Schur convexity of a function whose fourth-order derivative is non-negative and related inequalities

Yiting Wu*
Department of Mathematics
China Jiliang University
Hangzhou, 310018
People's Republic of China
yitingly@sina.com

Qing Meng
Department of Mathematics
China Jiliang University
Hangzhou, 310018
People's Republic of China
mengqing2233@163.com

Abstract. In this paper, we study the Schur convexity of a function containing variable upper and lower limit of integration, we prove that the function is Schur-convex if its fourth-order derivative is non-negative. Finally, we use the obtained result to derive an inequality of Hermite-Hadamard type.

Keywords: Schur-convex, majorization, fourth-order derivative, Hermite-Hadamard-type inequality.

1. Introduction

Schur convexity is an important notion in the theory of convex functions, which was introduced by Schur in 1923 (see [1]). Over the past half a century, Schur convexity has aroused the interest of many researchers due to its powerful applications in the theory of inequalities, we refer the reader to [2–19] and references cited therein.

In [20], Elezović and Pečarić proved the Schur convexity of the following function.

Claim 1.1. Suppose $f : I \to \mathbb{R}$ is a continuous function. Then, the function

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt, & x \neq y,\ x,y \in I \\ f(x), & x = y,\ x,y \in I \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if f is convex (concave) on I.

*. Corresponding author

In [21], Chu, Wang and Zhang showed the Schur convexity of the following two functions.

Claim 1.2. Suppose $f : I \to \mathbb{R}$ is a continuous function. Then, the function

$$M(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt - f\left(\frac{x+y}{2}\right), & x \neq y,\ x,y \in I \\ 0, & x = y,\ x,y \in I \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if f is convex (concave) on I, and the function

$$T(x,y) = \begin{cases} \frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(t)dt, & x \neq y,\ x,y \in I \\ 0, & x = y,\ x,y \in I \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if f is convex (concave) on I.

In [22], Franjić and Pečarić verified the Schur convexity of the function below.

Claim 1.3. Suppose $f : I \to \mathbb{R}$ is a continuous function. Then, the function

$$S(x,y) = \begin{cases} \frac{1}{6}f(x) + \frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(y) - \frac{1}{y-x}\int_x^y f(t)dt, & x \neq y,\ x,y \in I \\ 0, & x = y,\ x,y \in I \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if $f^{(4)} \geq 0$ ($f^{(4)} \leq 0$) on I.

Inspired by the research results described in [20-22] above, in this paper we study the Schur convexity of a function which contains variable upper and lower limit of integration, i.e.,

$$U(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt - \frac{f(x)+f(y)}{2} + \frac{1}{12}\left(f'(y) - f'(x)\right)(y-x), & x \neq y,\ x,y \in I \\ 0, & x = y,\ x,y \in I. \end{cases}$$

The remaining parts of this paper are organized as follows. In Section 2, we present some definitions and lemmas which are essential in the proof of the main results. In Sections 3 and 4, we give our main result and an application.

2. Preliminaries

Let us recall some definitions and lemmas, which will be used in the proofs of the main results in subsequent sections.

Definition 2.1 ([2, 23])**.** *Let* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ *and* $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(i) $\boldsymbol{x}$ *is said to be majorized by* $\boldsymbol{y}$ *(in symbols* $\boldsymbol{x} \prec \boldsymbol{y}$*) if* $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ *for* $k = 1, 2, \ldots, n-1$ *and* $\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}$*, where* $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ *and* $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ *are rearrangements of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *in a descending order.*

(ii) Let $\Omega \subset \mathbb{R}^n$, $\varphi : \Omega \to \mathbb{R}$ *is said to be a Schur-convex function on* Ω *if* $\boldsymbol{x} \prec \boldsymbol{y}$ *on* Ω *implies* $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$*. And* φ *is said to be a Schur-concave function on* Ω *if and only if* $-\varphi$ *is a Schur-convex function on* Ω*.*

Definition 2.2 ([2, 23])**.** *Let* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$.

(i) A set $\Omega \subset \mathbb{R}^n$ *is called a symmetric set, if* $\boldsymbol{x} \in \Omega$ *implies* $\boldsymbol{x}P \in \Omega$ *for every* $n \times n$ *permutation matrix* P.

(ii) A function $\varphi : \Omega \to \mathbb{R}$ *is called a symmetric function if for every permutation matrix* P, $\varphi(\boldsymbol{x}P) = \varphi(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \Omega$.

Lemma 2.1 ([2, 23])**.** *Let* $\Omega \subset \mathbb{R}^n$ *be symmetric and have a nonempty interior convex set.* Ω° *is the interior of* Ω. $\varphi : \Omega \to \mathbb{R}$ *is continuous on* Ω *and differentiable in* Ω°. *Then,* φ *is Schur-convex on* Ω *if and only if* φ *is symmetric on* Ω *and*

$$(1) \qquad (x_i - x_j)\left(\frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial x_j}\right) \geq 0 \qquad (i \neq j,\ i, j = 1, 2, \ldots, n)$$

for any $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega^\circ$. *Furthermore,* φ *is Schur-concave on* Ω *if and only if the reversed inequality above holds.*

Lemma 2.2 ([24])**.** *Let* $x \leq y$, $u(t) = ty + (1-t)x$, $v(t) = tx + (1-t)y$, $0 \leq t_1 \leq t_2 \leq \frac{1}{2}$ *or* $\frac{1}{2} \leq t_2 \leq t_1 \leq 1$. *Then*

$$(2) \qquad \left(\frac{x+y}{2}, \frac{x+y}{2}\right) \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (x, y).$$

Lemma 2.3 ([25])**.** (Simpson formula) *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *and* $x, y \in I$. *If* $f^{(4)}$ *is continuous on* I, *then*

$$(3) \quad \frac{1}{y-x}\int_x^y f(t)dt - \frac{1}{6}\left(f(x) + 4f\left(\frac{x+y}{2}\right) + f(y)\right) = -\frac{(y-x)^4}{2880} f^{(4)}(\xi),$$

where ξ *is some number between* x *and* y.

3. Main result

Our main result is stated in the following theorem.

Theorem 3.1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a continuous function. If* $f^{(4)} \geq 0$ $(f^{(4)} \leq 0)$ *on* I, *then the function*

$$U(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt - \frac{f(x)+f(y)}{2} + \frac{1}{12}\left(f'(y) - f'(x)\right)(y-x), & x \neq y,\ x, y \in I \\ 0, & x = y,\ x, y \in I \end{cases}$$

is Schur-convex (Schur-concave) on I^2.

Proof. Note that, $U(x,y)$ is symmetric about x, y on I, without loss of generality, we may assume that $y \geq x$. Below we divide the proof into two cases.

Case 1. If $x = y$, it follows from the definition of derivative and L'Hopital's rule that, for any $t_0 \in I$,

$$\begin{aligned}\frac{\partial U}{\partial x}|_{(t_0,t_0)} =& \lim_{\Delta t \to 0} \frac{U(t_0+\Delta t, t_0) - U(t_0,t_0)}{\Delta t} \\ =& \lim_{\Delta t \to 0} \frac{-\frac{1}{\Delta t}\int_{t_0+\Delta t}^{t_0} f(t)dt - \frac{f(t_0+\Delta t)+f(t_0)}{2} - \frac{\Delta t}{12}\left(f'(t_0) - f'(t_0+\Delta t)\right)}{\Delta t} \\ =& \lim_{\Delta t \to 0} \frac{-\int_{t_0+\Delta t}^{t_0} f(t)dt - \frac{\Delta t}{2}\left(f(t_0+\Delta t)+f(t_0)\right)}{(\Delta t)^2} \\ =& -\lim_{\Delta t \to 0} \frac{\Delta t f''(t_0+\Delta t)}{4} \\ =& 0.\end{aligned}$$

Similarly, we can obtain $\frac{\partial U}{\partial y}|_{(t_0,t_0)} = 0$. Hence we have, for any $x = y \in I$,

$$(y-x)\left(\frac{\partial U}{\partial y} - \frac{\partial U}{\partial x}\right) = 0.$$

Case 2. If $x \neq y$, differentiating $U(x,y)$ with respect to y and x respectively gives

$$\frac{\partial U}{\partial y} = -\frac{1}{(y-x)^2}\int_x^y f(t)dt + \frac{f(y)}{y-x} - \frac{f'(y)}{2} + \frac{f''(y)(y-x) + f'(y) - f'(x)}{12},$$

$$\frac{\partial U}{\partial x} = \frac{1}{(y-x)^2}\int_x^y f(t)dt - \frac{f(x)}{y-x} - \frac{f'(x)}{2} - \frac{f''(x)(y-x) + f'(y) - f'(x)}{12}.$$

Thus, we have

$$\begin{aligned}&(y-x)\left(\frac{\partial U}{\partial y} - \frac{\partial U}{\partial x}\right) \\ =& -\frac{2}{y-x}\int_x^y f(t)dt + (f(x)+f(y)) - \frac{(y-x)}{3}(f'(y) - f'(x)) \\ &+ \frac{(y-x)^2}{12}(f''(x) + f''(y)).\end{aligned} \tag{4}$$

Using the Simpson formula (Lemma 2.3) with $f^{(4)} \geq 0$, we obtain

$$\frac{1}{y-x}\int_x^y f(t)dt \leq \frac{1}{6}\left(f(x) + 4f\left(\frac{x+y}{2}\right) + f(y)\right). \tag{5}$$

Combining (4) and (5), we acquire that

$$\begin{aligned}&(y-x)\left(\frac{\partial U}{\partial y} - \frac{\partial U}{\partial x}\right) \\ \geq& -\frac{4}{3}f\left(\frac{x+y}{2}\right) + \frac{2}{3}(f(x)+f(y)) - \frac{(y-x)}{3}(f'(y) - f'(x)) \\ &+ \frac{(y-x)^2}{12}(f''(x) + f''(y)) \\ =:& Q(x,y).\end{aligned} \tag{6}$$

It is enough to prove $Q(x,y) \geq 0$ for any $x, y \in I$. Differentiating $Q(x,y)$ with respect to y and x respectively, we obtain

$$\frac{\partial Q}{\partial y} = -\frac{2}{3} f'(\frac{x+y}{2}) + \frac{f'(x)+f'(y)}{3} + \frac{(y-x)(f''(x)-f''(y))}{6} + \frac{(y-x)^2 f'''(y)}{12},$$

$$\frac{\partial Q}{\partial x} = -\frac{2}{3} f'(\frac{x+y}{2}) + \frac{f'(x)+f'(y)}{3} + \frac{(y-x)(f''(x)-f''(y))}{6} + \frac{(y-x)^2 f'''(x)}{12},$$

Then, by $f^{(4)} \geq 0$, we have

$$(y-x)\left(\frac{\partial Q}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{12}(y-x)^3 (f'''(y) - f'''(x)) \geq 0.$$

It follows from Lemma 2.1 that $Q(x,y)$ is Schur-convex on I^2. In addition, by Lemma 2.2, we have $\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \prec (x,y)$. Hence, we deduce from Definition 2.1 that

$$Q(x,y) \geq Q\left(\frac{x+y}{2}, \frac{x+y}{2}\right) = 0. \tag{7}$$

Combining (6) and (7), we conclude that, for any $x, y \in I$, $x \neq y$,

$$(y-x)\left(\frac{\partial U}{\partial y} - \frac{\partial U}{\partial x}\right) \geq Q(x,y) \geq 0.$$

Hence, we derive from Lemma 2.1 that $U(x,y)$ is Schur-convex on I^2.

By the same way as the proof of Theorem 3.1 for $f^{(4)} \geq 0$ above, we can prove that the $U(x,y)$ is Schur-concave for $f^{(4)} \leq 0$. This completes the proof of Theorem 4. □

4. An application

Theorem 4.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function with $f^{(4)} \geq 0$ on I. Then, for $x \neq y$, $x, y \in I$, $0 \leq t_1 \leq t_2 < \frac{1}{2}$ or $\frac{1}{2} < t_2 \leq t_1 \leq 1$, we have the following inequalities*

$$\begin{aligned}
&\frac{1}{y-x}\int_x^y f(t)dt - \frac{f(x)+f(y)}{2} + \frac{1}{12}\left(f'(y) - f'(x)\right)(y-x) \\
&\geq \frac{1}{(1-2t_1)(y-x)}\int_{t_1 y+(1-t_1)x}^{t_1 x+(1-t_1)y} f(t)dt \\
&- \frac{f\left(t_1 y + (1-t_1)x\right) + f\left(t_1 x + (1-t_1)y\right)}{2} \\
&+ \frac{1}{12}\left(f'\left(t_1 x + (1-t_1)y\right) - f'((t_1 y + (1-t_1)x)\right)(1-2t_1)(y-x) \\
&\geq \frac{1}{(1-2t_2)(y-x)}\int_{t_2 y+(1-t_2)x}^{t_2 x+(1-t_2)y} f(t)dt \qquad (8)\\
&- \frac{f\left(t_2 y + (1-t_2)x\right) + f\left(t_2 x + (1-t_2)y\right)}{2}
\end{aligned}$$

$$+\frac{1}{12}\left(f'\left(t_2x+(1-t_2)y\right)-f'((t_2y+(1-t_2)x)\right)(1-2t_2)(y-x)\geq \quad 0.$$

Each of the inequalities in (8) *is reverse for* $f^{(4)}\leq 0$ *on* I.

Proof. Since each of the inequalities in (8) is symmetric about x, y, without loss of generality, we can assume that $y > x$.

Using Lemma 2.2, we have

$$\left(\frac{x+y}{2},\frac{x+y}{2}\right)\prec(u(t_2),v(t_2))\prec(u(t_1),v(t_1))\prec(x,y), \tag{9}$$

where $u(t) = ty + (1-t)x$, $v(t) = tx + (1-t)y$.

In addition, from Theorem 3.1, we find that

$$U(x,y)=\begin{cases}\frac{1}{y-x}\int_x^y f(t)dt-\frac{f(x)+f(y)}{2}+\frac{1}{12}\left(f'(y)-f'(x)\right)(y-x), & x\neq y, x,y\in I\\ 0, & x=y,\ x,y\in I\end{cases}$$

is Schur-convex on I^2 under the assumption that $f^{(4)}\geq 0$.

Thus, we derive from the Definition (2.1) that

$$U\left(\frac{x+y}{2},\frac{x+y}{2}\right)\leq U(u(t_2),v(t_2))\leq U(u(t_1),v(t_1))\leq U(x,y), \tag{10}$$

which implies the required inequalities in (8). Similarly, we can deduce the reversed inequalities of (8) under the assumption that $f^{(4)}\leq 0$. The proof of Theorem 4.1 is complete. □

As a direct consequence of Theorem 4.1, we obtain

Corollary 4.1. *Let* $f : I\subset\mathbb{R}\to\mathbb{R}$ *be a continuous function with* $f^{(4)}\geq 0$ *on* I. *Then, for* $x\neq y$, $x, y\in I$, *the following inequality holds.*

$$\frac{1}{y-x}\int_x^y f(t)dt\geq\frac{f(x)+f(y)}{2}-\frac{1}{12}\left(f'(y)-f'(x)\right)(y-x). \tag{11}$$

Inequality (11) *is reverse for* $f^{(4)}\leq 0$ *on* I.

Acknowledgements

This work is supported by the Natural Science Foundation of Zhejiang Province under Grant No. LY21A010016.

References

[1] I. Schur, *Über eine klasse von mittelbildungen mit anwendungen die determinanten*, Theorie Sitzungsber, Berlin. Math. Gesellschaft, 22 (1923), 9-20.

[2] A. W. Marshall, I. Olkin, *Inequalities: the theory of majorization and its applications*, Academic Press, New York, 1979.

[3] J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Academic Press, New York, 1992.

[4] H. N. Shi, S. H. Wu, *Majorized proof and improvement of the discrete Steffensen's inequality*, Taiwanese J. Math., 11 (2007), 1203-1208.

[5] S. H. Wu, L. Debnath, *Inequalities for convex sequences and their applications*, Comput. Math. Appl., 54 (2007), 525-534.

[6] S. H. Wu, H. N. Shi, *A relation of weak majorization and its applications to certain inequalities for means*, Math. Slovaca, 61 (2011), 561-570.

[7] H. N. Shi, S. H. Wu, *Schur m-power convexity of geometric Bonferroni mean*, Italian J. Pure Appl. Math., 38 (2017), 769-776.

[8] H. N. Shi, S. H. Wu, *A concise proof of a double inequality involving the exponential and logarithmic functions*, Italian J. Pure Appl. Math., 41 (2019), 284-289.

[9] S. H. Wu, H. N. Shi, D. S. Wang, *Schur convexity of generalized geometric Bonferroni mean involving three parameters*, Italian J. Pure Appl. Math., 42 (2019), 196-207.

[10] H. N. Shi, S. H. Wu, *Schur convexity of the dual form of complete symmetric function involving exponent parameter*, Italian J. Pure Appl. Math., 44 (2020), 836-845.

[11] H. N. Shi, S. H. Wu, F. Qi, *An alternative note on the Schur-convexity of the extended mean values*, Math. Inequal. Appl., 9 (2006), 219-224.

[12] H. N. Shi, Y. M. Jiang and W. D. Jiang, *Schur-convexity and Schur-geometrically concavity of Gini mean*, Comput. Math. Appl., 57 (2009), 266-274.

[13] H. N. Shi, J. Zhang, *Schur-convexity, Schur geometric and Schur harmonic convexities of dual form of a class symmetric functions*, J. Math. Inequal., 8 (2014), 349-358.

[14] D. Wang, C. R. Fu, H. N. Shi, *Schur m-power convexity for a mean of two variables with three parameters*, J. Nonlinear Sci. Appl., 9 (2016), 2298-2304.

[15] H. P. Yin, H. N. Shi, F. Qi, *On Schur m-power convexity for ratios of some means*, J. Math. Inequal., 9 (2015), 145-153.

[16] Z. H. Yang, *Schur power convexity of Stolarsky means*, Publ. Math. Debrecen, 80 (2012), 43-66.

[17] Z. H. Yang, *Schur power convexity of Gini means*, Bull. Korean Math. Soc., 50 (2013), 485-498.

[18] Z. H. Yang, *Schur power convexity of the Daróczy means*, Math. Inequal. Appl., 16 (2013), 751-762.

[19] W. Wang, S. Yang, *Schur m-power convexity of generalized Hamy symmetric function*, J. Math. Inequal., 8 (2014), 661-667.

[20] N. Elezović, J. Pečarić, *A note on Schur-convex functions*, Rocky Mountain J. Math., 30 (2000), 853-856.

[21] Y. Chu, G. Wang, X. Zhang, *Schur convexity and Hadamard's inequality*, Math. Inequal. Appl., 13 (2010), 725-731.

[22] I. Franjić, J. Pečarić, *Schur-convexity and the Simpson formula*, Appl. Math. Lett., 24 (2011), 1565-1568.

[23] H. N. Shi, *Schur-convex functions and inequalities*, Harbin Institute of Technology Press, Harbin, 2017 (Chinese).

[24] H. N. Shi, *Theory of majorization and analytic inequalities*, Harbin Industrial University Press, Harbin, 2012 (Chinese).

[25] Y. Li, *Schur convexity and the dual Simpson's formula*, J. Appl. Math. Phys., 4 (2016), 623-629.

Accepted: November 22, 2022