Recursion formulas for Humbert's matrix functions

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Abstract. Special matrix functions have become a major area of study for mathematicians and physicists over the last two decades. The famous Humbert's matrix functions have received considerable attention by many authors from different points of view [5, 16, 24]. Inspired by the recent work by Abd-Elmageed *et.al.* [1], who established recursion formulas satisfied by the first Appell matrix function, namely F_1. In this paper, we find the recursion formulas for Humbert's matrix functions. This enriches the theory of special matrix functions. The obtained results are believed to be newly presented.

Keywords: Matrix functional calculus, Recursion formula, Humbert's matrix function.

1. Introduction

The theory of special functions and its generalisations appear frequently in physics, probability theory, engineering, and Lie theory, amongst other fields. Recursion formulas for the Appell functions have been studied in the literature, see [17, 28]. Recursion formulas forl multivariable hypergeometric functions were presented in [3, 19, 20, 21, 22]. Humbert's functions constitute a set of seven hy-

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pergeometric functions of two variables that are confluent cases of two variable Appell hypergeometric functions and generalize the Kummer's confluent hypergeometric function ${}_1F_1$ of one variable. The class of classical Humbert functions has been recently studied for reduction and summation formulas [4, 6, 25].

The matrix theory is used in orthogonal polynomials and special functions, and it is widely used in mathematics in general. Due to their applications in physics, engineering, probability theory, and Lie theory, special matrix functions have received a lot of attention [7, 10]. Special matrix functions connected to the matrix version of Laguerre, Hermite and Legendre differential equations and the corresponding polynomial families in [11, 12, 13]. Recently, Abd-Elmageed *et. al.* and Verma [1, 26] have obtained recursion formulas satisfied by the first Appell matrix function, namely F_1 and Srivastava's triple hypergeometric matrix functions. In [23, 27], recursion formulas for the Gauss hypergeometric matrix function and Lauricella matrix functions are presented. Motivated by this study, we obtain recursion formulas for Humbert's matrix functions.

The paper is organized as follows. In Section 2, we give a review of basic definitions that are needed in the sequel. In Section 3, we obtain the recursion formulas for Humbert's matrix function.

2. Preliminaries

Let $\mathbb{C}^{r\times r}$ be the vector space of r-square matrices with complex entries. For any matrix $A \in \mathbb{C}^{r\times r}$, its spectrum $\sigma(A)$ is the set of eigenvalues of A. The spectral abscissa of A is given by $\alpha(A)=$ max $[\Re(z)|z \in \sigma(A)]$, where $\Re(z)$ denotes the real part of a complex number z. If $\beta(A)=$ min $[\Re(z)|z \in \sigma(A)]$, then $\beta(A) = -\alpha(-A)$. A square matrix A in $\mathbb{C}^{r\times r}$ is said to be positive stable if $\beta(A) > 0$. The 2-norm of A is denoted by $||A||$ and defined by

$$||A|| = \max_{x\neq 0}\frac{||Ax||_2}{||x||_2} = \max[\sqrt(\lambda)|\lambda \in (A^\star A)], \tag{1}$$

where for any vector x in the r-dimensional complex space, $||x||_2 = (x^\star x)^{\frac{1}{2}}$ is the Euclidean norm of x and $A^\star$ denotes the transposed conjugate of A. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and A is a matrix in $\mathbb{C}^{r\times r}$ with $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus [8], it follows that

$$f(A)g(A) = g(A)f(A). \tag{2}$$

Furthermore, if $B \in \mathbb{C}^{r\times r}$ is a matrix for which $\sigma(B) \subset \Omega$, and if $AB = BA$, then

$$f(A)g(B) = g(B)f(A). \tag{3}$$

If A is a positive stable matrix in $\mathbb{C}^{r\times r}$, then $\Gamma(A)$ can be expressed as [15]

$$\Gamma(A) = \int_0^\infty e^{-t}\, t^{A-I}\, dt, \tag{4}$$

where, $t^{A-I} = \exp((A-I)\ln t)$ and ln is the principal branch of the logarithmic function.

Furthermore, if $A+kI$ is invertible for all integers $k \geq 0$, then the reciprocal gamma matrix function is defined as [15]

$$\Gamma^{-1}(A) = A(A+I)\dots(A+(n-1)I)\Gamma^{-1}(A+nI),\ n \geq 1. \tag{5}$$

By application of the matrix functional calculus, the Pochhammer symbol for $A \in \mathbb{C}^{r\times r}$ is given by [15]

$$(A)_n = \begin{cases} I, & \text{if } n = 0, \\ A(A+I)\dots(A+(n-1)I), & \text{if } n \geq 1. \end{cases} \tag{6}$$

This gives

$$(A)_n = \Gamma^{-1}(A)\ \Gamma(A+nI), \qquad n \geq 1. \tag{7}$$

Humbert's matrix functions are defined as follows [2, 5, 18]:

$$\Phi_1(A,B;C;x,y) = \sum_{m,n=0}^{\infty} (A)_{m+n}(B)_n(C)^{-1}_{m+n}\,\frac{x^m\,y^n}{m!\,n!}, \tag{8}$$

$$\Phi_2(A,A';C;x,y) = \sum_{m,n=0}^{\infty} (A)_m(A')_n(C)^{-1}_{m+n}\,\frac{x^m\,y^n}{m!\,n!}, \tag{9}$$

$$\Phi_3(A;C;x,y) = \sum_{m,n=0}^{\infty} (A)_m(C)^{-1}_{m+n}\,\frac{x^m\,y^n}{m!\,n!}, \tag{10}$$

$$\Psi_1(A,B;C,C';x,y) = \sum_{m,n=0}^{\infty} (A)_{m+n}(B)_m(C)^{-1}_m(C')^{-1}_n\,\frac{x^m\,y^n}{m!\,n!}, \tag{11}$$

$$\Psi_2(A;C,C';x,y) = \sum_{m,n=0}^{\infty} (A)_{m+n}(C)^{-1}_m(C')^{-1}_n\,\frac{x^m\,y^n}{m!\,n!}, \tag{12}$$

$$\Xi_1(A,A',B;C;x,y) = \sum_{m,n=0}^{\infty} (A)_m(A')_n(B)_m(C)^{-1}_{m+n}\,\frac{x^m\,y^n}{m!\,n!}, \tag{13}$$

$$\Xi_2(A,B;C;x,y) = \sum_{m,n=0}^{\infty} (A)_m(B)_m(C)^{-1}_{m+n}\,\frac{x^m\,y^n}{m!\,n!}, \tag{14}$$

where A, A', B, C and C' are matrices in $\mathbb{C}^{r\times r}$ such that $C+kI$ and $C'+kI$ are invertible for all integers $k \geq 0$.

3. Recursion formulas for Humbert's matrix functions

In this section, we obtain the recursion formulas for Humbert's matrix functions. Throughout the paper, I denotes the identity matrix and s denotes a non-negative integer.

Theorem 3.1. *Let $A+sI$ be an invertible matrix for all integers $s \geq 0$ and $BC = CB$. Then the following recursion formula holds true for Humbert's matrix function Φ_1:*

$$\begin{aligned}
&\Phi_1(A+sI,B;C;x,y)\\
&= \Phi_1(A,B;C;x,y) + x\left[\sum_{k=1}^{s} \Phi_1(A+kI,B;C+I;x,y)\right]C^{-1}\\
&+ y\left[\sum_{k=1}^{s} \Phi_1(A+kI,B+I;C+I;x,y)\right]BC^{-1}, \qquad (15)
\end{aligned}$$

$$\begin{aligned}
&\Phi_1(A+sI,B;C;x,y)\\
&= \sum_{k_1+k_2\leq s} \binom{s}{k_1,k_2} x^{k_1} y^{k_2}\\
&\times \Big[\Phi_1(A+(k_1+k_2)I,B+k_2I;C+(k_1+k_2)I;x,y)\Big](B)_{k_2}(C)^{-1}_{k_1+k_2}. \qquad (16)
\end{aligned}$$

Furthermore, if $A-kI$ is invertible for integers $k \leq s$, then

$$\begin{aligned}
&\Phi_1(A-sI,B;C;x,y)\\
&= \Phi_1(A,B;C;x,y) - x\left[\sum_{k=0}^{s-1} \Phi_1(A-kI,B;C+I;x,y)\right]C^{-1}\\
&- y\left[\sum_{k=0}^{s-1} \Phi_1(A-kI,B+I;C+I;x,y)\right]BC^{-1}, \qquad (17)
\end{aligned}$$

$$\begin{aligned}
&\Phi_1(A-sI,B;C;x,y)\\
&= \sum_{k_1+k_2\leq s} \binom{s}{k_1,k_2} (-x)^{k_1} (-y)^{k_2}\\
&\times \Big[\Phi_1(A,B+k_2I;C+(k_1+k_2)I;x,y)\Big](B)_{k_2}(C)^{-1}_{k_1+k_2}, \qquad (18)
\end{aligned}$$

where $\binom{s}{k_1,k_2} = \frac{s!}{k_1!k_2!(s-k_1-k_2)!}$.

Proof. From the definition of Humbert's matrix function Φ_1 and the transformation

$$(A+I)_{m+n} = A^{-1}(A)_{m+n}\,(A+mI+nI)$$

we get the following contiguous matrix relation:

$$\begin{aligned}
&\Phi_1(A+I,B;C;x,y)\\
&= \Phi_1(A,B;C;x,y) + x\Big[\Phi_1(A+I,B;C+I;x,y)\Big]C^{-1}\\
&+ y\Big[\Phi_1(A+I,B+I;C+I;x,y)\Big]BC^{-1}. \qquad (19)
\end{aligned}$$

To calculate contiguous matrix relation for $\Phi_1(A+2I,B,B';C;x,y)$, we replace A with $A+I$ in (19) and use in (19). This gives

$$\begin{aligned}&\Phi_1(A+2I,B;C;x,y)=\Phi_1(A,B;C;x,y)\\&+x\Big[\Phi_1(A+I,B;C+I;x,y)+\Phi_1(A+2I,B;C+I;x,y)\Big]C^{-1}\\&+y\Big[\Phi_1(A+I,B+I;C+I;x,y)+\Phi_1(A+2I,B+I;C+I;x,y)\Big]BC^{-1}.\end{aligned}\tag{20}$$

Iterating this process s times, we obtain (15). For the proof of (17), replace the matrix A with $A-I$ in (19). As $A-I$ is invertible, this gives

$$\begin{aligned}&\Phi_1(A-I,B;C;x,y)=\Phi_1(A,B;C;x,y)-x\Big[\Phi_1(A,B;C+I;x,y)\Big]C^{-1}\\&-y\Big[\Phi_1(A,B+I;C+I;x,y)\Big]BC^{-1}.\end{aligned}\tag{21}$$

Iteratively, we get (17).

The proof of (16) is based upon the principle of mathematical induction on $s\in\mathbb{N}$. For $s=1$, the result (16) is true obviously. Suppose (16) is true for $s=t$, that is,

$$\begin{aligned}&\Phi_1(A+tI,B;C;x,y)=\sum_{k_1+k_2\le t}\binom{t}{k_1,k_2}x^{k_1}y^{k_2}\\&\times\Phi_1(A+(k_1+k_2)I,B+k_2I;C+(k_1+k_2)I;x,y)(B)_{k_2}(C)^{-1}_{k_1+k_2},\end{aligned}\tag{22}$$

Replacing A with $A+I$ in (22) and using the contiguous matrix relation (19), we get

$$\begin{aligned}&\Phi_1(A+tI+I,B;C;x,y)=\sum_{k_1+k_2\le t}\binom{t}{k_1,k_2}x^{k_1}y^{k_2}\\&\times\Big[\Phi_1(A+(k_1+k_2)I,B+k_2I;C+(k_1+k_2)I;x,y)+x\\&\times\Phi_1(A+(k_1+k_2)I+I,B+k_2I;C+(k_1+k_2)I+I;x,y)(C+(k_1+k_2)I)^{-1}\\&\times+y\Phi_1(A+(k_1+k_2)I+I,B+k_2I+I;C+(k_1+k_2)I+I;x,y)\\&\times(B+k_2I)(C+(k_1+k_2)I)^{-1}\Big](B)_{k_2}(C)^{-1}_{k_1+k_2}.\end{aligned}\tag{23}$$

Simplifying, (23) takes the form

$$\begin{aligned}&\Phi_1(A+tI+I,B;C;x,y)\\&=\sum_{k_1+k_2\le t}\binom{t}{k_1,k_2}x^{k_1}y^{k_2}\\&\times\Phi_1(A+(k_1+k_2)I,B+k_2I;C+(k_1+k_2)I;x,y)(B)_{k_2}(C)^{-1}_{k_1+k_2}\\&+\sum_{k_1+k_2\le t+1}\binom{t}{k_1-1,k_2}x^{k_1}y^{k_2}\\&\times\Phi_1(A+(k_1+k_2)I,B+k_2I;C+(k_1+k_2)I;x,y)(B)_{k_2}(C)^{-1}_{k_1+k_2}\end{aligned}$$

$$+\sum_{k_1+k_2\le t+1}\binom{t}{k_1,k_2-1}x^{k_1}y^{k_2}$$

$$\times\,\Phi_1(A+(k_1+k_2)I,B+k_2I;C+(k_1+k_2)I;x,y)(B)_{k_2}(C)^{-1}_{k_1+k_2}. \tag{24}$$

Using Pascal's identity in (24), we have

$$\Phi_1(A+(t+1)I,B;C;x,y)=\sum_{k_1+k_2\le t+1}\binom{t+1}{k_1,k_2}x^{k_1}y^{k_2}$$

$$\times\,\Phi_1(A+(k_1+k_2)I,B+k_2I;C+(k_1+k_2)I;x,y)(B)_{k_2}(C)^{-1}_{k_1+k_2}. \tag{25}$$

This establishes (16) for $s=t+1$. Hence by induction, result given in (16) is true for all values of s. The second recursion formula (18) can be proved in a similar manner. □

Now, we present the recursion formulas for the matrix B of the Humbert's matrix function Φ_1. We omit the proofs of the given below theorems.

Theorem 3.2. *Let $B+sI$ be invertible matrix for all integers $s\ge 0$. Then the following recursion formulas hold true for Humbert's matrix function Φ_1:*

$$\Phi_1(A,B+sI;C;x,y)$$

$$=\Phi_1(A,B;C;x,y)+yA\Big[\sum_{k=1}^{s}\Phi_1(A+I,B+kI;C+I;x,y)\Big]C^{-1}, \tag{26}$$

$$\Phi_1(A,B-sI;C;x,y)$$

$$=\Phi_1(A,B;C;x,y)-yA\Big[\sum_{k=0}^{s-1}\Phi_1(A+I,B-kI;C+I;x,y)\Big]C^{-1}. \tag{27}$$

Theorem 3.3. *Let $B+sI$ be invertible matrix for all integers $s\ge 0$ then the following recursion formulas hold true for Humbert's matrix function Φ_1:*

$$\Phi_1(A,B+sI;C;x,y)$$

$$=\sum_{k_1=0}^{s}\binom{s}{k_1}(A)_{k_1}\,y^{k_1}\Big[\ \Phi_1(A+k_1I,B+k_1I;C+k_1I;x,y)\Big](C)^{-1}_{k_1}. \tag{28}$$

Furthermore, if $B-kI$ are invertible for $k\le s$, then

$$\Phi_1(A,B-sI;C;x,y)$$

$$=\sum_{k_1=0}^{s}\binom{s}{k_1}(A)_{k_1}\,(-y)^{k_1}\Big[\Phi_1(A+k_1I,B;C+k_1I;x,y)\Big](C)^{-1}_{k_1}. \tag{29}$$

Theorem 3.4. *Let $C - sI$ be an invertible matrix for all integers $s \geq 0$ and let $AB = BA$, then the following recursion formula holds true for Humbert's matrix function Φ_1:*

$$\begin{aligned}
&\Phi_1(A, B; C - sI; x, y) = \Phi_1(A, B; C; x, y) \\
&+ xA \sum_{k=1}^{s} \Big[\Phi_1(A + I, B; C + (2 - k)I; x, y)\Big] \\
&\times (C - kI)^{-1}(C - (k - 1)I)^{-1} \\
&+ yAB \sum_{k=1}^{s} \Big[\Phi_1(A + I, B + I; C + (2 - k)I; x, y)\Big] \\
&\times (C - kI)^{-1}(C - (k - 1)I)^{-1}. \qquad (30)
\end{aligned}$$

Proof. Applying the definition of Humbert's matrix function Φ_1 and the relation $(C - I)^{-1}_{m+n} = (C)^{-1}_{m+n}\left[1 + m(C - I)^{-1} + n(C - I)^{-1}\right]$, we obtain the following contiguous matrix relation:

$$\begin{aligned}
&\Phi_1(A, B, B'; C - I; x, y) \\
&= \Phi_1(A, B, B'; C; x, y) + xA\Big[\Phi_1(A + I, B; C + I; x, y)\Big]C^{-1}(C - I)^{-1} \\
&+ yAB\Big[\Phi_1(A + I, B + I; C + I; x, y)\Big]C^{-1}(C - I)^{-1}. \qquad (31)
\end{aligned}$$

We get (30) by using this contiguous matrix relation in Humbert's matrix function Φ_1 with the matrix $C - sI$ for s times. □

We state without proofs recursion formulas for remaining Humbert's matrix functions.

Theorem 3.5. *Let $A + sI$ and $A' + sI$ be an invertible matrix for all integers $s \geq 0$. Then the following recursion formula holds true for Humbert's matrix function Φ_2:*

$$\begin{aligned}
&\Phi_2(A + sI, A'; C; x, y) \\
&= \Phi_2(A, A'; C; x, y) + x\Big[\sum_{k=1}^{s} \Phi_2(A + kI, A'; C + I; x, y)\Big]C^{-1}, \qquad (32)
\end{aligned}$$

$$\begin{aligned}
&\Phi_2(A + sI, A'; C; x, y) \\
&= \sum_{k_1 \leq s} \binom{s}{k_1} x^{k_1} \Big[\Phi_2(A + k_1I, A'; C + k_1I; x, y)\Big] (C)^{-1}_{k_1}, \qquad (33)
\end{aligned}$$

$$\begin{aligned}
&\Phi_2(A, A' + sI; C; x, y) \\
&= \Phi_2(A, A'; C; x, y) + y\Big[\sum_{k=1}^{s} \Phi_2(A, A' + kI; C + I; x, y)\Big]C^{-1}, \qquad (34)
\end{aligned}$$

$$\Phi_2(A, A'+sI; C; x, y) = \sum_{k_1 \le s} \binom{s}{k_1} y^{k_1} \Big[\Phi_2(A, A'+k_1 I; C+k_1 I; x, y)\Big] (C)_{k_1}^{-1}. \tag{35}$$

If $A-kI$ and $A'-kI$ are invertible for $k \le s$, then

$$\Phi_2(A-sI, A'; C; x, y) = \Phi_2(A, A'; C; x, y) - x\Big[\sum_{k=0}^{s-1} \Phi_2(A-kI, A'; C+I; x, y)\Big] C^{-1}, \tag{36}$$

$$\Phi_2(A-sI, A'; C; x, y) = \sum_{k_1 \le s} \binom{s}{k_1} (-x)^{k_1} \Big[\Phi_2(A, A'; C+k_1 I; x, y)\Big] (C)_{k_1}^{-1}, \tag{37}$$

$$\Phi_2(A, A'-sI; C; x, y) = \Phi_2(A, A'; C; x, y) - y\Big[\sum_{k=0}^{s-1} \Phi_2(A, A'-kI; C+I; x, y)\Big] C^{-1}, \tag{38}$$

$$\Phi_2(A, A'-sI; C; x, y) = \sum_{k_1 \le s} \binom{s}{k_1} (-y)^{k_1} \Big[\Phi_2(A, A'; C+k_1 I; x, y)\Big] (C)_{k_1}^{-1}. \tag{39}$$

Theorem 3.6. *Let $C-sI$ be invertible matrices for all integers $s \ge 0$ and $AA' = A'A$. Then the following recursion formulas hold true for Humbert's matrix function Φ_2:*

$$\begin{aligned} &\Phi_2(A, A'; C-sI; x, y) \\ &= \Phi_2(A, B, B'; C, C'; x, y) + xA\Big[\sum_{k=1}^{s} \Phi_2(A+I, A'; C+(2-k)I; x, y) \\ &\times (C-kI)^{-1}(C-(k-1)I)^{-1}\Big] + yA'\Big[\sum_{k=1}^{s} \Phi_2(A, A'+I; C+(2-k)I; x, y) \\ &\times (C-kI)^{-1}(C-(k-1)I)^{-1}\Big]. \end{aligned} \tag{40}$$

Theorem 3.7. *Let $A+sI$ be invertible matrices for all integers $s \ge 0$. Then the following recursion formulas hold true for Humbert's matrix function Φ_3:*

$$\Phi_3(A+sI; C; x, y) = \Phi_3(A; C; x, y) + x\Big[\sum_{k=1}^{s} \Phi_3(A+kI; C+I; x, y)\Big] C^{-1}, \tag{41}$$

$$\Phi_3(A+sI;C;x,y) = \sum_{k_1=0}^{s}\binom{s}{k_1}x^{k_1}\Big[\Phi_3(A+k_1I;C+k_1I;x,y)\Big](C)_{k_1}^{-1}. \tag{42}$$

Furthermore, if $A-kI$ and $A'-kI$ are invertible for $k\le s$, then

$$\Phi_3(A-sI;C;x,y) = \Phi_3(A;C;x,y) - x\Big[\sum_{k=0}^{s-1}\Phi_3(A-kI;C+I;x,y)\Big]C^{-1}; \tag{43}$$

$$\Phi_3(A-sI;C;x,y) = \sum_{k_1=0}^{s}\binom{s}{k_1}(-x)^{k_1}\Big[\Phi_3(A;C+k_1I;x,y)\Big](C)_{k_1}^{-1}. \tag{44}$$

Theorem 3.8. *Let $C-sI$ be an invertible matrix for all integers $s\ge 0$. Then the following recursion formula holds true for Humbert's matrix function Φ_3:*

$$\begin{aligned}\Phi_3(A;C-sI;x,y) &= \Phi_3(A;C;x,y) + xA\Big[\sum_{k=1}^{s}\Phi_3(A+I;C+(2-k)I;x,y)\Big]\\ &\times (C-kI)^{-1}(C-(k-1)I)^{-1}\\ &+ y\Big[\sum_{k=1}^{s}\Phi_3(A;C+(2-k)I;x,y)(C-kI)^{-1}(C-(k-1)I)^{-1}\Big].\end{aligned} \tag{45}$$

Theorem 3.9. *Let $A+sI$ be an invertible matrix for all integers $s\ge 0$ and let $AB=BA$; $CC'=C'C$. Then the following recursion formula holds true for Humbert's matrix function Ψ_1:*

$$\begin{aligned}\Psi_1(A+sI,B;C,C';x,y) &= \Psi_1(A,B;C,C';x,y) + xB\Big[\sum_{k=1}^{s}\Psi_1(A+kI,B+I;C+I,C';x,y)\Big]C^{-1}\\ &+ y\Big[\sum_{k=1}^{s}\Psi_1(A+kI,B;C,C'+I;x,y)\Big]C'^{-1},\end{aligned} \tag{46}$$

$$\begin{aligned}\Psi_1(A+sI,B;C,C';x,y) &= \sum_{k_1+k_2\le s}\binom{s}{k_1,k_2}(B)_{k_1}x^{k_1}y^{k_2}\\ &\times\Big[\Psi_1(A+(k_1+k_2)I,B+k_1I;C+k_1I,C'+k_2I;x,y)\Big](C)_{k_1}^{-1}(C')_{k_2}^{-1}.\end{aligned} \tag{47}$$

Furthermore, if $A - kI$ is invertible for $k \leq s$, then

$$\begin{aligned}
&\Psi_1(A - sI, B; C, C'; x, y) \\
&= \Psi_1(A, B; C, C'; x, y) - xB\Big[\sum_{k=0}^{s-1} \Psi_1(A - kI, B + I; C + I, C'; x, y)\Big]C^{-1} \\
&- y\Big[\sum_{k=0}^{s-1} \Psi_1(A - kI, B; C, C' + I; x, y)\Big]C'^{-1},
\end{aligned} \tag{48}$$

$$\begin{aligned}
&\Psi_1(A - sI, B; C, C'; x, y) \\
&= \sum_{k_1 + k_2 \leq s} \binom{s}{k_1, k_2} (B)_{k_1} (-x)^{k_1} (-y)^{k_2} \\
&\times \Big[\Psi_1(A, B + k_1 I; C + k_1 I, C' + k_2 I; x, y)\Big](C)^{-1}_{k_1} (C')^{-1}_{k_2}.
\end{aligned} \tag{49}$$

Theorem 3.10. *Let $B + sI$ be an invertible matrix for all integers $s \geq 0$ and let $CC' = C'C$. Then the following recursion formula holds true for Humbert's matrix function Ψ_1:*

$$\begin{aligned}
&\Psi_1(A, B + sI; C, C'; x, y) = \Psi_1(A, B; C, C'; x, y) \\
&+ xA\Big[\sum_{k=1}^{s} \Psi_1(A + I, B + kI; C + I, C'; x, y)\Big]C^{-1},
\end{aligned} \tag{50}$$

$$\begin{aligned}
&\Psi_1(A, B + sI; C, C'; x, y) \\
&= \sum_{k_1 + k_2 \leq s} \binom{s}{k_1} (A)_{k_1} x^{k_1} \\
&\times \Big[\Psi_1(A + k_1 I, B + k_1 I; C + k_1 I, C'; x, y)\Big](C)^{-1}_{k_1}.
\end{aligned} \tag{51}$$

Furthermore, if $B - kI$ is invertible for $k \leq s$, then

$$\begin{aligned}
&\Psi_1(A, B - sI; C, C'; x, y) = \Psi_1(A, B; C, C'; x, y) \\
&- xA\Big[\sum_{k=0}^{s-1} \Psi_1(A + I, B - kI; C + I, C'; x, y)\Big]C^{-1},
\end{aligned} \tag{52}$$

$$\begin{aligned}
&\Psi_1(A, B - sI; C, C'; x, y) \\
&= \sum_{k_1 + k_2 \leq s} \binom{s}{k_1} (A)_{k_1} (-x)^{k_1} \\
&\times \Big[\Psi_1(A + k_1 I, B; C + k_1 I, C'; x, y)\Big](C)^{-1}_{k_1}.
\end{aligned} \tag{53}$$

Theorem 3.11. *Let $C - sI$ and $C' - sI$ be invertible matrices for all integers $s \geq 0$ and let $AB = BA$; $CC' = C'C$. Then following recursion formulas hold* *true for Humbert's matrix function* Ψ_1*:*

$$\begin{aligned}&\Psi_1(A,B;C-sI,C';x,y)=\Psi_1(A,B;C,C';x,y)\\&+xAB\Big[\sum_{k=1}^{s}\Psi_1(A+I,B+I;C\\&+(2-k)I,C';x,y)(C-kI)^{-1}(C-(k-1)I)^{-1}\Big],\end{aligned}\tag{54}$$

$$\begin{aligned}&\Psi_1(A,B;C,C'-sI;x,y)=\Psi_1(A,B;C,C';x,y)\\&+yA\Big[\sum_{k=1}^{s}\Psi_1(A+I,B;\,C,C'\\&+(2-k)I;x,y)(C'-kI)^{-1}(C'-(k-1)I)^{-1}\Big].\end{aligned}\tag{55}$$

Theorem 3.12. *Let* $A+sI$ *be an invertible matrix for all integers* $s\geq 0$ *and let* $C'C=CC'$*. Then the following recursion formula holds true for Humbert's matrix function* Ψ_2*:*

$$\begin{aligned}&\Psi_2(A+sI;C,C';x,y)\\&=\Psi_2(A;C,C';x,y)+x\Big[\sum_{k=1}^{s}\Psi_2(A+kI;C+I,C';x,y)\Big]C^{-1}\\&+y\Big[\sum_{k=1}^{s}\Psi_2(A+kI;C,C'+I;x,y)\Big]C'^{-1},\end{aligned}\tag{56}$$

$$\begin{aligned}&\Psi_2(A+sI;C,C';x,y)\\&=\sum_{k_1+k_2\leq s}\binom{s}{k_1,k_2}x^{k_1}y^{k_2}\\&\times\Big[\Psi_2(A+(k_1+k_2)I;C+k_1I,C'+k_2I;x,y)\Big](C)^{-1}_{k_1}(C')^{-1}_{k_2}.\end{aligned}\tag{57}$$

Furthermore, if $A-kI$ *is invertible for* $k\leq s$*, then*

$$\begin{aligned}&\Psi_2(A-sI;C,C';x,y)\\&=\Psi_2(A;C,C';x,y)-x\Big[\sum_{k=0}^{s-1}\Psi_2(A-kI;C+I,C';x,y)\Big]C^{-1}\\&-y\Big[\sum_{k=0}^{s-1}\Psi_2(A-kI;C,C'+I;x,y)\Big]C'^{-1},\end{aligned}\tag{58}$$

$$\begin{aligned}&\Psi_2(A-sI;C,C';x,y)\\&=\sum_{k_1+k_2\leq s}\binom{s}{k_1,k_2}(-x)^{k_1}(-y)^{k_2}\\&\times\Big[\Psi_2(A;C+k_1I,C'+k_2I;x,y)\Big](C)^{-1}_{k_1}(C')^{-1}_{k_2}.\end{aligned}\tag{59}$$

Theorem 3.13. *Let $C - sI$ and $C' - sI$ be invertible matrices for all integers $s \geq 0$. Then the following recursion formulas hold true for Humbert's matrix function Ψ_2:*

$$\begin{aligned}
&\Psi_2(A; C - sI, C'; x, y) = \Psi_2(A; C, C'; x, y) \\
&+ xA\Big[\sum_{k=1}^{s} \Psi_2(A + I; C \\
&+ (2 - k)I, C'; x, y)(C - kI)^{-1}(C - (k - 1)I)^{-1}\Big], C'C = CC', \qquad (60)
\end{aligned}$$

$$\begin{aligned}
&\Psi_2(A; C, C' - sI; x, y) = \Psi_2(A; C, C'; x, y) \\
&+ yA\Big[\sum_{k=1}^{s} \Psi_2(A + I; C, C' \\
&+ (2 - k)I; x, y)(C' - kI)^{-1}(C' - (k - 1)I)^{-1}\Big]. \qquad (61)
\end{aligned}$$

Theorem 3.14. *Let $A + sI$ be an invertible matrix for all integers $s \geq 0$ and let $BC = CB$. Then the following recursion formula holds true for Humbert's matrix function Ξ_1:*

$$\begin{aligned}
&\Xi_1(A + sI, A', B; C; x, y) \\
&= \Xi_1(A, A', B; C; x, y) + x\Big[\sum_{k=1}^{s} \Xi_1(A + kI, A', B + I; C + I; x, y)\Big]BC^{-1}, \qquad (62)
\end{aligned}$$

$$\begin{aligned}
&\Xi_1(A + sI, A', B; C; x, y) \\
&= \sum_{k_1 \leq s} \binom{s}{k_1} x^{k_1} \Big[\Xi_1(A + k_1 I, A', B + k_1 I; C + k_1 I; x, y)\Big] (B)_{k_1} (C)_{k_1}^{-1}. \qquad (63)
\end{aligned}$$

Furthermore, if $A - kI$ is invertible for $k \leq s$, then

$$\begin{aligned}
&\Xi_1(A - sI, A', B; C; x, y) \\
&= \Xi_1(A, A', B; C; x, y) - x\Big[\sum_{k=0}^{s-1} \Xi_1(A - kI, A', B + I; C + I; x, y)\Big]BC^{-1}, \qquad (64)
\end{aligned}$$

$$\begin{aligned}
&\Xi_1(A - sI, A', B; C; x, y) \\
&= \sum_{k_1 \leq s} \binom{s}{k_1} (-x)^{k_1} \Big[\Xi_1(A, A', B + k_1 I; C + k_1 I; x, y)\Big](B)_{k_1} (C)_{k_1}^{-1}. \qquad (65)
\end{aligned}$$

Theorem 3.15. *Let $A' + sI$ be an invertible matrix for all integers $s \geq 0$. Then the following recursion formula holds true for Humbert's matrix function Ξ_1:*

$$\begin{aligned}
&\Xi_1(A, A' + sI, B; C; x, y) \\
&= \Xi_1(A, A', B; C; x, y) + y\Big[\sum_{k=1}^{s} \Xi_1(A, A' + kI, B; C + I; x, y)\Big]C^{-1}, \qquad (66)
\end{aligned}$$

$$\begin{aligned}&\Xi_1(A, A'+sI, B; C, C'; x, y)\\ &= \sum_{k_1 \le s} \binom{s}{k_1} y^{k_1} \Big[\Xi_1(A, A'+k_1I, B; C+k_1I; x, y)\Big] (C)_{k_1}^{-1}.\end{aligned} \tag{67}$$

Furthermore, if $A'-kI$ is invertible for $k \le s$, then

$$\begin{aligned}&\Xi_1(A, A'-sI, B; C; x, y)\\ &= \Xi_1(A, A', B; C; x, y) - y\Big[\sum_{k=0}^{s-1} \Xi_1(A, A'-kI, B; C+I; x, y)\Big] C^{-1},\end{aligned} \tag{68}$$

$$\begin{aligned}&\Xi_1(A, A'-sI, B; C, C'; x, y)\\ &= \sum_{k_1 \le s} \binom{s}{k_1} (-y)^{k_1} \Big[\Xi_1(A, A', B; C+k_1I; x, y)\Big] (C)_{k_1}^{-1}.\end{aligned} \tag{69}$$

Theorem 3.16. *Let $B+sI$ be an invertible matrix for all integers $s \ge 0$. Then the following recursion formula holds true for Humbert's matrix function Ξ_1:*

$$\begin{aligned}&\Xi_1(A, A', B+sI; C; x, y)\\ &= \Xi_1(A, A', B; C; x, y) + xA\Big[\sum_{k=1}^{s} \Xi_1(A+I, A', B+kI; C+I; x, y)\Big] C^{-1},\end{aligned} \tag{70}$$

$$\begin{aligned}&\Xi_1(A, A', B+sI; C, C'; x, y)\\ &= \sum_{k_1 \le s} \binom{s}{k_1} x^{k_1} (A)_{k_1} \Big[\Xi_1(A+k_1I, A', B+k_1I; C+k_1I; x, y)\Big] (C)_{k_1}^{-1}.\end{aligned} \tag{71}$$

Furthermore, if $B-kI$ is invertible for $k \le s$, then

$$\begin{aligned}&\Xi_1(A, A', B-sI; C; x, y)\\ &= \Xi_1(A, A', B; C; x, y) - xA\Big[\sum_{k=0}^{s-1} \Xi_1(A+I, A', B+kI; C+I; x, y)\Big] C^{-1},\end{aligned} \tag{72}$$

$$\begin{aligned}&\Xi_1(A, A', B-sI; C, C'; x, y)\\ &= \sum_{k_1 \le s} \binom{s}{k_1} (-x)^{k_1} (A)_{k_1} \Big[\Xi_1(A+k_1I, A', B; C+k_1I; x, y)\Big] (C)_{k_1}^{-1}.\end{aligned} \tag{73}$$

Theorem 3.17. *Let $C-sI$ be an invertible matrix for all integers $s \ge 0$ and let $AA' = A'A$; $BC = CB$. Then the following recursion formula holds true for*

Humbert's matrix function Ξ_1:

$$\begin{aligned}
&\Xi_1(A,A',B;C-sI;x,y)\\
&=\Xi_1(A,A',B;C;x,y)+xA\Big[\sum_{k=1}^{s}\Xi_1(A+I,A',B+I;C+(2-k)I;x,y)\Big]\\
&\times B(C-kI)^{-1}(C-(k-1)I)^{-1}\\
&+yA'\Big[\sum_{k=1}^{s}\Xi_1(A,A'+I,B;C+(2-k)I;x,y)(C-kI)^{-1}(C-(k-1)I)^{-1}\Big]. \qquad (74)
\end{aligned}$$

Theorem 3.18. *Let $A+sI$ and $B+sI$ be an invertible matrix for all integers $s\geq 0$ and let $AB=BA$. Then the following recursion formula holds true for Humbert's matrix function Ξ_2:*

$$\begin{aligned}
&\Xi_2(A+sI,B;C;x,y)\\
&=\Xi_2(A,B;C;x,y)+xB\Big[\sum_{k=1}^{s}\Xi_2(A+kI,B+I;C+I;x,y)\Big]C^{-1}, \qquad (75)\\
&\Xi_2(A,B+sI;C;x,y)\\
&=\Xi_2(A,B;C;x,y)+xA\Big[\sum_{k=1}^{s}\Xi_2(A+I,B+kI;C+I;x,y)\Big]C^{-1}, \qquad (76)\\
&\Xi_2(A+sI,B;C;x,y)\\
&=\sum_{k_1\leq s}\binom{s}{k_1}x^{k_1}(B)_{k_1}\left[\Xi_2(A+k_1I,B+k_1I;C+k_1I;x,y)\right](C)_{k_1}^{-1}, \qquad (77)\\
&\Xi_2(A,B+sI;C,C';x,y)\\
&=\sum_{k_1\leq s}\binom{s}{k_1}x^{k_1}(A)_{k_1}\left[\Xi_2(A+k_1I,B+k_1I;C+k_1I;x,y)\right](C)_{k_1}^{-1}. \qquad (78)
\end{aligned}$$

Furthermore, if $A-kI$ and $B-kI$ is invertible for $k\leq s$, then

$$\begin{aligned}
&\Xi_2(A-sI,B;C;x,y)\\
&=\Xi_2(A,B;C;x,y)-xB\Big[\sum_{k=0}^{s-1}\Xi_2(A-kI,B+I;C+I;x,y)\Big]C^{-1}, \qquad (79)\\
&\Xi_2(A,B-sI;C;x,y)\\
&=\Xi_2(A,B;C;x,y)-xA\Big[\sum_{k=0}^{s-1}\Xi_2(A+I,B-kI;C+I;x,y)\Big]C^{-1}, \qquad (80)
\end{aligned}$$

$$\begin{aligned}&\Xi_2(A-sI,B;C;x,y)\\ &=\sum_{k_1\leq s}\binom{s}{k_1}(-x)^{k_1}\,(B)_{k_1}\Big[\Xi_2(A,B+k_1I;C+k_1I;x,y)\Big](C)^{-1}_{k_1},\end{aligned}\tag{81}$$

$$\begin{aligned}&\Xi_2(A,B-sI;C;x,y)\\ &=\sum_{k_1\leq s}\binom{s}{k_1}(-x)^{k_1}\,(A)_{k_1}\Big[\Xi_2(A+k_1I,B;C+k_1I;x,y)\Big](C)^{-1}_{k_1}.\end{aligned}\tag{82}$$

Theorem 3.19. *Let $C-sI$ be an invertible matrix for all integers $s\geq 0$ and $AB=BA$. Then the following recursion formula holds true for Humbert's matrix function Ξ_2:*

$$\begin{aligned}&\Xi_2(A,B;C-sI;x,y)\\ &=\Xi_2(A,B;C;x,y)+xAB\Big[\sum_{k=1}^{s}\Xi_2(A+I,B+I;C+(2-k)I;x,y)\\ &\times(C-kI)^{-1}(C-(k-1)I)^{-1}\Big]\\ &+y\Big[\sum_{k=1}^{s}\Xi_2(A,B;C+(2-k)I;x,y)(C-kI)^{-1}(C-(k-1)I)^{-1}\Big].\end{aligned}\tag{83}$$

4. Conclusion

We have studied the recursion formulas for Humbert's matrix function. These matrix formulas will contribute to the literature on special function theory and have the potential to find new applications in mathematics and physics.

Acknowledgments

The authors thank the referees for their valuable suggestions that led to a better presentation of the paper. The financial assistance provided to the third author in the form of a minor research project File no. $(133/PU/IQAC/2022)$ from the Veer Bahadur Singh Purvanchal University Jaunpur, India is gratefully acknowledged.

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Accepted: April 1, 2023