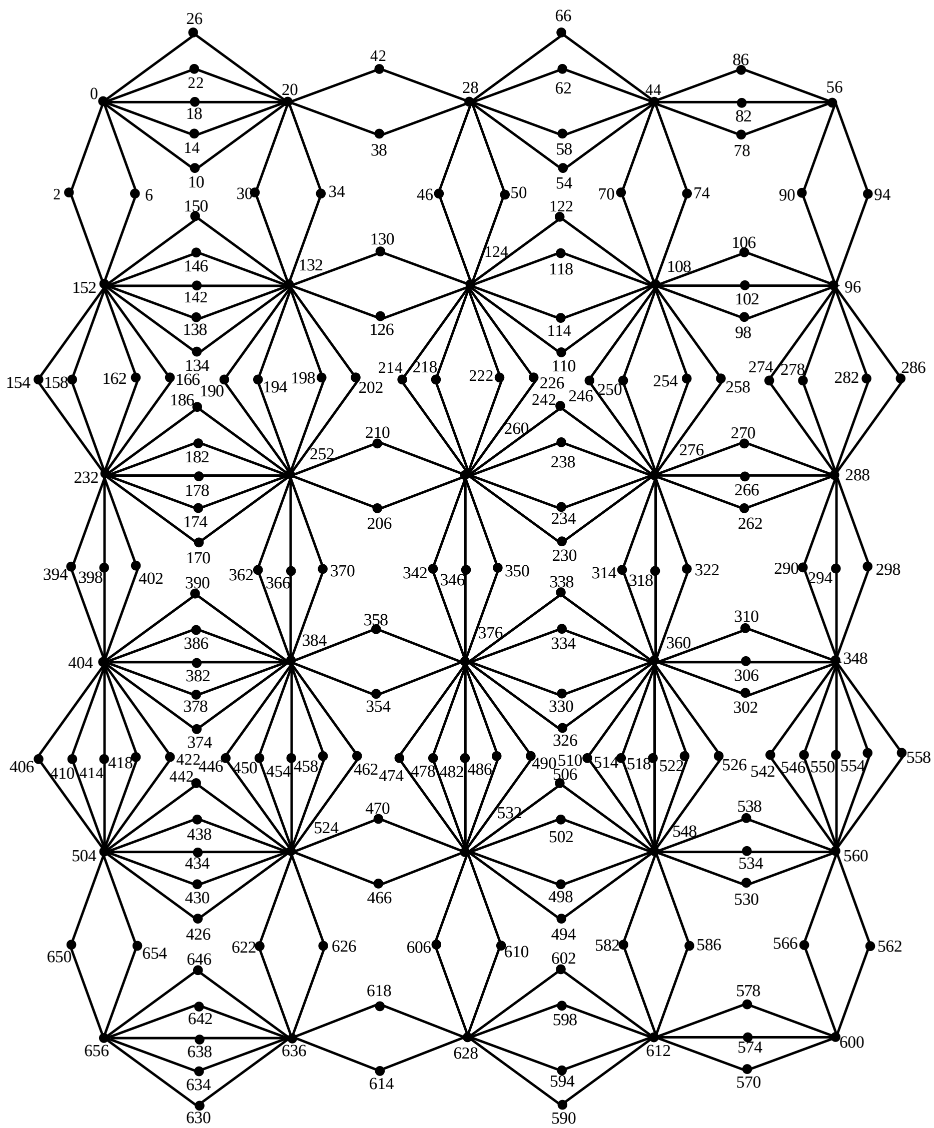

Figure 6: An even-odd mean graph of $ASS(P_6 \times P_5)$

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