# A Mehrotra-type algorithm with logarithmic updating technique for $P_*(\kappa)$ linear complementarity problems

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**Abstract.** A Mehrotra-type predictor-corrector algorithm for $P_*(\kappa)$ linear complementarity problems is presented. In this algorithm, the corrector step takes a new direction, and the barrier parameter is the smaller positive root of a logarithmic equation. The iteration complexity of the new algorithm matches the currently best-known results. Numerical results show that the algorithm is efficient.

**Keywords:** interior-point algorithm, linear complementarity problems, Mehrotra-type algorithm, iteration complexity.

## 1. Introduction

Mehrotra's predictor-corrector algorithm [1, 2] and its variants have become the backbones of some optimization solvers [3-7]. The superior practical perforance of Mehrotra-type predictor-corrector algorithms motivated scholars to explore their theoretical properties. Jarre and Wechs [8] investigated an interior point method in which the search direction is based on corrector directions of Mehrotra's algorithm. To avoid small steps, Salahi et al. [9] introduced a safeguard

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strategy for a Mehrotra-type algorithm. After that, Salahi and Terlaky [10] proposed a new variant of Mehrotra-type algorithm without any safeguards and proved the iteration complexity bound coincides with the result in [9]. Recently, Salahi [11] introduced a new adaptive updating technique of the barrier parameter in Mehrotra-type algorithm for linear optimization (LO), which allowed them to prove the polynomial iteration complexity without employing any safeguards. Infeasible versions of Mehrotra-type algorithm [12, 13] and second order Mehrotra-type algorithms [14, 15] are also studied by scholars. Since efficiency in computation, Mehrotra-type predictor-corrector algorithm are extended to linear complementarity problems (LCPs) [12, 16], semidefinite programming [17-19], nonlinear complementarity problems [20] and many other problems.

LCPs are closely associated with linear programming and quadratic programming. The class of $P_*(\kappa)$ LCP is an important branch of LCPs. Interior point algorithms for $P_*(\kappa)$ LCPs have been widely studied in the last few decades [21]. Large update technique [22], full-Newton step [23, 24] and interior point method based on kernel function [25] are also presented for $P_*(\kappa)$ LCPs.

In this paper, a new Mehrotra-type algorithm for $P_*(\kappa)$ LCPs is presented, in which it takes a different corrector search direction and an adaptive updating technique of the barrier parameter. It extends the algorithm in [11] for LO to $P_*(\kappa)$ LCPs. In $P_*(\kappa)$ LCPs, the search directions $\Delta x$ and $\Delta s$ are not orthogonal any more, while they are orthogonal in LO, this leads a different technique to analyze the iteration complexity. Taking a specific default value as the predictor step size, we prove that the algorithm stops after at most $O(\sqrt{(1+4\kappa)(1+2\kappa)}n\log((x^0)^Ts^0/\epsilon))$ iterations. If $\kappa=0$, the iteration bound coincides with the result of LO in [11].

The rest of this paper is organized as follows. In Section 2, we recall some basic concepts and state a new Mehrotra-type algorithm for $P_*(\kappa)$ LCPs. Section 3 includes several important technical results, and subsequently the iteration bound of this algorithm is derived. Two illustrative numerical results of this algorithm are presented in Section 4. Finally, conclusion and final remarks are shown in Section 5.

For simplicity, we use the following notations throughout the paper:

$$e=(1,1,\cdots,1)^T.$$
$$I=\{1,2,\cdots,n\}, I_+=\{i\in I|\Delta x_i^a\Delta s_i^a\geq 0\}, I_-=\{i\in I|\Delta x_i^a\Delta s_i^a<0\}.$$
$$\mathcal{F}=\big\{(x,s)\in\mathbb{R}^n\times\mathbb{R}^n|s=Mx+q,(x,s)\geq 0\big\}.$$
$$\mathcal{F}^0=\big\{(x,s)\in\mathcal{F}|(x,s)>0\big\}.$$
$$X=\operatorname{diag}(x), S=\operatorname{diag}(s).$$
$$xs=Xs=(x_1s_1,x_2s_2,\cdots,x_ns_n)^T.$$

## 2. The algorithm

A matrix $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$ matrix if there is a constant $\kappa \geq 0$ such that

$$(1+4\kappa)\sum_{i\in I_+(x)} x_i(Mx)_i + \sum_{i\in I_-(x)} x_i(Mx)_i \geq 0,\ \forall\ x \in \mathbb{R}^n,$$

or equivalently

$$x^T Mx \geq -4\kappa \sum_{i\in I_+(x)} x_i(Mx)_i,\ \forall x \in \mathbb{R}^n,$$

where $I_+(x) = \{i|x_i(Mx)_i \geq 0, i \in I\}$ and $I_-(x) = \{i|x_i(Mx)_i < 0, i \in I\}$. Note that, $M$ is a positive semidefinite matrix if $\kappa = 0$. Thus, the class of $P_*(\kappa)$ matrices includes positive semi-definite matrices. The goal of a $P_*(\kappa)$ LCP is to find solutions $(x, s) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$Mx + q = s, xs = 0, (x, s) \geq 0, \tag{1}$$

where $M$ is a $P_*(\kappa)$ matrix, $q \in \mathbb{R}^n$ and $n \geq 2$.

To find an approximate solution of (1), a parameterized system is established as follows:

$$Mx + q = s, xs = \mu e, (x, s) \geq 0, \tag{2}$$

where $\mu > 0$. We assume that system (1) satisfies the interior point condition (IPC), i.e., there exists a point $(x^0, s^0)$ such that

$$s^0 = Mx^0 + q,\quad x^0 > 0,\quad s^0 > 0.$$

For a given $\mu > 0$, if the IPC holds, then system (2) has a unique solution $(x(\mu), s(\mu))$, which is called the $\mu$-center of (1). The set of all $\mu$-centers is called the central path of (1). As $\mu$ goes to 0, the limit of $(x(\mu), s(\mu))$ exists and approaches the solution of (1).

In the following, a feasible version of Mehrotra-type predictor-corrector algorithm for $P_*(\kappa)$ LCPs will be presented, which works in a negative infinity neighborhood defined as

$$\mathcal{N}_\infty^-(\gamma) = \{(x, s) \in \mathcal{F}^0|\ x_i s_i \geq \gamma\mu_g, \forall i \in I\},$$

where $\gamma \in (0, 1)$ is a constant independent of $n$. The neighborhood $\mathcal{N}_\infty^-(\gamma)$ is also widely used in the implementation of other interior point algorithms.

The predictor direction $(\Delta x^a, \Delta s^a)$ is determined by the following system:

$$\begin{aligned} M\Delta x^a &= \Delta s^a,\\ s\Delta x^a + x\Delta s^a &= -xs, \end{aligned} \tag{3}$$

and the predictor step size $\alpha_a$ is defined by

$$\alpha_a = \max\{\alpha|(x + \alpha\Delta x^a, s + \alpha\Delta s^a) \in \mathcal{F}, 0 < \alpha \leq 1\}. \tag{4}$$

However, the our algorithm does not take a predictor step right away. By using information about the predictor step, the algorithm derives the corrector direction from the following system:

$$\begin{aligned} M\Delta x &= \Delta s, \\ s\Delta x + x\Delta s &= \mu e - xs - \alpha_a^2 \Delta x^a \Delta s^a. \end{aligned} \tag{5}$$

The corrector direction in system (5) is different from that in [9] where it is determined by the equations $M\Delta x = \Delta s$ and $s\Delta x + x\Delta s = \mu e - xs - \Delta x^a \Delta s^a$. Motivation of the modification is based on the following observation. Since $0 < \alpha_a \leq 1$, it can be found that $\alpha_a^2 |\Delta x^a \Delta s^a| \leq |\Delta x^a \Delta s^a|$, thus $\mu e - xs - \alpha_a^2 \Delta x^a \Delta s^a$ is much closer to $\mu e - xs$ than $\mu e - xs - \Delta x^a \Delta s^a$.

In each iteration of a primal-dual interior point algorithm, the barrier parameter $\mu$ needs to be updated. In this paper, we focus on the updating technique in [11]. A classical logarithmic barrier proximity function is used to measure the distance from the current iterate to the central path, and it is defined as

$$\Phi(x, s, \mu) := \frac{x^T s}{2\mu} - \frac{n}{2} + \frac{n}{2}\log\mu - \frac{1}{2}\sum_{i=1}^{n} \log(x_i s_i). \tag{6}$$

Obviously, for given $(x, s)$, the function $\Phi(x, s, \mu)$ is minimum if $\mu = \mu_g = \frac{x^T s}{n}$. We denote $\mu_h = \sqrt[n]{x_1 s_1 \cdots x_n s_n}$. From the Arithmetic Mean−Geometric Mean inequality, it is clear that $\mu_h \leq \mu_g$. We consider the following equation with respect to $\mu$,

$$\Phi(x, s, \mu) = \frac{(\sigma - 1)n}{2}, \tag{7}$$

where the constant $\sigma > 4\kappa + 4$. From (6) and (7), it can be found that equation (7) is equivalent to

$$\frac{\mu_g}{\mu} + \log\frac{\mu}{\mu_h} - \sigma = 0. \tag{8}$$

Follows from Corollary 2.5 of [11], equation (8) has two positive roots. The smaller one is defined as the target barrier parameter denoted by $\mu_t$.

The barrier parameter $\mu_t$ is used to compute the corrector search direction $(\Delta x, \Delta s)$ by the following equations

$$\begin{aligned} M\Delta x &= \Delta s, \\ s\Delta x + x\Delta s &= \mu_t e - xs - \alpha_a^2 \Delta x^a \Delta s^a. \end{aligned} \tag{9}$$

The new iterate is denoted as $(x(\alpha_c), s(\alpha_c)) = (x + \alpha_c \Delta x, s + \alpha_c \Delta s)$ where the corrector step size $\alpha_c$ is defined by

$$\alpha_c = \max\{\alpha | (x(\alpha), s(\alpha) \in \mathcal{N}_\infty^-(\gamma), 0 < \alpha \leq 1\}. \tag{10}$$

Based on the previous analysis, a new Mehrotra-type predictor-corrector algorithm for $P_*(\kappa)$ LCP is stated as Algorithm 1.

**Algorithm 1**
**Input**:
A parameter $\sigma > 4\kappa + 4$, a starting point $(x^0, s^0) \in \mathcal{N}_\infty^-(\gamma)$ with $\gamma = \frac{1}{\sigma}$, an accuracy parameter $\epsilon > 0$.
**begin**
Set $x := x^0; s = s^0$;
**while** $x^T s \geq \epsilon$ **do**
**begin** **Predictor step**
Solve (3) and calculate the predictor step size $\alpha_a$ from (4);
**end**
**begin** **Corrector step**
Solve (8) to derive the smaller positive root $\mu_t$;
Solve (9) and calculate the corrector step size $\alpha_c$ from (10);
Set $(x, s) := (x(\alpha_c), s(\alpha_c))$.
**end**
**end**

## 3. Complexity analysis

In this section, we establish the polynomial complexity for Algorithm 1. In the following, we give the bounds of $\mu_t$, $\|\Delta x \Delta s\|$, $\Delta x^T \Delta s$ and step sizes of Algorithm 1. The bounds are important in the complexity analysis.

**Lemma 3.1** ([11]))**.** *For all iterates* $(x, s)$ *of Algorithm* 1*, we have* $\sigma \leq \frac{\mu_g}{\mu_t} \leq 2\sigma$.

**Lemma 3.2.** *Let* $(\Delta x^a, \Delta s^a)$ *be the solution of* (3)*. Then:*

$$\text{(i)} \quad \Delta x_i^a \Delta s_i^a \leq \frac{x_i s_i}{4},\ i \in I_+; \quad -\Delta x_i^a \Delta s_i^a \leq \frac{1}{\alpha_a}(\frac{1}{\alpha_a} - 1) x_i s_i,\ i \in I_-;$$

$$\text{(ii)} \quad \sum_{i \in I_+} \Delta x_i^a \Delta s_i^a \leq \frac{x^T s}{4}; \quad \sum_{i \in I_-} |\Delta x_i^a \Delta s_i^a| \leq \frac{4\kappa + 1}{4} x^T s;$$

$$\text{(iii)} \quad -\kappa x^T s \leq (\Delta x^a)^T \Delta s^a \leq \frac{x^T s}{4}.$$

**Proof.** (i) The proof is similar to that of Lemma A.1 and Proposition 4.1 in [9], and it is omitted here.

(ii) The first conclusion is a direct consequence of (i). We will prove the second conclusion in the following. Since $M$ is a $P_*(\kappa)$ matrix, following from the first conclusion, we have

$$0 > \sum_{i \in I_-} \Delta x_i^a \Delta s_i^a \geq -(1 + 4\kappa) \sum_{i \in I_+} \Delta x_i^a \Delta s_i^a \geq -\frac{1 + 4\kappa}{4} x^T s,$$

that is $\sum_{i \in I_-} |\Delta x_i^a \Delta s_i^a| \leq \frac{1+4\kappa}{4} x^T s$.

(iii) From statement (ii), it follows that $(\Delta x^a)^T \Delta s^a \leq \sum_{i\in I_+} \Delta x_i^a \Delta s_i^a \leq \frac{x^T s}{4}$. Since $\Delta s^a = M\Delta x^a$ and $M$ is a $P_*(\kappa)$ matrix, we get

$$(\Delta x^a)^T \Delta s^a \geq -4\kappa \sum_{i\in I_+} \Delta x_i^a \Delta s_i^a \geq -\kappa x^T s.$$

This completes the proof. □

**Theorem 3.1** ([16])**.** *If the current iterate $(x, s) \in \mathcal{N}_\infty^-(\gamma)$ and $\alpha_a$ is the predictor step size, then*

$$\alpha_a \geq \sqrt{\frac{\gamma}{(4\kappa+1)n}}.$$

In what follows, we consider the lower bound as a default value for predictor step size, that is

$$\alpha_a = \sqrt{\frac{\gamma}{(4\kappa+1)n}}. \tag{11}$$

**Lemma 3.3.** *Let $(x, s) \in \mathcal{N}_\infty^-(\gamma)$ and $(\Delta x, \Delta s)$ be the solution of* (5) *with $\mu > 0$, then*

$$||\Delta x \Delta s|| \leq \sqrt{(\frac{1}{4}+\kappa)(\frac{1}{2}+\kappa)}||w||^2, \sum_{i\in I_+} \Delta x_i \Delta s_i \leq \frac{1}{4}||w||^2,$$

*where $w = (xs)^{-\frac{1}{2}}(\mu e - xs - \alpha_a^2 \Delta x^a \Delta s^a)$.*

**Proof.** The proof is similar to that of Lemma 8 in [26], and we omit it here. □

**Lemma 3.4.** *Let $(x, s) \in \mathcal{N}_\infty^-(\gamma)$ and $(\Delta x, \Delta s)$ be the solution of* (5) *with $\mu > 0$, then*

$$||w||^2 \leq \frac{n\mu^2}{\gamma\mu_g} - 2n\mu + \frac{(4\kappa+1)\alpha_a^2 n\mu}{2\gamma} + \frac{\alpha_a^4 + 8\alpha_a^2 + 4\alpha_a^2(1-\alpha_a)(4\kappa+1)+16}{16} n\mu_g.$$

**Proof.** From Lemma 3.3, one has

$$\begin{aligned} ||w||^2 =& \mu^2 \sum_{i\in I} \frac{1}{x_i s_i} + \sum_{i\in I} x_i s_i - 2n\mu + \alpha_a^4 \sum_{i\in I} \frac{(\Delta x_i^a \Delta s_i^a)^2}{x_i s_i} \\ &- 2\alpha_a^2 \mu \sum_{i\in I} \frac{\Delta x_i^a \Delta s_i^a}{x_i s_i} + 2\alpha_a^2 \sum_{i\in I} \Delta x_i^a \Delta s_i^a. \end{aligned}$$

Due to $(x, s) \in \mathcal{N}_\infty^-(\gamma)$, we have $\mu^2 \sum_{i\in I} \frac{1}{x_i s_i} \leq \frac{n\mu^2}{\gamma\mu_g}$. Using (i) and (ii) in Lemma 3.2, we obtain

$$\begin{aligned}
\alpha_a^4 \sum_{i\in I} \frac{(\Delta x_i^a \Delta s_i^a)^2}{x_i s_i} &= \alpha_a^4 \sum_{i\in I_+} \frac{(\Delta x_i^a \Delta s_i^a)^2}{x_i s_i} + \alpha_a^4 \sum_{i\in I_-} \frac{(\Delta x_i^a \Delta s_i^a)^2}{x_i s_i} \\
&\leq \alpha_a^4 \sum_{i\in I_+} \frac{(\frac{x_i s_i}{4})^2}{x_i s_i} + \alpha_a^4 \sum_{i\in I_-} \frac{-\Delta x_i^a \Delta s_i^a}{x_i s_i}(-\Delta x_i^a \Delta s_i^a) \\
&\leq \alpha_a^4 \sum_{i\in I_+} \frac{x_i s_i}{16} + \frac{\alpha_a^4}{\alpha_a}(\frac{1}{\alpha_a} - 1) \sum_{i\in I_-} |\Delta x_i^a \Delta s_i^a| \\
&\leq \frac{\alpha_a^4}{16} x^T s + \alpha_a^2 (1-\alpha_a) \frac{4\kappa+1}{4} x^T s \\
&= \frac{\alpha_a^4 + 4\alpha_a^2(1-\alpha_a)(4\kappa+1)}{16} n\mu_g,
\end{aligned}$$

and

$$-2\alpha_a^2 \mu \sum_{i\in I} \frac{\Delta x_i^a \Delta s_i^a}{x_i s_i} \leq 2\alpha_a^2 \mu \sum_{i\in I_-} \frac{|\Delta x_i^a \Delta s_i^a|}{x_i s_i} \leq \frac{2\alpha_a^2 \mu (4\kappa+1)}{4\gamma\mu_g} x^T s \leq \frac{(4\kappa+1)\alpha_a^2 n\mu}{2\gamma},$$

where the second inequality follows from $(x, s) \in \mathcal{N}_\infty^-(\gamma)$. Moreover,

$$2\alpha_a^2 \sum_{i\in I} \Delta x_i^a \Delta s_i^a \leq 2\alpha_a^2 \sum_{i\in I_+} \Delta x_i^a \Delta s_i^a \leq \frac{\alpha_a^2}{2} n\mu_g.$$

Combining the above results yields that

$$\begin{aligned}
&||w||^2 \\
\leq &\frac{n\mu^2}{\gamma\mu_g} + n\mu_g - 2n\mu + \frac{\alpha_a^4 + 4\alpha_a^2(1-\alpha_a)(4\kappa+1)}{16} n\mu_g + \frac{(4\kappa+1)\alpha_a^2 n\mu}{2\gamma} + \frac{\alpha_a^2}{2} n\mu_g \\
= &\frac{n\mu^2}{\gamma\mu_g} - 2n\mu + \frac{(4\kappa+1)\alpha_a^2 n\mu}{2\gamma} + \frac{\alpha_a^4 + 8\alpha_a^2 + 4\alpha_a^2(1-\alpha_a)(4\kappa+1) + 16}{16} n\mu_g.
\end{aligned}$$

This completes the proof. □

**Lemma 3.5.** *Let* $(x, s) \in \mathcal{N}_\infty^-(\gamma)$ *and* $(\Delta x, \Delta s)$ *be the solution of* (5) *with* $\mu = \mu_t$, *then*

$$||\Delta x \Delta s|| \leq p_1 n\mu_g, \Delta x^T \Delta s \leq p_2 n\mu_g,$$

*where* $p_1 = \frac{37}{128}\sqrt{(1+4\kappa)(2+4\kappa)}$, $p_2 = \frac{37}{128}$.

**Proof.** Lemma 3.1 implies that $\frac{\gamma}{2} = \frac{1}{2\sigma} \leq \frac{\mu_t}{\mu_g} \leq \frac{1}{\sigma} = \gamma$. Following from Lemma 3.4, one has

$$\begin{aligned} ||w||^2 \leq & \frac{n\mu_t^2}{\gamma\mu_g} - 2n\mu_t + \frac{(4\kappa+1)\alpha_a^2 n\mu_t}{2\gamma} + \frac{\alpha_a^4 + 8\alpha_a^2 + 4\alpha_a^2(1-\alpha_a)(4\kappa+1) + 16}{16} n\mu_g \\ = & \Big[\frac{1}{\gamma}\big(\frac{\mu_t}{\mu_g}\big)^2 - 2\frac{\mu_t}{\mu_g} + \frac{(4\kappa+1)\alpha_a^2}{2\gamma}\frac{\mu_t}{\mu_g} + \frac{\alpha_a^4 + 8\alpha_a^2 + 4\alpha_a^2(1-\alpha_a)(4\kappa+1) + 16}{16}\Big] n\mu_g \\ \leq & \Big(\frac{1}{\gamma}\gamma^2 - 2\frac{\gamma}{2} + \frac{\gamma}{4\gamma}\gamma + \frac{6\gamma+16}{16}\Big) n\mu_g \\ \leq & \frac{37}{32} n\mu_g, \end{aligned}$$

where the second inequality is due to $n \geq 2$, $\kappa \geq 0$ and $\alpha_a = \sqrt{\frac{\gamma}{(4\kappa+1)n}} \leq 1$ by Theorem 3.1. The third inequality comes from $\gamma = \frac{1}{\sigma} < \frac{1}{4\kappa+4} \leq \frac{1}{4}$.

From Lemma 3.3, it follows that

$$||\Delta x \Delta s|| \leq \frac{37}{32}\sqrt{(\frac{1}{4}+\kappa)(\frac{1}{2}+\kappa)} n\mu_g = \frac{37}{128}\sqrt{(1+4\kappa)(2+4\kappa)} n\mu_g = p_1 n\mu_g,$$

and $\Delta x^T \Delta s \leq \frac{37}{128} n\mu_g$, which completes the proof. □

In order to simplify the analysis, we define

$$t = \max_{i \in I_+} \left\{ \frac{\Delta x_i^a \Delta s_i^a}{x_i s_i} \right\}, \tag{12}$$

that is, $\Delta x_i^a \Delta s_i^a \leq t x_i s_i$ if $i \in I_+$. Since $M$ is a $P_*(\kappa)$ matrix, one has $I_+ \neq \emptyset$ and $t \leq \frac{1}{4}$ from Lemma 3.2.

**Theorem 3.2.** *Let $(x, s) \in \mathcal{N}_\infty^-(\gamma)$, where $\gamma = \frac{1}{\sigma}$ and $\sigma > 4 + 4\kappa$. If $(\Delta x, \Delta s)$ is the solution of (5) with $\mu = \mu_t$ and $\alpha_c$ is the corrector step size, then*

$$\alpha_c \geq \frac{14\gamma}{37n\sqrt{(1+4\kappa)(2+4\kappa)}}. \tag{13}$$

**Proof.** The goal is to determine a maximum step size $\alpha \in (0, 1]$ in the corrector step such that

$$x_i(\alpha)s_i(\alpha) \geq \gamma\mu_g(\alpha), \quad \forall i \in I, \tag{14}$$

where $\mu_g(\alpha) = \frac{x(\alpha)^T s(\alpha)}{n}$ and

$$\begin{aligned} x_i(\alpha)s_i(\alpha) &= x_i s_i + \alpha(x_i \Delta s_i + s_i \Delta x_i) + \alpha^2 \Delta x_i \Delta s_i \\ &= x_i s_i + \alpha(\mu_t - x_i s_i - \alpha_a^2 \Delta x_i^a \Delta s_i^a) + \alpha^2 \Delta x_i \Delta s_i \\ &= (1-\alpha)x_i s_i + \alpha\mu_t - \alpha\alpha_a^2 \Delta x_i^a \Delta s_i^a + \alpha^2 \Delta x_i \Delta s_i. \end{aligned}$$

Since we consider the lower bound of $\Delta x_i^a \Delta s_i^a$, we should give more focus on the case of $\Delta x_i^a \Delta s_i^a > 0$ than $\Delta x_i^a \Delta s_i^a \leq 0$. Thus, we have to prove $x_i(\alpha)s_i(\alpha) \geq \gamma\mu_g(\alpha)$ for all $i \in I_+$. From Lemma 3.5 and equation (12), it follows that, for any $i \in I_+$,

$$\begin{aligned} x_i(\alpha)s_i(\alpha) &= (1-\alpha)x_is_i + \alpha\mu_t - \alpha\alpha_a^2\Delta x_i^a\Delta s_i^a + \alpha^2\Delta x_i\Delta s_i \\ &\geq [1-(1+\alpha_a^2 t)\alpha]x_is_i + \alpha\mu_t - \alpha^2 p_1 n\mu_g \\ &\geq [1-(1+\frac{\alpha_a^2}{4})\alpha]x_is_i + \frac{\alpha}{2}\gamma\mu_g - \alpha^2 p_1 n\mu_g, \end{aligned}$$

where the last inequality follows from $t \leq \frac{1}{4}$ and $\frac{\mu_g}{\mu_t} \leq 2\sigma$.

Since $(x,s) \in \mathcal{N}_\infty^-(\gamma)$, it is clear that $[1-(1+\frac{\alpha_a^2}{4})\alpha]x_is_i \geq [1-(1+\frac{\alpha_a^2}{4})\alpha]\gamma\mu_g$ if $\alpha \leq \frac{4}{4+\alpha_a^2}$. Thus,

$$x_i(\alpha)s_i(\alpha) \geq [1-(1+\frac{\alpha_a^2}{4})\alpha]\gamma\mu_g + \frac{\alpha}{2}\gamma\mu_g - \alpha^2 p_1 n\mu_g \tag{15}$$

if $\alpha \leq \frac{4}{4+\alpha_a^2}$.

On the other hand, we have

$$\mu_g(\alpha) = \frac{x(\alpha)^T s(\alpha)}{n} = \frac{x^Ts + \alpha[n\mu_t - x^Ts - \alpha_a^2(\Delta x^a)^T\Delta s^a] + \alpha^2\Delta x^T\Delta s}{n}.$$

From Lemma 3.1, 3.2 and 3.5, we get

$$\begin{aligned} \mu_g(\alpha) &\leq \frac{x^Ts + \alpha(\frac{n\mu_g}{\sigma} - x^Ts + \alpha_a^2\kappa x^Ts) + \alpha^2 np_2\mu_g}{n} \\ &= (1-\alpha)\mu_g + \alpha\gamma\mu_g + \alpha\alpha_a^2\kappa\mu_g + \alpha^2 p_2\mu_g. \end{aligned} \tag{16}$$

Combining (15) and (16) yields that the new iterate is certainly in the neighborhood $\mathcal{N}_\infty^-(\gamma)$ if

$$\big[1-(1+\frac{\alpha_a^2}{4})\alpha\big]\gamma\mu_g + \frac{\alpha}{2}\gamma\mu_g - \alpha^2 p_1 n\mu_g \geq (1-\alpha)\gamma\mu_g + \alpha\gamma^2\mu_g + \alpha\alpha_a^2\kappa\gamma\mu_g + \alpha^2\gamma p_2\mu_g.$$

This is equivalent to $(\frac{1}{2} - \gamma - \frac{\alpha_a^2}{4} - \alpha_a^2\kappa)\gamma \geq (\gamma p_2 + np_1)\alpha$, that is,

$$\alpha \leq \frac{(\frac{1}{2} - \gamma - \frac{\alpha_a^2}{4} - \alpha_a^2\kappa)\gamma}{\gamma p_2 + np_1}.$$

Furthermore,

$$\begin{aligned} \frac{\frac{1}{2} - \gamma - \frac{\alpha_a^2}{4} - \alpha_a^2\kappa}{\gamma p_2 + np_1} &= \frac{\frac{1}{2} - \gamma - \frac{\gamma}{4n}}{\frac{37}{128}\gamma + \frac{37}{128}n\sqrt{(1+4\kappa)(2+4\kappa)}} \\ &\geq \frac{\frac{7}{32}}{\frac{37}{64}n\sqrt{(1+4\kappa)(2+4\kappa)}} \\ &= \frac{14}{37n\sqrt{(1+4\kappa)(2+4\kappa)}}, \end{aligned}$$

where the inequality follows from $\gamma < \frac{1}{4\kappa+4} \le \frac{1}{4} < n\sqrt{(1+4\kappa)(2+4\kappa)}$ and $n \ge 2$. Therefore inequality (14) holds if $\alpha \le \frac{14\gamma}{37n\sqrt{(1+4\kappa)(2+4\kappa)}}$. Thus, the maximal step size satisfies

$$\alpha \ge \min\left\{\frac{4}{4+\alpha_a^2}, \frac{14\gamma}{37n\sqrt{(1+4\kappa)(2+4\kappa)}}\right\}.$$

Since $\alpha_a \le 1$, $\gamma < \frac{1}{4}$, $n \ge 2$ and $\kappa \ge 0$, we have $\frac{4}{4+\alpha_a^2} \ge \frac{4}{5} > \frac{14\gamma}{37n} > \frac{14\gamma}{37n\sqrt{(1+4\kappa)(2+4\kappa)}}$. Consequently, the corrector step size $\alpha_c$ satisfies

$$\alpha_c \ge \frac{14\gamma}{37n\sqrt{(1+4\kappa)(2+4\kappa)}}.$$

This completes the proof. □

The following theorem gives the upper bound of iteration number in which Algorithm 1 stops with an $\epsilon$-approximate solution.

**Theorem 3.3.** *After at most*

$$O\Big(\sqrt{(1+4\kappa)(2+4\kappa)}n\log\frac{(x^0)^Ts^0}{\epsilon}\Big)$$

*iterations, Algorithm 1 stops with a solution for which* $x^Ts \le \epsilon$.

**Proof.** After each iteration, the dual gap is $\mu_g(\alpha_c)$. From (16), it follows that

$$\begin{aligned}\mu_g(\alpha_c) &\le \big[1-(1-\gamma-\alpha_a^2\kappa)\alpha_c + p_2\alpha_c^2\big]\mu_g\\ &\le \Big[1-\Big(1-\frac{1}{4}-\frac{1}{32}\Big)\alpha_c + \frac{37}{128}\alpha_c\Big]\mu_g\\ &= \Big(1-\frac{55}{128}\alpha_c\Big)\mu_g\\ &\le \Big[1-\frac{385\gamma}{2368n\sqrt{(1+4\kappa)(2+4\kappa)}}\Big]\mu_g,\end{aligned}$$

where the second inequality is due to $\alpha_a = \sqrt{\frac{\gamma}{(4\kappa+1)n}}$ and $\gamma < \frac{1}{4}$. This completes the proof by Theorem 3.2 of [27]. □

## 4. Numerical results

It is difficult to know the value of parameter $\kappa$ of a $P_*(\kappa)$ matrix [26], however, it is well known that a positive semi-definite matrix is a $P_*(0)$ matrix. In the following, Algorithm 1 is applied to $P_*(0)$ LCPs.

**Example 4.1.** Let $M = (m_{ij})_{n\times n}$, $q = (q_i)_{n\times 1}$, where $q_i = n+1-i$ and

$$m_{ij} = \begin{cases} 2, & \text{if } i = j;\\ -1, & \text{if } |i-j| = 1;\\ 0, & \text{else.}\end{cases}$$

Table 1: Iteration numbers of Example 4.1

| | n=5 | n=10 | n=50 | n=100 | n=200 | n=400 | n=800 | n=1000 |
|---|---|---|---|---|---|---|---|---|
| $\sigma = 4.5$ | 11 | 11 | 14 | 15 | 16 | 17 | 18 | 18 |
| $\sigma = 5$ | 10 | 11 | 13 | 14 | 15 | 16 | 17 | 17 |
| $\sigma = 5.5$ | 10 | 11 | 13 | 14 | 15 | 15 | 16 | 17 |
| $\sigma = 6$ | 10 | 10 | 12 | 13 | 14 | 15 | 16 | 16 |
| $\sigma = 6.5$ | 9 | 10 | 12 | 13 | 14 | 15 | 16 | 16 |
| $\sigma = 7$ | 9 | 10 | 12 | 13 | 14 | 14 | 15 | 16 |
| $\sigma = 7.5$ | 9 | 10 | 12 | 13 | 13 | 14 | 15 | 16 |
| $\sigma = 8$ | 9 | 10 | 12 | 12 | 13 | 14 | 15 | 15 |

Table 2: Average iteration numbers of Example 4.2

| | n=5 | n=10 | n=50 | n=100 | n=200 | n=400 | n=800 | n=1000 |
|---|---|---|---|---|---|---|---|---|
| $\sigma = 4.5$ | 10.98 | 12.00 | 15.00 | 16.91 | 18.00 | 19.00 | 21.00 | 21.01 |
| $\sigma = 5$ | 10.40 | 11.85 | 15.00 | 16.00 | 17.09 | 19.00 | 20.00 | 21.00 |
| $\sigma = 5.5$ | 10.04 | 11.05 | 14.05 | 16.00 | 17.00 | 18.00 | 20.00 | 20.02 |
| $\sigma = 6$ | 9.98 | 11.00 | 14.00 | 15.10 | 17.00 | 18.00 | 19.01 | 20.00 |
| $\sigma = 6.5$ | 9.79 | 11.00 | 14.00 | 15.00 | 16.82 | 18.00 | 19.00 | 20.00 |
| $\sigma = 7$ | 9.39 | 10.80 | 14.00 | 15.00 | 16.00 | 18.00 | 19.00 | 19.00 |
| $\sigma = 7.5$ | 9.15 | 10.30 | 13.35 | 15.00 | 16.00 | 17.08 | 19.00 | 19.00 |
| $\sigma = 8$ | 9.07 | 10.15 | 13.01 | 15.00 | 16.00 | 17.00 | 19.00 | 19.00 |

**Example 4.2.** Let $M = RR^T$, where $R = (r_{ij})_{n\times n}$ is randomly generated and $r_{ij} \in [0, 1]$. The vector $q = (q_i)_{n\times 1}$ is also randomly generated, where $q_i \in [0, 5]$.

In both examples, the accuracy parameter is set as $\epsilon = 10^{-6}$. Table 1 shows the iteration numbers to obtain an $\epsilon$-solution for Example 4.1. In Example 4.2, for each $n$ and every $\sigma$, one hundred random $P_*(0)$ LCPs are considered. Iteration numbers in Table 2 are the average iteration numbers of the one hundred LCPs. From Table 1 and Table 2, we can find that, for a given $n$, the iteration number decreases if $\sigma$ increases. This is because that if $\sigma$ is larger, then the neighborhood $N_\infty^-(\gamma)$ is bigger, and Algorithm 1 has a larger corrector step size and fewer steps. The numerical results show that Algorithm 1 is efficient.

## 5. Concluding remarks

In this paper, we present a modified Mehrotra-type predictor-corrector algorithm for $P_*(\kappa)$ LCPs and discuss the polynomial complexity of this algorithm. It should be pointed out that the corrector direction in our algorithm is different from other algorithms. The iteration bound is $O(\sqrt{(1+4\kappa)(2+4\kappa)}n \log \frac{(x^0)^T s^0}{\epsilon})$. If $\kappa = 0$, this bound coincides with the iteration bound for LO.

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