On improved Heinz inequalities for matrices

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Abstract. In this paper, we improve some Heinz inequalities for matrices by using the convexity of function. Theoretical analysis shows that new inequalities are refinement of the result in the related literature.
Keywords: Heinz inequalities, convex function, positive semidefinite matrix.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. So, $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. The singular values $s_j(A)(j = 1, \ldots, n)$ of A are the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$. The Schatten p-norm $\|\cdot\|_p$ is defined as

$$\|A\|_p = (\sum_{j=1}^{n} s_j^p(A))^{\frac{1}{p}}, 1 \le p < \infty$$

and the Ky Fan k-norm $\|\cdot\|_{(k)}$ is defined as

$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A), k = 1, \ldots, n.$$

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It is well known that the Schatten p-norm $\|\cdot\|_p$ and the Ky Fan k-norm $\|\cdot\|_{(k)}$ are unitarily invariant [1].

Bhatia and Davis [2] have proved the following inequality

$$(1.1)\quad 2\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\| \le \left\|A^vXB^{1-v}+A^{1-v}XB^v\right\| \le \|AX+XB\|, 0\le v\le 1,$$

where $A,B,X\in M_n$ with A and B are positive semidefinite matrices.

Kittaneh [3] proved that if $A,B,X\in M_n$ such that A and B are positive semidefinite, then

$$(1.2)\quad \left\|A^vXB^{1-v}+A^{1-v}XB^v\right\| \le (1-2r_0)\|AX+XB\| + 4r_0\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|,$$

where $0\le v\le 1, r_0=\min\{v,1-v\}$. The inequality (1.2) is a refinement of the second inequality in (1.1).

He et al. [4] proved that if $A,B,X\in M_n$ such that A and B are positive semidefinite, then

$$(1.3)\quad \left\|A^vXB^{1-v}+A^{1-v}XB^v\right\|^2 \le (1-2r_0)\|AX+XB\|^2 + 8r_0\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|^2,$$

where $0\le v\le 1, r_0=\min\{v,1-v\}$.

Improvements of Heinz inequalities have been done by many researchers. We refer the reader to [5-8]. In this paper, we will improve the inequalities (1.2) and (1.3) using the convexity of function.

2. Main results

Applying the convexity of function, we obtain the following theorem.

Theorem 1. *Let $A,B,X\in M_n$ such that A and B are positive semidefinite. Then for every unitarily invariant norm*

$$\begin{aligned}\left\|A^vXB^{1-v}+A^{1-v}XB^v\right\| &\le (1-6r_0)\|AX+XB\| \\ &\quad + 6r_0\left\|A^{\frac{1}{6}}XB^{\frac{5}{6}}+A^{\frac{5}{6}}XB^{\frac{1}{6}}\right\|, v\in[0,\tfrac{1}{6}]\cup(\tfrac{5}{6},1], \\ \left\|A^vXB^{1-v}+A^{1-v}XB^v\right\| &\le (6r_0-1)\left\|A^{\frac{1}{3}}XB^{\frac{2}{3}}+A^{\frac{2}{3}}XB^{\frac{1}{3}}\right\| \\ &\quad + 2(1-3r_0)\left\|A^{\frac{1}{6}}XB^{\frac{5}{6}}+A^{\frac{5}{6}}XB^{\frac{1}{6}}\right\|, \\ &\quad v\in(\tfrac{1}{6},\tfrac{1}{3}]\cup(\tfrac{2}{3},\tfrac{5}{6}]\end{aligned}$$

and

$$\begin{aligned}\left\|A^vXB^{1-v}+A^{1-v}XB^v\right\| &\le 4(3r_0-1)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\| \\ &\quad + 3(1-2r_0)\left\|A^{\frac{1}{3}}XB^{\frac{2}{3}}+A^{\frac{2}{3}}XB^{\frac{1}{3}}\right\|, v\in(\tfrac{1}{3},\tfrac{2}{3}],\end{aligned}$$

where $r_0=\min\{v,1-v\}$.

Proof. For $v = 0$, the Theorem 1 is obvious. For $0 < v \leq \frac{1}{6}$, since $f(v) = \left\|A^v X B^{1-v} + A^{1-v} X B^v\right\|$ is convex on $[0,1]$, it follows by a slope argument that

$$\frac{f(v) - f(0)}{v - 0} \leq \frac{f(\frac{1}{6}) - f(0)}{\frac{1}{6} - 0},$$

and so

$$f(v) \leq (1 - 6v) f(0) + 6v f(\frac{1}{6}),$$

that is

$$\begin{aligned} \left\|A^v X B^{1-v} + A^{1-v} X B^v\right\| &\leq (1 - 6v) \left\|AX + XB\right\| \\ &\quad + 6v \left\|A^{\frac{1}{6}} X B^{\frac{5}{6}} + A^{\frac{5}{6}} X B^{\frac{1}{6}}\right\|. \end{aligned} \tag{2.1}$$

For $\frac{1}{6} < v \leq \frac{1}{3}$, similarly, we have

$$\frac{f(v) - f(\frac{1}{6})}{v - \frac{1}{6}} \leq \frac{f(\frac{1}{3}) - f(\frac{1}{6})}{\frac{1}{6}},$$

and so

$$f(v) \leq (6v - 1) f(\frac{1}{3}) + (2 - 6v) f(\frac{1}{6}),$$

that is

$$\begin{aligned} \left\|A^v X B^{1-v} + A^{1-v} X B^v\right\| &\leq (6v - 1) \left\|A^{\frac{1}{3}} X B^{\frac{2}{3}} + A^{\frac{2}{3}} X B^{\frac{1}{3}}\right\| \\ &\quad + 2(1 - 3v) \left\|A^{\frac{1}{6}} X B^{\frac{5}{6}} + A^{\frac{5}{6}} X B^{\frac{1}{6}}\right\|. \end{aligned} \tag{2.2}$$

For $\frac{1}{3} < v \leq \frac{1}{2}$, similarly, we have

$$\begin{aligned} \left\|A^v X B^{1-v} + A^{1-v} X B^v\right\| &\leq 4(3v - 1) \left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\| \\ &\quad + 3(1 - 2v) \left\|A^{\frac{1}{3}} X B^{\frac{2}{3}} + A^{\frac{2}{3}} X B^{\frac{1}{3}}\right\|. \end{aligned} \tag{2.3}$$

For $\frac{1}{2} < v \leq \frac{2}{3}$, it follows by applying (2.3) to $1 - v$ that

$$\begin{aligned} \left\|A^v X B^{1-v} + A^{1-v} X B^v\right\| &\leq 4(2 - 3v) \left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\| \\ &\quad + 3(2v - 1) \left\|A^{\frac{1}{3}} X B^{\frac{2}{3}} + A^{\frac{2}{3}} X B^{\frac{1}{3}}\right\|. \end{aligned}$$

For $\frac{2}{3} < v \leq \frac{5}{6}$, by applying (2.2) to $1 - v$, we have

$$\begin{aligned} \left\|A^v X B^{1-v} + A^{1-v} X B^v\right\| &\leq (5 - 6v) \left\|A^{\frac{1}{3}} X B^{\frac{2}{3}} + A^{\frac{2}{3}} X B^{\frac{1}{3}}\right\| \\ &\quad + 2(3v - 2) \left\|A^{\frac{1}{6}} X B^{\frac{5}{6}} + A^{\frac{5}{6}} X B^{\frac{1}{6}}\right\|. \end{aligned}$$

For $\frac{5}{6} < v \leq 1$, by applying (2.1) to $1-v$, we have

$$\left\|A^v XB^{1-v} + A^{1-v} XB^v\right\| \leq (6v-5)\left\|AX+XB\right\| + 6(1-v)\left\|A^{\frac{1}{6}} XB^{\frac{5}{6}} + A^{\frac{5}{6}} XB^{\frac{1}{6}}\right\|.$$

This completes the proof. □

Remark 1. Theorem 1 is better than inequality (1.2). For $v \in [0, \frac{1}{6}] \cup (\frac{5}{6}, 1]$, we have

$$\begin{aligned}
&(1-6r_0)\left\|AX+XB\right\| + 6r_0\left\|A^{\frac{1}{6}} XB^{\frac{5}{6}} + A^{\frac{5}{6}} XB^{\frac{1}{6}}\right\| \\
&\leq (1-6r_0)\left\|AX+XB\right\| + 6r_0(\frac{2}{3}\left\|AX+XB\right\| + \frac{2}{3}\left\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right\|) \\
&= (1-2r_0)\left\|AX+XB\right\| + 4r_0\left\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right\|.
\end{aligned}$$

For $v \in (\frac{1}{6}, \frac{1}{3}] \cup (\frac{2}{3}, \frac{5}{6}]$, we have

$$\begin{aligned}
&(6r_0-1)\left\|A^{\frac{1}{3}} XB^{\frac{2}{3}} + A^{\frac{2}{3}} XB^{\frac{1}{3}}\right\| + 2(1-3r_0)\left\|A^{\frac{1}{6}} XB^{\frac{5}{6}} + A^{\frac{5}{6}} XB^{\frac{1}{6}}\right\| \\
&\leq (6r_0-1)[\frac{1}{3}\left\|AX+XB\right\| + \frac{4}{3}\left\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right\|] \\
&+ 2(1-3r_0)[\frac{2}{3}\left\|AX+XB\right\| + \frac{2}{3}\left\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right\|] \\
&= (1-2r_0)\left\|AX+XB\right\| + 4r_0\left\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right\|.
\end{aligned}$$

For $v \in (\frac{1}{3}, \frac{2}{3}]$, we have

$$\begin{aligned}
&4(3r_0-1)\left\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right\| + 3(1-2r_0)\left\|A^{\frac{1}{3}} XB^{\frac{2}{3}} + A^{\frac{2}{3}} XB^{\frac{1}{3}}\right\| \\
&\leq 4(3r_0-1)\left\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right\| + 3(1-2r_0)[\frac{1}{3}\left\|AX+XB\right\| + \frac{4}{3}\left\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right\|] \\
&= (1-2r_0)\left\|AX+XB\right\| + 4r_0\left\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right\|.
\end{aligned}$$

The following result implies that the inequality in Theorem 2 is a refinement of the inequality (1.3).

Theorem 2. *Let $A, B, X \in M_n$ such that A and B are positive semidefinite. Then for $0 \leq v \leq 1$ and for every unitarily invariant norm*

$$\begin{aligned}
&\left\|A^v XB^{1-v} + A^{1-v} XB^v\right\|^2 \\
&+ 2r_0(\left\|A^v XB^{1-v} + A^{1-v} XB^v\right\| - 2\left\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right\|)(\left\|AX+XB\right\| - 2\left\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right\|) \\
&\leq (1-2r_0)\left\|AX+XB\right\|^2 + 8r_0\left\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right\|^2,
\end{aligned}$$

where $r_0 = \min\{v, 1-v\}$.

Proof. For $v = 0, 1$, the result in Theorem 2 is obvious. For $0 < v \leq \frac{1}{2}$, since $f(v) = \left\|A^v X B^{1-v} + A^{1-v} X B^v\right\|$ is convex on $[0, 1]$, it follows that

$$\frac{f(v) - f(0)}{v - 0} \leq \frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0},$$

and so

$$2v(f(0) - f(\frac{1}{2}))(f(0) + f(v)) \leq f^2(0) - f^2(v),$$

that is

$$f^2(v) + 2v(f(v) - f(\frac{1}{2}))(f(0) - f(\frac{1}{2})) \leq (1 - 2v)f^2(0) + 2vf^2(\frac{1}{2}). \tag{2.4}$$

For $\frac{1}{2} < v < 1$, similarly, we have

$$\begin{aligned} & f^2(v) + 2(1-v)(f(v) - f(\frac{1}{2}))(f(0) - f(\frac{1}{2})) \\ & \leq (1 - 2(1-v))f^2(0) + 2(1-v)f^2(\frac{1}{2}). \end{aligned} \tag{2.5}$$

From (2.4) and (2.5), we obtain

$$f^2(v) + 2r_0(f(v) - f(\frac{1}{2}))(f(0) - f(\frac{1}{2})) \leq (1 - 2r_0)f^2(0) + 2r_0 f^2(\frac{1}{2}),$$

that is

$$\begin{aligned} & \left\|A^v X B^{1-v} + A^{1-v} X B^v\right\|^2 \\ & + 2r_0(\left\|A^v X B^{1-v} + A^{1-v} X B^v\right\| - 2\left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\|)(\|AX + XB\| - 2\left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\|) \\ & \leq (1 - 2r_0)\|AX + XB\|^2 + 8r_0\left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\|^2. \end{aligned}$$

This completes the proof. □

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