Finite groups of order p^3qr in which the number of elements of maximal order is p^4q

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Abstract. Suppose that G is a finite group. As is known to all, the order of G and the number of elements of maximal order in G are closely related to the structure of G. This topic involves Thompson's problem. In this paper we classify the finite groups of order p^3qr in which the number of elements of maximal order is p^4q, where $p < q < r$ are different primes.

Keywords: finite groups, group order, the number of elements of maximal order, isomorphic classification.

1. Introduction

All groups considered in our paper are finite. Let n be an integer. We denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Then, $\pi(|G|)$ is denoted by $\pi(G)$. The set of orders of elements of G is denoted by $\pi_e(G)$. We denote by $k(G)$ and $m(G)$ the maximal order of elements in G and the number of elements of order $k(G)$ in G, respectively. We write H char G if H is characteristic in G. $G = N \rtimes Q$ stands for the split extension of a normal subgroup N of G by a complement Q. By $M \lesssim G$ we denote M is isomorphic to a subgroup of G. And we denote by Z_n a cyclic group of order n. All unexplained notations are standard and can be found in [6].

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For a finite group G, $|G|$ and $m(G)$ have an important influence on the structure of G. The authors in [13, 3, 9] proved that finite groups G with $m(G) = lp$ are soluble, where $l = 2$, 4, or 18. In [8] it was proved that finite groups G with $m(G) = 2p^2$ are soluble. The authors in [2, 7] gave a classification of the finite groups G with $m(G) = 30$ and $m(G) = 24$. The authors in [10] showed that if G is a finite group which has $4p^2q$ elements of maximal order, where p, q are primes and $7 \leq p \leq q$, then either G is soluble or G has a section who is isomorphic to one of $L_2(7)$, $L_2(8)$ or $U_3(3)$. These studies are closely related to the following problem.

Thompson's problem. Let H be a finite group. For a positive integer d, define $H(d) = |\{x \in H||x| = d\}|$. Suppose that $H(d) = G(d)$ for $d = 1, 2, \ldots$, where G is a soluble group. Is it true that H is also necessarily soluble?

The problem we consider is also closely related to Thompson's problem. In this paper we classify the finite groups of order p^3qr in which the number of elements of maximal order is p^4q, where $p < q < r$ are primes (Let us denote this property by (*) for brevity). We find that this isomorphic classification problem is complex. Our results are:

Theorem 1.1. *A group G has property (*) if and only if one of the following statements holds:*

(1) $G \cong M \ltimes Z_r$ *and* $r - 1 = 16q$. *Moreover,* $C_M(Z_r) \cong Z_2$, $M/C_M(Z_r) \lesssim Aut(Z_r)$ *and* $|M/C_M(Z_r)| = 4q$;

(2) $G \cong K \ltimes Z_r$ *and* $r - 1 = 8q$. *Moreover,* $C_K(Z_r) \cong Z_4$, $K/C_K(Z_r) \lesssim Aut(Z_r)$ *and* $|K/C_K(Z_r)| = 2q$;

(3) $G \cong L \ltimes Z_r$ *and* $r - 1 = 8q$. *Moreover,* $C_L(Z_r) \cong D_8$, $L/C_L(Z_r) \lesssim Aut(Z_r)$ *and* $|L/C_L(Z_r)| = q$;

(4) $G \cong R \ltimes Z_r$ *and* $r - 1 = 4q$. *Moreover,* $C_R(Z_r) \cong Z_4 \times Z_2$, $R/C_R(Z_r) \lesssim Aut(Z_r)$ *and* $|R/C_R(Z_r)| = q$;

(5) $G \cong Z_q \ltimes Z_{8r}$ *and* $r - 1 = 4q$. *Moreover,* $C_{Z_q}(Z_{8r}) = 1$;

(6) $G \cong M \ltimes Z_7$ *and* $C_M(Z_7) \cong A_4$, *where* $M \cong A_4 \times Z_2$ *or* S_4;

(7) $G \cong Z_{168}$;

(8) $G \cong Q_8 \times Z_{15}$;

(9) $G \cong D_8 \times Z_{qr}$, *where* $q = 3$ *and* $r = 13$ *or* $q = 5$ *and* $r = 11$;

(10) $G \cong (Z_4 \times Z_2) \times Z_{21}$;

(11) $G \cong M \ltimes Z_{qr}$, $q = 3$ *and* $r = 13$ *or* $q = 5$ *and* $r = 11$, *where* M *is a group of order 8. Moreover,* $C_M(Z_{qr}) \cong Z_4$;

(12) $G \cong (A_4 \times Z_2) \times Z_7$;

(13) $G \cong SL_2(F_3) \times Z_7$;

(14) *G is a Frobenius group and* $G \cong Z_{8q} \ltimes Z_r$. *Moreover,* $r - 1 = 16q$;

(15) $G \cong L_2(7)$;

(16) *G is a 2-Frobenius group and* $G \cong Z_3 \ltimes (Z_7 \ltimes P)$, *where P is an elementary abelian 2-group of order 8,* $P \trianglelefteq G$ *and* $G/P \cong Z_3 \ltimes Z_7$. *Moreover,* $\pi_e(G) = \{1, 2, 3, 6, 7\}$.

Corollary 1.2. *All of the groups with property* $(*)$ *are of even order.*

Corollary 1.3. *Suppose that* G *is a non-soluble group with property* $(*)$*. Then,* $G \cong L_2(7)$*.*

Corollary 1.4. *The answer to Thompson's problem is yes for finite groups* (1)-(14) *and* (16) *of Theorem 1.1.*

2. Preliminaries

We need the following lemmas to prove our results.

Lemma 2.1 ([12])**.** *Let* G *be a finite group. Then, the number of elements whose orders are multiples of* n *is either zero, or a multiple of the greatest divisor of* $|G|$ *that is prime to* n*.*

Lemma 2.2 ([3])**.** *Let* G *be a finite group. We denote by* A_i $(1 \leq i \leq s)$ *a complete representative system of conjugate classes of cyclic subgroups of order* $k(G)$*, respectively. Then, we have:*

(1) $m(G) = \varphi(k(G)) \sum n_i$*, where* $\varphi(k(G))$ *is Euler function,* $n_i = |G : N_G(A_i)|$ *and* $1 \leq i \leq s$*;*

(2) $|G| = |G : N_G(A_i)||N_G(A_i) : C_G(A_i)||C_G(A_i)|$*, where* $1 \leq i \leq s$*;*

(3) $|N_G(A_i) : C_G(A_i)||\varphi(k(G))$*, where* $1 \leq i \leq s$*;*

(4) $\pi(C_G(A_i)) = \pi(A_i)$*, where* $1 \leq i \leq s$*.*

Lemma 2.3 ([4])**.** *Let* G *be a soluble group of order* mn*, where* m *is prime to* n*. Then, the number of subgroups of* G *of order* m *may be expressed as a product of factors, each of which (i) is congruent to 1 modulo some prime factor of* m*, (ii) is a power of a prime and divides the order of some chief factor of* G*.*

Lemma 2.4 ([1])**.** *Let* H *be a finite group and* $\pi_e(H) = \{1, 2, 3, 4\}$*. Then,* $H = N \rtimes Q$ *and one of the following conclusions holds:*

(*i*) N *has exponent 4 and class* ≤ 2*,* $Q \cong Z_3$*.*

(*ii*) $N = {Z_2}^{2t}$ *and* $Q \cong S_3$*, where* ${Z_2}^{2t}$ *stands for the direct product of* $2t$ *copies of* Z_2*.*

(*iii*) $N = {Z_3}^{2t}$ *and* $Q \cong Z_4$ *or* Q_8 *and* H *is a Frobenius group, where* Q_8 *is the generalized quaternion group.*

Lemma 2.5 ([14])**.** *Let* G *be a finite group satisfying* $|G| = 2^3 \cdot 3 \cdot 7$ *and* $m(G) = 48$*.*

(1) *If* $k(G) = 42$*, then* $G \cong (A_4 \times Z_2) \times Z_7$ *or* $G \cong SL_2(F_3) \times Z_7$*.*

(2) *If* $k(G) = 21$*, then* $G \cong M \ltimes Z_7$ *and* $C_M(Z_7) \cong A_4$*, where* $M \cong A_4 \times Z_2$ *or* S_4*.*

Lemma 2.6 ([5])**.** *Let* G *be a finite simple group. If* $|\pi(G)| = 3$*, then we call* G *a simple* K_3*-group. If* G *is a simple* K_3*-group, then* G *is isomorphic to one of the following groups:* A_5*,* A_6*,* $L_2(7)$*,* $L_2(8)$*,* $L_2(17)$*,* $L_3(3)$*,* $U_3(3)$ *and* $U_4(2)$*.*

Lemma 2.7 ([15]). *Let G be a finite group. Then, $|G| = |L_2(7)|$ and $k(G) = k(L_2(7))$ if and only if $G \cong L_2(7)$ or G is a 2-Frobenius group, at this moment, $G \cong Z_3 \ltimes (Z_7 \ltimes P)$, where P is an elementary abelian 2-group of order 8, $P \trianglelefteq G$ and $G/P \cong Z_3 \ltimes Z_7$. Moreover, $\pi_e(G) = \{1, 2, 3, 6, 7\}$.*

3. Proof of the Results

Proof of Theorem 1.1

It is not hard to see that all the groups from items (1)-(16) of Theorem 1.1 have property (*).

Now, we assume that G has property (*). Namely, $|G| = p^3qr$ and $m(G) = p^4q$. From Lemma 2.1 we get that $\pi(G) \subseteq \pi(m(G)) \bigcup \pi(k(G))$. Then, $r \in \pi(k(G))$. Since $\varphi(k(G)) \mid m(G)$ by Lemma 2.2, we obtain that $\varphi(r) = r-1|p^4q$. From $2|r-1$ it follows that $p = 2$. In the following we discuss four cases.

Case 1. If $\pi(k(G)) = \{2, r\}$, then $k(G) = 2r$, $4r$ or $8r$.

Suppose that $k(G) = 2r$. Choose an arbitrary element x of order $k(G)$ in G and let $\langle x\rangle = A$. It is clear that $x^2 \in Z(C_G(A))$ and so G has a Sylow r-subgroup P_r such that $P_r \leq Z(C_G(A))$. Therefore P_r char $C_G(A)$ and it follows that $P_r \trianglelefteq N_G(A)$ since $C_G(A) \trianglelefteq N_G(A)$. Therefore $N_G(A) \leq N_G(P_r)$ and thus $|G : N_G(P_r)|||G : N_G(A)|$. By Lemma 2.2 we get that $|G : N_G(A)||4q$. So $|G : N_G(P_r)||4q$.

If $P_r \ntrianglelefteq G$, then $|G : N_G(P_r)| = 2q$ or $4q$ by Sylow's theorem. If $|G : N_G(P_r)| = 4q$, then $|G : N_G(A)| = 4q$ and so $4q|n$ by Lemma 2.2, where n is the number of cyclic subgroups of order $k(G)$ in G. Note that $n = \frac{m(G)}{\varphi(2r)} = \frac{16q}{r-1}$, thus $r-1 = 4$ and so $r = 5$. It follows that $q = 3$. Hence, $|G : N_G(P_5)| = 12$, which is contradict to Sylow's theorem. If $|G : N_G(P_r)| = 2q$, then $|N_G(P_r)| = 4r$ and $|C_G(P_r)| = 2^\alpha r$, where $1 \leq \alpha \leq 2$. Moreover, $C_G(P_r)$ contains exactly $\frac{m(G)}{2q} = 8$ elements of order $2r$. On the other hand, we get that $C_G(P_r) = H \times P_r$ by Schur-Zassenhaus's theorem since $P_r \leq Z(C_G(P_r))$, where H is a group satisfying $|H| = 2^\alpha$. It follows that $C_G(P_r)$ contains exactly $(2^\alpha - 1)(r-1)$ elements of order $2r$. Thus, $(2^\alpha - 1)(r-1) = 8$, which is impossible obviously since $1 \leq \alpha \leq 2$.

If $P_r \trianglelefteq G$, then $C_G(P_r)$ contains all the elements of order $k(G)$ in G since $A \leq C_G(A) \leq C_G(P_r)$. Note that $P_r \leq Z(C_G(P_r))$, thus $|C_G(P_r)| = 2^l r$, where $1 \leq l \leq 3$. Moreover, $C_G(P_r) = H_1 \times P_r$ by Schur-Zassenhaus's theorem, where H_1 is a group of order 2^l. If $l = 2$, then H_1 is an elementary abelian group of order 4. Thus, $3(r-1) = 16q$ and it follows that $q = 3$ and $r = 17$. Since $|G/C_G(P_{17})|||Aut(P_{17})|$, we get that $6|16$, which is a contradiction. Similarly, we can show that $l \neq 3$. If $l = 1$, then $r - 1 = 16q$. Note that $P_r \cong Z_r$, then by Schur-Zassenhaus's theorem we get that $G \cong M \ltimes Z_r$. Moreover, $C_M(Z_r) \cong Z_2$, $M/C_M(Z_r) \lesssim \mathrm{Aut}(Z_r)$ and $|M/C_M(Z_r)| = 4q$. Hence, (1) holds.

Suppose that $k(G) = 4r$. Choose an arbitrary element x of order $k(G)$ in G and let $\langle x\rangle = A$. Similar to the above, we can get that G has a Sylow r-

subgroup P_r such that $P_r \leq Z(C_G(A))$, $|G : N_G(P_r)|\,|2q$ and $|G : N_G(P_r)| = |G : N_G(A)| = 2q$ by Sylow's theorem if $P_r \ntrianglelefteq G$. Hence, $C_G(P_r)$ contains exactly $\frac{m(G)}{2q} = 8$ elements of order $4r$. Note that $P_r \leq Z(C_G(P_r))$, thus $r-1|8$. It follows that $r = 5$ and so $q = 3$. Therefore $|G : N_G(P_5)| = 6$ and so $|N_G(P_5)| = |C_G(P_5)| = 20$. Hence, G is 5-nilpotent by Burnside's theorem. Then, G is soluble. By Lemma 2.3 it follows that $2 \equiv 1 (\bmod 5)$ and $3 \equiv 1 (\bmod 5)$, which is impossible.

If $P_r \trianglelefteq G$, then $C_G(P_r)$ contains all the elements of order $4r$ in G. Furthermore, $|C_G(P_r)| = 2^\alpha \cdot q^\beta \cdot r$, where $2 \leq \alpha \leq 3$ and $0 \leq \beta \leq 1$. Note that $P_r \leq Z(C_G(P_r))$, then by Schur-Zassenhaus's theorem we have $C_G(P_r) = H \times P_r$, where H is a group of order $2^\alpha \cdot q^\beta$.

Suppose that $\beta = 1$. Then, $q = 3$ since $k(G) = 4r$ is the maximal element order of G. If $\alpha = 2$, then H is a group of order 12 and $\pi_e(H) = \{1, 2, 3, 4\}$. It follows that $H \cong Z_4 \rtimes Z_3$ by Lemma 2.4. Hence, $2(r-1) = m(G) = 48$. It follows that $r = 25$, which is impossible. If $\alpha = 3$, then $C_G(P_r) = G$ and so $P_r \leq Z(G)$. Consequently, $G = M \times P_r$ by Schur-Zassenhaus's theorem, where M is a group of order 24. Note that $\pi_e(M) = \{1, 2, 3, 4\}$, thus $M \cong (Z_2 \times Z_2) \rtimes S_3$ or $N \rtimes Z_3$ by Lemma 2.4. If $M \cong (Z_2 \times Z_2) \rtimes S_3$, then $6(r-1) = m(G) = 48$ and thus $r = 9$, which is a contradiction. If $M \cong N \rtimes Z_3$, then the conjugate action of Z_3 on N is fixed-point-free. Thus, $|Z_3|\,||N| - 1$ and it follows that $3|7$, which is impossible.

Suppose that $\beta = 0$. Then, $|C_G(P_r)| = 4r$ or $8r$. If $|C_G(P_r)| = 4r$, then $C_G(P_r) \cong Z_4 \times Z_r$. It follows that $2(r-1) = m(G) = 16q$ and so $r - 1 = 8q$. Moreover, $G \cong K \ltimes Z_r$ by Schur-Zassenhaus's theorem, $C_K(Z_r) \cong Z_4$, $K/C_K(Z_r) \lesssim \mathrm{Aut}(Z_r)$ and $|K/C_K(Z_r)| = 2q$. Hence, (2) holds. If $|C_G(P_r)| = 8r$, then H is isomorphic to the dihedral group D_8, the generalized quaternion group Q_8 or $Z_4 \times Z_2$ since $k(H) = 4$. If $H \cong Q_8$, then $6(r-1) = m(G) = 16q$ and so $r = 9$, which is a contradiction. If $H \cong D_8$, then $2(r-1) = m(G) = 16q$ and so $r - 1 = 8q$. Moreover, $G \cong L \ltimes Z_r$ by Schur-Zassenhaus's theorem, $C_L(Z_r) \cong D_8$, $L/C_L(Z_r) \lesssim \mathrm{Aut}(Z_r)$ and $|L/C_L(Z_r)| = q$. Hence, (3) holds. If $H \cong Z_4 \times Z_2$, then $4(r-1) = m(G) = 16q$ and so $r - 1 = 4q$. Moreover, $G \cong R \ltimes Z_r$, $C_R(Z_r) \cong Z_4 \times Z_2$, $R/C_R(Z_r) \lesssim \mathrm{Aut}(Z_r)$ and $|R/C_R(Z_r)| = q$. Hence, (4) holds.

Suppose that $k(G) = 8r$. Choose an arbitrary element x of order $k(G)$ in G and let $\langle x \rangle = A$. It is clear that $x^8 \in Z(C_G(A))$ and so G has a Sylow r-subgroup P_r such that $P_r \leq Z(C_G(A))$. Since $A \leq C_G(A) \leq C_G(P_r)$, we have $|C_G(P_r)| = 8q^\gamma r$, where $0 \leq \gamma \leq 1$. Note that $P_r \leq Z(C_G(P_r))$, thus $C_G(P_r) = H \times P_r$ by Schur-Zassenhaus's theorem, where H is a group of order $8q^\gamma$ and $k(H) = 8$.

Suppose that $\gamma = 1$. Since $k(G) = 8r$, we have $q = 3$, 5 or 7. Note that the Sylow 2-subgroup P_2 of H is cyclic, thus H is 2-nilpotent and so the Sylow q-subgroup Q of H is normal in H. If $q = 5$ or 7, then the conjugate action of P_2 on Q is fixed-point-free since $k(H) = 8$. Therefore $8|q - 1$, which is impossible.

If $q = 3$, then H is a group of order 24 satisfying $k(H) = 8$. Now, we get a contradiction since such group H does not exist by [11].

Suppose that $\gamma = 0$. Then, $|C_G(P_r)| = 8r$. Since the Sylow 2-subgroup of G is cyclic, we get that G is 2-nilpotent. It follows that the subgroup of order qr of G is normal in G. Then, $P_r \trianglelefteq G$ by Sylow's theorem and so $C_G(P_r) \trianglelefteq G$. Hence, $C_G(P_r)$ contains all the elements of order $8r$ and $G \cong Z_q \ltimes Z_{8r}$ by Schur-Zassenhaus's theorem. Moreover, $4(r-1) = m(G) = 16q$ and $C_{Z_q}(Z_{8r}) = 1$. Hence, (5) holds.

Case 2. If $\pi(k(G)) = \{q, r\}$, then $k(G) = qr$.

Choose an arbitrary element x of order $k(G)$ in G and let $\langle x \rangle = A$. Similar to Case 1, we can get that G has a Sylow r-subgroup P_r such that $P_r \le Z(C_G(A))$ and $|G : N_G(P_r)| = 1$ or 8.

If $|G : N_G(P_r)| = 1$, then $P_r \trianglelefteq G$ and $C_G(P_r)$ contains all the elements of order qr in G since $A \le C_G(A) \le C_G(P_r)$. Moreover, G is soluble. By Lemma 2.3 it follows that $|G : N_G(A)| = 1$, 4 or 8. If $|G : N_G(A)| = 8$, then $8(q-1)(r-1) = m(G) = 16q$. If follows that $q = 3$ and $r = 4$, which is a contradiction. If $|G : N_G(A)| = 4$, then $4 \equiv 1 \pmod q$ by Lemma 2.3. Therefore $q = 3$ and thus $4(3-1)(r-1) = m(G) = 48$. Hence, $r = 7$. Therefore by (2) of Lemma 2.5 we have $G \cong M \ltimes Z_7$ and $C_M(Z_7) \cong A_4$, where $M \cong A_4 \times Z_2$ or S_4. Hence, (6) holds. If $|G : N_G(A)| = 1$, then $(q-1)(r-1) = 16q$, which is impossible we can find by simple calculation.

If $|G : N_G(P_r)| = 8$, then $C_G(P_r)$ contains exactly $\frac{m(G)}{8} = 2q$ elements of order qr. On the other hand, we know that $A \le C_G(A) \le C_G(P_r)$, thus $C_G(P_r)$ contains at least $\varphi(qr) = (q-1)(r-1)$ elements of order qr. Now, we get a contradiction since $(q-1)(r-1) > 2q$.

Case 3. If $\pi(k(G)) = \{2, q, r\}$, then $k(G) = 8qr$, $4qr$ or $2qr$.

If $k(G) = 8qr$, then $\varphi(8qr) = 4(q-1)(r-1) = 16q$. Consequently, $\frac{q-1}{2} \cdot \frac{r-1}{2} = q$. Since $\frac{r-1}{2} > 1$, we have $\frac{q-1}{2} = 1$ and so $q = 3$. It follows that $r = 7$. Hence, $G \cong Z_{168}$ and thus (7) holds.

Suppose that $k(G) = 4qr$. Choose an arbitrary element x of order $k(G)$ in G and let $\langle x \rangle = A$. It is clear that $Z(C_G(A))$ contains elements of order qr, and so G has a subgroup H of order qr such that $H \le Z(C_G(A))$. Therefore H char $C_G(A)$ and it follows that $H \trianglelefteq N_G(A)$ since $C_G(A) \trianglelefteq N_G(A)$. So $N_G(A) \le N_G(H)$. Then, $|G : N_G(H)| \,\big|\, |G : N_G(A)|$. Note that $|G : N_G(A)| = 1$, thus $|G : N_G(H)| = 1$ and so $H \trianglelefteq G$. Therefore $C_G(H)$ contains all the elements of order $k(G)$ in G and so $|C_G(H)| = 2^\alpha qr$, where $2 \le \alpha \le 3$.

If $\alpha = 3$, then $C_G(H) = G$ and so $H \le Z(G)$. Thus, $G = K \times H$ by Schur-Zassenhaus's theorem. Obviously, K is isomorphic to the dihedral group D_8, the generalized quaternion group Q_8 or $Z_4 \times Z_2$. If $K \cong Q_8$, then $6(q-1)(r-1) = m(G) = 16q$. Hence, $q = 3$ and $r = 5$. Therefore $G \cong Q_8 \times Z_{15}$. Hence, (8) holds. If $K \cong D_8$, then similarly we can get that $G \cong D_8 \times Z_{qr}$, where $q = 3$ and

$r = 13$ or $q = 5$ and $r = 11$. Hence, (9) holds. If $K \cong Z_4 \times Z_2$, then similarly we can get that $G \cong (Z_4 \times Z_2) \times Z_{21}$. Hence, (10) holds.

If $\alpha = 2$, then $C_G(H) \cong Z_4 \times Z_{qr}$. So $2(q-1)(r-1) = 16q$. It follows that $q = 3$ and $r = 13$ or $q = 5$ and $r = 11$. Furthermore, $G \cong M \ltimes Z_{qr}$ by Schur-Zassenhaus's theorem and $C_M(Z_{qr}) \cong Z_4$, where M is a group of order 8. Hence, (11) holds.

Suppose that $k(G) = 2qr$. Choose an arbitrary element x of order $k(G)$ in G and let $\langle x \rangle = A$. From the fact that $Z(C_G(A))$ contains elements of order qr we get that G has a cyclic subgroup H of order qr such that $H \leq Z(C_G(A))$. Similar to the above, we get that $|G : N_G(H)| = 1$, 2 or 4. Moreover, $|C_G(H)| = 2^\alpha qr$, where $1 \leq \alpha \leq 3$.

If $|G : N_G(H)| = 1$, then $H \trianglelefteq G$. It follows that $C_G(H)$ contains all elements of order $2qr$ since $A \leq C_G(A) \leq C_G(H)$. Since $H \leq Z(C_G(H))$, by Schur-Zassenhaus's theorem we have $C_G(H) = M \times H$, where M is an elementary abelian group of order 2^α. Hence, $(2^\alpha - 1)(q-1)(r-1) = m(G) = 16q$, which is impossible we can find by simple calculation. If $|G : N_G(H)| = 2$, then G is non-soluble by Lemma 2.3. Note that $N_G(H) \trianglelefteq G$, thus $N_G(H) \cong A_5$ by Lemma 2.6, which is a contradiction since $2qr \in \pi_e(N_G(H))$ and $2qr \notin \pi_e(A_5)$. If $|G : N_G(H)| = 4$, then $|G : N_G(A)| = 4$. Thus, $4|n$ by Lemma 2.2, where n is the number of the cyclic subgroups of order $2qr$ of G. Note that $n = \frac{m(G)}{\varphi(k(G)} = \frac{16q}{(q-1)(r-1)}$, thus $q = 3$ and $r = 7$. Therefore $G \cong (A_4 \times Z_2) \times Z_7$ or $G \cong SL_2(F_3) \times Z_7$ by (1) of Lemma 2.5. Hence, (12) and (13) hold.

Case 4. If $\pi(k(G)) = \{r\}$, then $k(G) = r$.

We know that the number n_r of Sylow r-subgroups of G is equal to 1, $2q$, $4q$, $8q$ or 8 by Sylow's theorem.

If $n_r = 1$, then the Sylow r-subgroup P_r of G is normal in G and $r - 1 = m(G) = 16q$. Moreover, G has an r-complement H of order $8q$ by Schur-Zassenhaus's theorem. Note that the conjugate action of H on P_r is fixed-point-free, thus G is a Frobenius group with Frobenius kernel P_r and Frobenius complement H. Note that $P_r \cong Z_r$ and H is a cyclic group since $H \lesssim \mathrm{Aut}(P_r)$, thus $G \cong Z_{8q} \ltimes Z_r$. Hence, (14) holds.

If $n_r = 2q$, then $2q(r-1) = m(G) = 16q$. It follows that $r = 9$, which is impossible.

If $n_r = 4q$, then $4q(r-1) = m(G) = 16q$. It follows that $r = 5$ and $q = 3$, which is contradict to Sylow's theorem.

If $n_r = 8q$, then $8q(r-1) = m(G) = 16q$. It follows that $r = 3$, which is impossible.

If $n_r = 8$, then $r = 7$ by Sylow's theorem and so $q = 3$ or 5. If $q = 5$, then $|N_G(P_7)| = 35$. Since $N_G(P_7)/C_G(P_7) \lesssim \mathrm{Aut}(P_7)$, we have $|N_G(P_7)/C_G(P_7)|$ divides $|Aut(P_7)|$. Note that $|C_G(P_7)| = 7$, thus $5|6$, which is a contradiction. If $q = 3$, then by Lemma 2.7 $G \cong L_2(7)$ or G is a 2-Frobenius group, at this moment, $G \cong Z_3 \ltimes (Z_7 \ltimes P)$, where P is an elementary abelian 2-group of order

8, $P \trianglelefteq G$ and $G/P \cong Z_3 \ltimes Z_7$. Moreover, $\pi_e(G) = \{1, 2, 3, 6, 7\}$. Hence, (15) and (16) hold.

Proof of Corollaries 1.2 and 1.3. It is evident by Theorem 1.1.

Proof of Corollary 1.4. Assume that G is a group, which is isomorphic to one of the finite groups (1-14) and (16) of Theorem 1.1. Suppose that H is a group satisfying $H(d) = G(d)$. Then, $|H| = |G|$ and $m(H) = m(G)$. Thus, H is soluble by Theorems 1.1. Hence, Corollary 1.4 holds.

Now, the proofs of our results are complete.

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