Extended of generalized power series reversible rings

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Abstract. Let R be a ring and $(S, \leq)$ a strictly ordered monoid. This paper aims to introduce and study generalized power series *nil*-reversible rings. The researchers obtains various necessary or sufficient conditions for a generalized power series *nil*-reversible rings are 2-primal, *nil*-semicommutative and *nil*-symmetric. Examples are given to show that, a generalized power series *nil*-reversible which is neither generalized power series semicommutative nor generalized power series reversible. Also, we proved that a multiplicatively closed subset of R consisting of central non-zero divisors is generalized power series *nil*-reversible if and only if R is generalized power series *nil*-reversible. Moreover, other standard ring-theoretic properties are given.

Keywords: Armendariz ring, generalized power series reversible, ordered monoid $(S, \leq)$, semicommutative ring.
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1. Introduction

Throughout this paper, any ring is associative and has an identity unless stated. We write $P(R)$, $nil(R)$, $Mat_n(R)$, $T_n(R,)$ $S_n(R)$ and $R[x]$ respectively for the prime radical, the set of all nilpotent elements of R, the ring of all $n \times n$ matrices, the ring of all $n \times n$ upper triangular matrices for a positive integer n with entries in R, the subring consisting of all upper triangular matrices over a ring R and the polynomial ring over a ring R.

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In [1], Cohn introduced the notion of a reversible ring. A ring R is said to be reversible, if whenever $a, b \in R$ satisfy $ab = 0$, then $ba = 0$. Anderson-Camillo [2] used the term ZC_2 for what is called reversible. While Krempa-Niewieczerzal [3] took the term C_0 for it.

In [4], a ring R is called semicommutative if for all $a, b \in R, ab = 0$ implies $aRb = 0$. This is equivalent to the definition that any left (right) annihilator of R is an ideal of R. According to [5], semicommutativity of rings is generalized to nil-semicommutativity of rings. A ring R is called nil-semicommutative if $a, b \in R$ satisfy that ab is nilpotent, then $ahb \in nil(R)$, for any $h \in R$. Clearly, every semicommutative ring is nil-semicommutative. Reduced rings (*i.e.*, rings with no nonzero nilpotent elements in R) are symmetric by [6, P. 361], symmetric rings are clearly reversible, and reversible rings are semicommutative by Proposition 1.3 [6], but the converses are not true. Kim and Lee showed that polynomial rings over reversible rings need not be reversible Example 2.1 [7]. In [8], they consider these reversible rings over which polynomial rings are reversible and call them be strongly reversible, *i.e.*, a ring R is called strongly reversible, if whenever polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $g(x)f(x) = 0$. Reversible Armendariz rings are such rings Proposition 2.4 [7], so reduced rings are strongly reversible, but the converse is not true in general. A ring R is said to be 2- primal if $nil(R)$ coincides with $P(R)$. A ring R is called an NI-ring if the upper nilradical $Nil^*(R)$ coincides with the set of nilpotent elements $nil(R)$. Note that R is an NI-ring if and only if $nil(R)$ forms an ideal and 2-primal rings are NI.

The notion of Armendariz ring is introduced by Rege and Chhawchharia in [4]. A ring R to be an Armendariz if $f(x)g(x) = 0$ implies $a_ib_j = 0$, for all polynomials $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_mx^m$, $g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n \in R[x]$.

This paper introduce and study generalized power series nil-reversible rings. Under some various necessary or sufficient conditions for a generalized power series nil-reversible rings to be nil-semicommutative and nil-symmetric. Also, we proved that, a multiplicatively closed subset of R consisting of central non-zero divisors is generalized power series nil-reversible if and only if R is generalized power series nil-reversible. Moreover, some results of generalized power series nil-reversible are given.

We will write monoids multiplicatively unless otherwise indicated. If R is a ring and X is a nonempty subset of R, then the left (right) annihilator of X in R is denoted by $\ell_R(X)(r_R(X))$.

We use the following terminology. If A and B are non-empty subsets of a monoid S, then an element $u_0 \in AB = \{ab : a \in A, b \in B\}$ is said to be a unique product element (*u.p.* element) in the product of AB if it is uniquely presented in form $u = ab$ where $a \in A$ and $b \in B$. For a partially ordered set Y, we write $min(Y)$ to denote the set of minimal elements of Y.

Recall that a monoid S is called unique product monoid (*u.p.*- monoid) if for any two non-empty finite subsets $A, B \in S$ there exist $a \in A$ and $b \in B$ such that ab is *u.p.* element in the product of AB.

We continue by recalling the structure of the generalized power series ring construction, introduced in [9]. Suppose that $(S, \leq)$ is an ordered set, then $(S, \leq)$ is artinian if every strictly decreasing sequence of elements of S is finite, and $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of S is finite. Thus, $(S, \leq)$ is artinian and narrow if and only if every nonempty subset of S has at least one but only a finite number of minimal elements. Let S be a commutative monoid. Unless stated otherwise, the operation of S will be denoted additively, and the neutral element by 0. Following definition is due to Ribenboim and Elliott [14].

Let $(S, \leq)$ is a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if $s, s', d \in S$ and $s < s'$, then $s + d < s' + d$), and R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f : S \to R$ such that $supp(f) = \{s \in S | f(s) \neq 0\}$ is artinian and narrow. With pointwise addition, $[[R^{S,\leq}]]$ is an abelian additive group. For every $s \in S$ and $f, g \in [[R^{S,\leq}]]$, let $X_s(f, g) = \{(u, v) \in S \times S | u + v = s, f(u) \neq 0, g(v) \neq 0\}$. It follows from Ribenboim [10, 4.1] that $X_s(f, g)$ is finite. This fact allows one to define the operation of convolution:

$$(fg)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)g(v).$$

Clearly, $\text{supp}(fg) \subseteq supp(f) + supp(g)$, thus by Ribenboim [9, 3.4] $\text{supp}(fg)$ is artinian and narrow, hence $fg \in [[R^{S,\leq}]]$. With this operation, and pointwise addition, $[[R^{S,\leq}]]$ becomes an associative ring, with identity element e, namely $e(0) = 1, e(s) = 0$ for every $0 \neq s \in S$. Which is called the ring of generalized power series with coefficients in R and exponents in S. Many examples and results of rings of generalized power series are given in ([11]$-$[13]), Elliott and Ribenboim [14] and Varadarajan ([15], [16]). For example, if $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the usual order, then $[[R^{\mathbb{N}\cup\{0\},\leq}]] \cong R[[x]]$, the usual ring of power series. If S is a commutative monoid and $\leq$ is the trivial order, then $[[R^{S,\leq}]] \cong R[S]$, the monoid ring of S over R. Further examples are given in Ribenboim [9]. To any $r \in R$ and $s \in S$, we associate the maps $c_r, e_s \in [[R^{S,\leq}]]$ defined by

$$c_r(x) = \begin{cases} r, & x = 0, \\ 0, & \text{otherwise}, \end{cases} \quad e_s(x) = \begin{cases} 1, & x = s, \\ 0, & \text{otherwise}. \end{cases}$$

It is clear that $r \mapsto c_r$ is a ring embedding of R into $[[R^{S,\leq}]], s \mapsto e_s$, is a monoid embedding of S into the multiplicative monoid of the ring $[[R^{S,\leq}]]$, and $c_r e_s = e_s c_r$. Recall that a monoid S is torsion-free if the following property holds: If $s, t \in S$, k is an integer, $k \geq 1$ and $ks = kt$, then $s = t$.

2. Generalized power series nil-reversible rings

In this section, we first give the following concept, so called generalized power series nil-reversible, that is a generalization of power series reversible rings and generalized power series reversible, we use this concept by studying the relations between nil generalized power series reversible and some certain classes of rings.

Definition 2.1. *Let $(S, \leq)$ be a strictly ordered monoid. A ring R is called generalized power series nil-reversible if whenever $f, g \in [[R^{S,\leq}]]$ satisfy $fg \in [[nil(R)^{S,\leq}]]$ implies $gf \in [[nil(R)^{S,\leq}]]$.*

Let $S = (\mathbb{N} \cup \{0\}, +)$, and $\leq$ is the usual order. Then, $[[R^{S,\leq}]] \cong R[[x]]$. So, the ring R is generalized power series nil-reversible if and only if R is power series nil-reversible. Hence, a generalized power series nil-reversible is a generalization of power series nil-reversible and power series reversible.

Remark 2.2. By definition, it is clear that generalized power series nil-reversible rings are closed under subrings.

Now, we can give example of nil-reversible rings of generalized power series which is neither generalized power series reversible nor generalized power series semicommutative. As we know, generalized power series reversible rings are both generalized power series semicommutative and generalized power series nil-reversible by definition. So, we may conjecture that generalized power series nil-reversible rings may be generalized power series semicommutative. But the following examples eliminate the possibility. We need the following Propositions.

Proposition 2.3. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. If R is a reduced ring with $nil(R)$ an ideal of R, then R is generalized power series nil-reversible ring.*

Proof. Assume that $f, g \in [[R^{S,\leq}]]$, satisfying that fg is nilpotent. So, there exists a positive integer n such that $(fg)^n = 0$, so $(f(u)g(v))^n = 0$, for any $u, v \in S$. Then, $f(u)g(v) \in nil(R)$. Hence, $g(v)f(u)$ is nilpotent by reducedness. Thus, gf is nilpotent. □

Proposition 2.4. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. A ring R is generalized power series nil-reversible ring if and only if, for any n, the n-by-n upper triangular matrix ring $T_n(R)$ is generalized power series nil-reversible.*

Proof. Assume that $f, g \in [[T_n(R)^{S,\leq}]]$, such that $fg \in [[nil(T_n(R))^{S,\leq}]]$. So, by [17], $nil(T_n(R)) = \begin{pmatrix} nil(R) & R & R & \dots & R \\ 0 & nil(R) & R & \dots & R \\ 0 & 0 & nil(R) & \dots & R \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & nil(R) \end{pmatrix}$.

Let R be a reduced ring. Then, $nil(R) = 0$ and so $nil(T_n(R))$ is an ideal. By Proposition 2.3, $T_n(R)$ is generalized power series nil-reversible. The if part follows Remark 2.2. □

Example 2.5. Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. Let R be generalized power series nil-reversible ring. Then

$$T = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid a_{ij} \in R \right\}.$$

is a generalized power series nil-reversible ring by Proposition 2.4. Note that $fg = 0$, where $f = c_{E_{23}} + c_{E_{13}} e_s$ and $g = c_{E_{12}} + c_{E_{22}} e_s$, But we have $gf \neq 0$. So, T is not generalized power series reversible. In fact, T is not generalized power series semiccomutative by the same as argument from the last sentence of Example 3.17 [18] (with $n = 3$).

Also let S be a generalized power series nil-reversible ring. Then, the ring

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a & a_{23} & \dots & a_{2n} \\ 0 & 0 & a & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a \end{pmatrix} \mid a, a_{ij} \in S; n \geq 3 \right\}.$$

is not generalized power series reversible by Example 1.5 [19]. But R_n is generalized power series nil-reversible by Proposition 2.4, since any subring of generalized power series nil-reversible ring is generalized power series nil-reversible. It is obvious that R_4 is not generalized power series semicommutative and it can be proved similarly that R_n is not generalized power series semicommutative for $n \geq 5$.

Proposition 2.6. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and R a generalized power series nil-reversible ring. If $f_1, f_2, \dots, f_n \in [[R^{S,\leq}]]$ such that $f_1 f_2 \dots f_n \in [[nil(R)^{S,\leq}]]$, then $f_1(u_1) f_2(u_2) \dots f_n(u_n) \in nil(R)$, for all $u_1, u_2, \dots, u_n \in S$.*

Proof. Suppose $f_1 f_2 \dots f_n \in [[nil(R)^{S,\leq}]]$. Then, for $f_1(f_2 \dots f_n) \in [[nil(R)^{S,\leq}]]$ it follows that $f_1(u_1)(f_2 \dots f_n)(v) \in nil(R)$, for all $u_1, v \in S$. Thus, $C_{f_1(u_1)}(f_2 \dots f_n)(v) \in nil(R)$, for any $v \in S$, and so $C_{f_1(u_1)} f_2 \dots f_n \in [[nil(R)^{S,\leq}]]$. Now, from $(C_{f_1(u_1)} f_2) f_3 \dots f_n \in [[nil(R)^{S,\leq}]]$, it follows that $(C_{f_1(u_1)} f_2)(u_2)(f_3 \dots f_n)(w) \in nil(R)$, since $u_2, w \in S$. $(C_{f_1(u_1)} f_2)(u_2) = f_1(u_1)(f_2(u_2))$, for any $u_1, u_2 \in S$, we have $f_1(u_1) f_2(u_2)(f_3 \dots f_n)(w) \in nil(R)$, for all $u_1, u_2, w \in S$. Hence,

$$C_{f_1(u_1)} C_{f_2(u_2)}(f_3 \dots f_n) \in [[nil(R)^{S,\leq}]].$$

Continuing this manner, we see that $f_1(u_1) f_2(u_2) \dots f_n(u_n) \in nil(R)$, for any $u_1, u_2, \dots, u_n \in S$. As we are desired $f_1(u_1) f_2(u_2) \dots f_n(u_n) \in nil(R)$, for any $u_1, u_2, \dots, u_n \in S$. □

Corollary 2.7. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. If R is generalized power series nil-reversible, then $nil([[R^{S,\leq}]]) \subseteq [[nil(R)^{S,\leq}]]$.*

Proposition 2.8. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. If R is generalized power series nil-reversible rings, then:*
(1) *R is abelian.*
(2) *R is 2-primal.*

Proof. Let R be a generalized power series *nil*-reversible ring.
(1) Let e be an idempotent element of R. For any $f(u) \in R, u \in S, c_e f(u) - c_e f(u)c_e \in nil(R)$. Note that $(c_e f(u) - c_e f(u)c_e)c_e = 0$. By hypothesis, $c_e(c_e f(u) - c_e f(u)c_e) = 0$, so $c_e f(u) = c_e f(u)c_e$. Again, $f(u)c_e - c_e f(u)c_e \in nil(R)$ and $c_e(f(u)c_e - c_e f(u)c_e) = 0$. So by generalized power series *nil*-reversibility of R, we have $(f(u)c_e - c_e f(u)c_e)c_e = 0$, that is, $f(u)c_e = c_e f(u)c_e$. Hence, $c_e f(u) = f(u)c_e$.
(2) Note that $P(R) \subseteq nil(R)$. Suppose $g(v) \in nil(R)$. Then, there is a positive integer $m \geq 2$ such that $(g(v))^m = 0$. Thus, $R(g(v))^{m-1}g(v) = 0$. This implies that $g(v)R(g(v))^{m-1} = 0$ as R is generalized power series *nil*-reversible. This yields $(Rg(v))^m = 0$, so $g(v) \in P(R)$. □

Proposition 2.9. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. Every generalized power series nil-reversible rings are generalized power series nil-Armendariz.*

Proof. Let $0 \neq f, g \in [[R^{S,\leq}]]$ be such that $fg \in [[nil(R)^{S,\leq}]]$. By Ribenboim [9], there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$. We will use transfinite induction on the strictly totally ordered set $(S, \leq)$ to show that $f(u)g(v) \in nil(R)$, for any $u \in supp(f)$ and $v \in supp(g)$. Let s and d denote the minimum elements of supp(f) and supp(g) in the $\leq'$ order, respectively. If $u \in supp(f)$ and $v \in supp(g)$ are such that $u + v = s + d$, then $s \leq' u$ and $d \leq' v$. If $s <' u$, then $s + d <' u + v = s + d$, a contradiction. Thus $u = s$. Similarly, $v = d$. Hence, $0 = (fg)(s + d) = \sum_{(u,v)\in X_{s+d}(f,g)} f(u)g(v) = f(s)g(d)$.

Now, suppose that $w \in S$ such that for any $u \in supp(f)$ and $v \in supp(g)$ with $u + v <' w, f(u)g(v) = 0$. We will show that $f(u)g(v) \in nil(R)$, for any $u \in supp(f)$ and $v \in supp(g)$ with $u + v = w$. We write $X_w(f, g) = \{(u, v) \mid u + v = w, u \in supp(f), v \in supp(g)\}$ as $\{(u_i, v_i) \mid i = 1, 2, \ldots, n\}$ such that

$$u_1 <' u_2 <' \ldots <' u_n.$$

Since S is cancellative, $u_1 = u_2$ and $u_1 + v_1 = u_2 + v_2 = w$ imply $v_1 = v_2$. Since $\leq'$ is a strict order, $u_1 <' u_2$ and $u_1 + v_1 = u_2 + v_2 = w$ imply $v_2 <' v_1$. Thus we have

$$v_n <' \ldots <' v_2 <' v_1.$$

Now,

$$0 = (fg)(w) = \sum_{(u,v)\in X_w(f,g)} f(u)g(v) = \sum_{i=1}^{n} f(u_i)g(v_i). \tag{2.1}$$

For any $i \geq 2, u_1 + v_i <' u_i + v_i = w$, and thus, by induction hypothesis, we have $f(u_1)g(v_i) \in nil(R)$. R is 2- primal by Proposition 2.8 this implies $f(u_1)g(v_i) \in nil(R)$. Hence, multiplying Eq. (2.1) on the right by $f(u_1)g(v_1)$, we obtain

$$\Big(\sum_{i=1}^{n} f(u_i)g(v_i)\Big) f(u_1)g(v_1) = f(u_1)g(v_1)f(u_1)g(v_1) = 0.$$

Then, $(f(u_1)rg(v_1))^2 = 0$ and so $f(u_1)g(v_1) \in nil(R)$. Now, Eq. (2.1) becomes

$$\sum_{i=2}^{n} f(u_i)g(v_i) = 0. \tag{2.2}$$

Multiplying $f(u_2)g(v_2)$ on Eq. (2.2) from the right-hand side, we obtain $f(u_2)g(v_2) = 0$ by the same way as the above. Continuing this process, we can prove $f(u_i)g(v_i) = 0$ for $i = 1, 2, \ldots, n$. Thus $f(u)g(v) \in nil(R)$, for any $u \in supp(f)$ and $v \in supp(g)$ with $u+v = w$. Therefore, by transfinite induction, $f(u)g(v) \in nil(R)$, for any $u \in supp(f)$ and $v \in supp(g)$. □

Lemma 2.10. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. For a ring R, consider the following conditions.*
(1) R is generalized power series nil-reversible.
(2) If AB is a nil set, then so is BA, for any subsets A, B of R.
(3) If IJ is nil, then JI is nil for all right (or left) ideals I, J of R.
Then, (1) ⇒ (2) ⇒ (3).

Proof. (1) ⇒ (2) Assume that R is nil generalized power series reversible. Let A, B be two nonempty subsets of R with AB is a nil set. For any $f \in A$ and $g \in B$ is nilpotent. Then, gf is nilpotent. This implies that BA is nil.
(2) ⇒ (3) Let I and J be any right ideals of R such that IJ is nil. Since $IR \subseteq I, IJ$ is nil. By (2), JI is nil. Since $JI \subseteq JRI$, we get JI is nil. Assume that I and J be any left ideals of R such that IJ is nil. Since $RJ \subset J$ and then $IRJ \subseteq IJ, IJ$ is nil. By (2), JRI is nil. Since $JI \subseteq JRI$, we get JI is nil. □

Lemma 2.11. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. Then, every generalized power series nil-reversible rings are generalized power series nil-semicommutative.*

Proof. Let $f, g \in [[R^{S,\leq}]]$ with $fg \in [[nil(R)^{S,\leq}]]$. Then, $gf \in [[nil(R)^{S,\leq}]]$ and $g(v)h(w)f(u) \in nil(R)$, for any $u, v, w \in S$ and $h(w) \in R$, so $f(v)h(w)g(u) \in nil(R)$. Thus, $fhg \in [[nil(R)^{S,\leq}]]$ by [7, Lemma 1.1]. Therefore, R is generalized power series *nil*-semicommutative. □

Let I be an index set and R_i be a ring for each $i \in I$. Let $(S, \leq)$ be a strictly ordered monoid, if there is an injective homomorphism $f : R \to \prod_{i\in I} R_i$ such that, for each $j \in I, \pi_j f : R \to R_j$ is a surjective homomorphism, where $\pi_j : \prod_{i\in I} R_i \to R_j$ is the jth projection. We have the following.

Proposition 2.12. *Let R_i be a ring, $(S, \leq)$ a strictly totally ordered monoid, for each i in a finite index set I. If R_i is generalized power series nil-reversible ring. for each i, then $R = \prod_{i\in I} R_i$ is generalized power series nil-reversible ring.*

Proof. Let $R = \prod_{i\in I} R_i$ be the direct product of rings $(R_i)_{i\in I}$ and R_i is generalized power series nil-reversible, for each $i \in I$. Denote the projection $R \to R_i$ as Π_i. Suppose that $f, g \in [[R^{S,\leq}]]$ such that $fg \in [[nil(R)^{S,\leq}]]$. Set $f_i = \prod_i f$, $g_i = \prod_i g$. Then, $f_i, g_i \in [[R_i^{S,\leq}]]$. For any $u, v \in S$, assume $f(u) = (a_i^u)_{i\in I}$, $g(v) = (b_i^v)_{i\in I}$. Now, for any $s \in S$,

$$\begin{aligned}(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)g(v) &= \sum_{(u,v)\in X_s(f,g)} (a_i^u)_{i\in I}(b_i^v)_{i\in I} \\ = \sum_{(u,v)\in X_s(f,g)} ((a_i^u)(b_i^v))_{i\in I} &= \sum_{(u,v)\in X_s(f,g)} (f_i(u)g_i(v))_{i\in I} \\ &= \Big(\sum_{(u,v)\in X_s(f_i,g_i)} f_i(u)g_i(v)\Big)_{i\in I} \\ &= ((f_i g_i)(s))_{i\in I}.\end{aligned}$$

Since $(fg)(s) \in nil(R)$, we have $(f_i g_i)(s) \in nil(R_i)$. Thus, $f_i g_i \in [[nil(R_i)^{S,\leq}]]$. Now, it follows $f_i(u)g_i(v) \in nil(R_i)$, for any $u, v \in S$ and any $i \in I$, since R_i is generalized power series nil-reversible. Hence, for any $u, v \in S$,

$$f(u)g(v) = (f_i(u)g_i(v))_{i\in I} \in nil(R),$$

since I is finite. Thus, $f(u)g(v) \in nil(R)$. Then, by reversible ring, we have

$$(g_i(v)f_i(u))_{i\in I} = g(v)f(u) \in nil(R).$$

This means that $gf \in [[nil(R)^{S,\leq}]]$. The proof is done. □

Proposition 2.13. *Let $(S, \leq)$ be a strictly ordered monoid. If R is finite subdirect product of generalized power series nil-reversible rings, then R is generalized power series nil-reversible ring.*

Proof. Let $I_k (k = 1, \ldots, l)$ be ideals of R such that R/I_k is generalized power series nil-reversible and $\bigcap_{k=1}^{l} I_k = 0$. Let f and g be in $[[R^{S,\leq}]]$ with $fg \in [[nil(R)^{S,\leq}]]$. Clearly $\bar{f}\bar{g} \in [[nil(R/I_k)^{S,\leq}]]$. Since R/I_k is generalized power series nil-reversible, we have $(f(u)g(v))^{r_{u,v,k}} \in I_k$, for each $u, v \in S$ and $k = 1, \ldots, l$. Assume that $r_{u,v} = \max\{r_{u,v,k} | k = 1, \ldots, l\}$. So, $(f(u)g(v))^{r_{u,v}} \in \bigcap_{k=1}^{l} I_k = 0$. Hence, $f(u)g(v) \in nil(R)$, for each $u, v \in S$, then $g(v)f(u) \in nil(R)$. Thus, $gf \in [[nil(R)^{S,\leq}]]$, and we are done. □

Proposition 2.14. *Let $(S,\leq)$ a strictly ordered monoid. Let R be a ring and $e^2=e\in R$. If R is generalized power series nil-reversible, then so is eRe.*

Proof. Let $c_efc_e, c_egc_e \in [[(eRe)^{S,\leq}]]$ with $(c_efc_e)(c_egc_e) \in [[nil(eRe)^{S,\leq}]]$. Let e be an idempotent of R. It is easy to see that c_e is an idempotent element of $[[(eRe)^{S,\leq}]]$ and $c_eg = gc_e$ for every $g \in [[(R)^{S,\leq}]]$. Then, $(c_ef)(c_eg) \in [[nil(eR)^{S,\leq}]]$. Since R is generalized power series nil-reversible, we have $fg \in [[nil(R)^{S,\leq}]]$, and so $gf \in [[nil(R)^{S,\leq}]]$. Then, there exists $m \in \mathbb{N}$ such that $((c_efc_e)(c_egc_e))^m = 0$. Hence, $(c_egc_e)(c_efc_e) \in [[nil(eRe)^{S,\leq}]]$. □

Corollary 2.15. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid. For a central idempotent e of a ring R, eR and $(1-e)R$ are generalized power series nil-reversible if and only if R is generalized power series nil-reversible.*

Proof. Assume that eR and $(1-e)R$ are generalized power series nil-reversible. Since the nil generalized power series reversibility property is closed under finite direct products, $R \cong eR \times (1-e)R$ is generalized power series nil-reversible. The converse is trivial by Proposition 2.14. □

In [20], A homomorphic image of a nil-reversible ring may not be nil-reversible, so as generalized power series nil-reversible by the next example.

Example 2.16. Let R be a ring, $(S,\leq)$ a strictly ordered monoid. Assume that $R = D[[S,\leq]]$, where D is a division ring and $I =< xy >$, where $xy \neq yx$. As R is a domain, R is generalized power series nil-reversible. Clearly $\overline{yx} \in nil(R/I)[[S,\leq]]$ and $\overline{x}(\overline{yx}) = \overline{xyx} = 0$. But, $(\overline{yx})\overline{x} = \overline{yx^2} \neq 0$. This implies R/I is not generalized power series nil-reversible.

Theorem 2.17. *Let R be a ring and $(S,\leq)$ a strictly ordered monoid. If R is a generalized power series nil-reversible and I an ideal consisting of nilpotent elements of bounded index $\leq n$ in R, then R/I is generalized power series nil-reversible.*

Proof. Let $\bar{f},\bar{g} \in [[(R/I)^{S,\leq}]]$ with $\bar{f}\bar{g} \in [[nil(R/I)^{S,\leq}]]$. By hypothesis, the order $(S,\leq)$ can be refined to a strict total order $\leq$ on S. We will use transfinite induction on the strictly totally ordered set $(S,\leq)$ to show that $\bar{g}\bar{f} \in [[nil(R/I)^{S,\leq}]]$. Firstly, by transfinite induction to show $g(t)f(s) \in nil(R)$, for any $s \in supp(f)$ and any $t \in supp(g)$. Since $supp(f)$ and $supp(g)$ are nonempty subsets of S, the set of minimal elements of $supp(f)$ and $supp(g)$, respectively, are finite and non-empty. Let s_0 and t_0 denote the minimum elements of $supp(f)$ and $supp(g)$ in the $\leq$ order, respectively. By analogy with the proof of Theorem 2.25 [21], we can show that $f(s_0)g(t_0) = 0$. Therefore, by transfinite induction, we can proof that $f(s)g(t) = 0$. Since $\bar{f}\bar{g} \in [[nil(R/I)^{S,\leq}]]$, then, there is a positive integer $n \in \mathbb{N}$ such that $(\bar{f}\bar{g})^n = \bar{0}$. So, $(f(s)g(t))^n \in I$, for any $s,t \in S$. Since $I \subseteq nil(R), (f(s)g(t))^n = 0$. Hence, $f(s)g(t) \in nil(R)$, so $g(t)f(s) \in nil(R)$, by R is generalized power series nil-reversible, $gf \in [[nil(R)^{S,\leq}]]$. Thus, $\bar{g}\bar{f} \in [[nil(R/I)^{S,\leq}]]$. Therefore, R/I is generalized power series nil-reversible. □

Now, we give some characterizations of nil generalized power series reversibility by using the prime radical of a ring.

Corollary 2.18. *Let R be a ring and $(S, \leq)$ a strictly ordered monoid. If a ring R is generalized power series nil-reversible, then $R/P(R)$ is generalized power series nil-reversible.*

Proof. Since every element of $P(R)$ is nilpotent, the proof follows from Theorem 2.17. □

Proposition 2.19. *Let R be a ring and $(S, \leq)$ a strictly ordered monoid. Let J be a reduced ideal of a ring R such that R/J is generalized power series nil-reversible. Then, R is generalized power series nil-reversible.*

Proof. Let $f, g \in [[R^{S,\leq}]]$ and suppose that $fg \in [[nil(R)^{S,\leq}]]$. Then, $\bar{f}\bar{g} \in [[nil(R/J)^{S,\leq}]]$ and so $\bar{g}\bar{f} \in [[nil(R/J)^{S,\leq}]]$, since R/J is nil generalized power series reversible. There exists $m \in \mathbb{N}$ such that $(\bar{f}\bar{g})^m = \bar{0}$. This shows that $(f(s)g(t))^m \in J$, for any $s, t \in S$. Since J is reduced, we have $f(s)g(t) = 0$ yields $g(t)f(s) = 0$. Thus, $gf \in [[nil(R)^{S,\leq}]]$. Therefore, R is generalized power series nil-reversible. □

A ring is called semiperfect if every finitely generated R-right-module has a projective cover by [22]. For abelian semiperfect, here we have.

Theorem 2.20. *Let R be a ring and $(S, \leq)$ a strictly ordered monoid. Consider the following statements.*
(1) *R is a finite direct sum of local generalized power series nil-reversible rings.*
(2) *R is a semiperfect generalized power series nil-reversible ring.*
Then, (1) ⇒ (2) *and the converse is true when R is abelian.*

Proof. (1) ⇒ (2) Assume that R is a finite direct sum of local generalized power series nil-reversible rings. Then, R is semiperfect because local rings are semiperfect and a finite direct sum of semiperfect rings is semiperfect, and moreover R is generalized power series nil-reversible by Proposition 2.12.
(2) ⇒ (1) Suppose that R is an abelian semiperfect generalized power series nil-reversible ring. Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \ldots, e_n\}$ of local idempotents whose sum is 1 by Theorem 27.6 [23], say $1 = e_1 + e_2 + \ldots + e_n$ such that each e_iRe_i is a local ring where $1 \leq i \leq n$. The ring R being abelian implies $e_iRe_i = e_iR$. Each e_iR is a generalized power series nil-reversible by Proposition 2.14. Hence, R is generalized power series nil-reversible by Proposition 2.12. □

3. Weak annihilator of generalized power series reversible and some rings property

In [24], Ouyang introduced the notion of weak annihilators and investigated their properties. For a subset X of a ring R put $Nr_R(X) = \{a \in R | Xa \in nil(R)\}$

and $Nl_R(X) = \{b \in R | bX \in nil(R)\}$. By a simple computation, we can see that $Nr_R(X) = Nl_R(X)$. The set $Nr_R(X)$ is called the weak annihilator of X. It is also easy to see that, $Nr_R(X)$ is an ideal of R in case R is a NI-ring. Furthermore when R is reduced, then $r_R(X) = Nr_R(X) = l_R(X) = Nl_R(X)$. For more details and results on weak annihilators see [25].

Now, we investigate the relations between weak annihilators in a ring R and weak annihilators in a generalized power series ring $R[[S, \leq]]$. Given a ring R and let $\gamma = C(f)$ be the content of f, i.e., $C(f) = \{f(u) | u \in supp(f)\} \subseteq R$. Since, $R \simeq c_R$ we can identify, the content of f with

$$c_{C(f)} = \{c_{f(u_i)} | u_i \in supp(f)\} \subseteq [[R^{S,\leq}]].$$

Then, we have two maps $\phi : NrAnn_R(id(R)) \to NrAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ and $\psi : NlAnn_R(id(R)) \to NlAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ defined by $\phi(I) = I[[R^{S,\leq}]]$ and $\psi(J) = [[R^{S,\leq}]]J$ for every $I \in NrAnn_R(id(R)) = \{Nr_R(U) | U$ is an ideal of $R\}$ and $J \in NlAnn_R(id(R)) = \{Nl_R(U) | U$ is an ideal of $R\}$, respectively. Obviously, ϕ is injective. In the following theorem, we show that ϕ and ψ are bijective maps if and only if R is generalized power series *nil*-reversible.

Theorem 3.1. *Let R be a ring and $(S, \leq)$ a strictly ordered monoid. If R is reduced and $nil(R)$ is a nilpotent ideal of R, then the following are equivalent:*
(1) *R is generalized power series nil-reversible ring.*
(2) *The function $\phi : NrAnn_R(id(R)) \to NrAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ is bijective, where $\phi(I) = I[[R^{S,\leq}]]$ for every $I \in NrAnn_R(id(R))$.*
(3) *The function $\psi : NlAnn_R(id(R)) \to NlAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ is bijective, where $\psi(J) = [[R^{S,\leq}]]J$ for every $J \in NlAnn_R(id(R))$.*

Proof. (1)⇒(2) Let $Y \subseteq [[R^{S,\leq}]]$ and $\gamma = \bigcup_{f \in Y} C(f)$. From Proposition 2.6 it is sufficient to show that $Nr_{[[R^{S,\leq}]]}(f) = Nr_R C(f)[[R^{S,\leq}]]$, for all $f \in Y$. In fact, let $g \in Nr_{[[R^{S,\leq}]]}(f)$. Then, $fg \in [[nil(R)^{S,\leq}]]$ and by assumption $f(u_i)g(v_j) \in nil(R)$ for each $u_i \in supp(f)$ and each $v_j \in supp(g)$. Then, for a fixed $u_i \in supp(f)$ and each $v_j \in supp(g), 0 = f(u_i)g(v_j) = (c_{f(u_i)}g)(v_j)$, it follows that

$$g \in Nr_R \bigcup_{u_i \in supp(f)} c_{f(u_i)}[[R^{S,\leq}]] = Nr_R C(f)[[R^{S,\leq}]].$$

So,

$$Nr_{[[R^{S,\leq}]]}(f) \subseteq Nr_R C(f)[[R^{S,\leq}]].$$

Conversely, let $g \in Nr_R C(f)[[R^{S,\leq}]]$, then $c_{f(u_i)}g \in [[nil(R)^{S,\leq}]]$ for each $u_i \in supp(f)$. Hence, $(c_{f(u_i)}g)(v_j) = f(u_i)g(v_j) \in nil(R)$ and $v_j \in supp(g)$. Thus,

$$(fg)(s) = \sum_{(u_i,v_j) \in X_s(f,g)} f(u_i)g(v_j) = 0$$

and it follows that $g \in Nr_{[[R^{S,\leq}]]}(f)$. Hence, $Nr_R C(f)[[R^{S,\leq}]] \subseteq Nr_{[[R^{S,\leq}]]}(f)$ and it follows that $Nr_R C(f)[[R^{S,\leq}]] = Nr_{[[R^{S,\leq}]]}(f)$. So,

$$Nr_{[[R^{S,\leq}]]}(Y) = \bigcap_{f \in Y} Nr_{[[R^{S,\leq}]]}(f) = \bigcap_{f \in Y} C(f)[[R^{S,\leq}]] = Nr_R(\gamma)[[R^{S,\leq}]].$$

(2)⇒(1) Suppose that $f, g \in [[R^{S,\leq}]]$ be such that $fg \in [[nil(R)^{S,\leq}]]$. Then, $g \in Nr_{[[R^{S,\leq}]]}(f)$ and by assumption $Nr_{[[R^{S,\leq}]]}(f) = \gamma[[R^{S,\leq}]]$ for some right ideal γ of R. Consequently, $0 = fc_{g(v_j)}$ and for any $u_i \in supp(f), (fc_{g(v_j)})(u_i) = f(u_i)g(v_j) \in nil(R)$ for each $u_i \in supp(f)$ and $v_j \in supp(g)$. Thus by reduced ring, $g(v_j)f(u_i) \in nil(R)$, then $gf \in [[nil(R)^{S,\leq}]]$. Hence, R is generalized power series nil-reversible. The proof of (1)⇔(3) is similar to the proof of (1)⇔(2). □

According to Liu [26], the ring R is called S-Armendariz if whenever $f, g \in [[R^{S,\leq}]]$ satisfy $fg = 0$, then $f(u)g(v) = 0$ for each $u, v \in S$. Now, we given a strong condition under which $[[R^{S,\leq}]]$ is nil-reversible.

Theorem 3.2. *Let R be a ring and $(S, \leq)$ a strictly ordered monoid. Assume that R is an S-Armendariz ring, then R is generalized power series nil-reversible if and only if $[[R^{S,\leq}]]$ is nil-reversible.*

Proof. Suppose R is generalized power series nil-reversible. Let $f, g \in [[R^{S,\leq}]]$ be such that $fg \in [[nil(R)^{S,\leq}]]$. By [27, Proposition 2.17], $[[nil(R)^{S,\leq}]] = nil([[R^{S,\leq}]])$. So, $f(u_i)g(v_j) \in nil(R)$, for any $u, v \in S$ and any i, j. Since R is S-Armendariz, $f(u_i)g(v_j) = 0$, for all i, j. By nil-reversibility, $g(v_j)f(u_i) \in nil(R)$, for all i, j. So, $gf = 0$. Therefore, $[[R^{S,\leq}]]$ is nil-reversible. The proof of the converse is trivial. □

Theorem 3.3. *Let R be a ring and $(S, \leq)$ a strictly ordered monoid. Let Δ denotes a multiplicatively closed subset of R consisting of central non-zero divisors. Then, R is generalized power series nil-reversible if and only if $\Delta^{-1}R$ is generalized power series nil-reversible.*

Proof. Suppose R is generalized power series nil-reversible and $p_i, d_j, u, v \in R$. Let $u^{-1}C_{p_i}, v^{-1}C_{d_j} \in \Delta^{-1}R[[S, \leq]]$, for all i, j satisfying that $u^{-1}C_{p_i}v^{-1}C_{d_j} \in nil(\Delta^{-1}R[[S, \leq]])$. Then, $(u^{-1}C_{p_i}v^{-1}C_{d_j})^n = 0$ for some positive integer n. This implies $(C_{p_i}C_{d_j})^n = 0$, so $p_i d_j \in nil(R)$. For any $u^{-1}C_{p_i}, v^{-1}C_{d_j} \in \Delta^{-1}R[[S, \leq]]$ having the property that $(u^{-1}C_{p_i})(v^{-1}C_{d_j}) = 0$, we have $(uv)^{-1}C_{p_i}C_{d_j} = 0, C_{p_i}C_{d_j} = 0$, for all i, j. Since R is generalized power series nil-reversible, $d_j p_i \in nil(R)$, so $(v^{-1}u^{-1})C_{d_j}C_{p_i} = 0$ which further yields $(v^{-1}C_{d_j})(u^{-1}C_{p_i}) \in nil(\Delta^{-1}R[[S, \leq]])$. Hence, $\Delta^{-1}R$ is generalized power series nil-reversible. The converse part is trivial. □

Following Lambek [28], a ring R is called symmetric if $abc = 0$ implies $acb = 0$, for all $a, b, c \in R$. It is obvious that commutative rings are symmetric and symmetric rings are reversible ring.

Theorem 3.4. *Let R be a reversible ring and $(S, \leq)$ a strictly ordered monoid with $nil(R)$ is a nilpotent ideal of R. Then, R is generalized power series nil-symmetric if and only if $R[[S, \leq]]$ is nil-symmetric.*

Proof. Assume that R is generalized power series nil-symmetric and $f, g, h \in R[[S, \leq]]$ are such that $fgh \in nil(R[[S, \leq]])$. Hence, by Proposition 2.6, $f(u)g(v)h(t) \in nil(R)$, for all $u, v, t \in S$. Since R is nil-symmetric, we have $f(u)h(t)g(v) \in nil(R)$. Now, for all $s \in S$, we have

$$(fhg)(s) = \sum_{(u,t,v) \in X_s(f,h,g)} f(u)h(t)g(v).$$

So, the reversibility of R imply that $fhg \in nil(R[[S, \leq]])$, hence $R[[S, \leq]]$ is nil-symmetric. Conversely, if $R[[S, \leq]]$ is nil-symmetric, then R is generalized power series nil-symmetric, as subrings of generalized power series nil-symmetric rings are also generalized power series nil-symmetric. □

4. Conclusion

In this paper, we have introduced the notion of generalized power series nil-reversible rings. The researchers obtains various necessary or sufficient conditions for a generalized power series nil-reversible rings to be some rings related. We use this concept by studying the relations between generalized power series reversible and some certain classes of rings. One can extend this work to study different rings on this structure. Further one can identify some real life applications in a monoid homomorphism and ideal rings. In our future work we will introduce the concept of skew generalized power series nil-reversible, that is a generalization of power series nil-reversible, when R is S-compatible, $(S, \leq)$ a strictly ordered monoid and connected by annihilator rings.

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