Real hypersurfaces in nonflat complex space forms with Lie derivative of structure tensor fields

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Abstract. In this paper, we obtain some non-existence theorems for real hypersurfaces in nonflat complex space forms such that the structure tensor fields are of Lie Codazzi, Lie Killing or Lie recurrent type.
Keywords: real hypersurface, complex space form, structure tensor field, Lie derivative.

1. Introduction

Let $M^n(c)$ be a complete and simply connected complex space form which is complex analytically isometric to

- a complex projective space $\mathbb{C}P^n(c)$ if $c > 0$;
- a complex Euclidean space $\mathbb{C}^n$ if $c = 0$;
- a complex hyperbolic space $\mathbb{C}H^n(c)$ if $c < 0$,

where c is the constant holomorphic sectional curvature. Let M be a real hypersurface of real dimension $2n-1$ immersed in $M^n(c)$, $n \geq 2$. On M there exists a natural almost contact metric structure (ϕ, ξ, η, g) induced from the complex structure on $M^n(c)$ and the normal vector field, respectively, where ξ and ϕ are called the structure vector field and the structure tensor field, respectively. If the structure vector field ξ on real hypersurfaces is principal at each point, then the hypersurface is said to be Hopf. In geometry of real hypersurface, the structure tensor field ϕ plays important roles in classification and characterization of Hopf hypersurfaces (see, many references in [2, 17]). Before stating our main study, we exhibit some well known results in this field.

A Hopf hypersurface in $\mathbb{C}P^n(c)$ has constant principal curvatures if and only if it is locally congruent to a type (A_1), (A_2), (B), (C), (D) or (E) hypersurfaces (see, [9, 21]). A Hopf hypersurface in $\mathbb{C}H^n(c)$ has constant principal curvatures if and only if it is locally congruent to a type (A_0), $(A_{1,0})$, $(A_{1,1})$, (A_2) or (B) hypersurfaces (see, [1]). All type (A_0), (A_1), $(A_{1,0})$, $(A_{1,1})$ and (A_2) hypersurfaces are referred to collectively as type (A).

Maeda and Udagawa in [16] first considered the Lie derivative of the structure tensor field ϕ and proved that the structure vector field ξ of a real hypersurface in $\mathbb{C}P^n$ is an infinitesimal automorphism of the structure tensor field ϕ if and only if the hypersurface is of type (A). Such a conclusion is still true even when the restriction was weakened to some other geometric conditions and this was first considered by Kwon and Suh in [12, Theorem] for a real hypersurface of dimension ≥ 5. Results in [16] have been generalized by Lim [13] by considering the coincidence of the Lie derivative and covariant derivative of the structure tensor field along ξ. Very recently, a new operator generated by the Lie derivative of the structure tensor field ϕ along the structure vector field ξ was extensively studied by Okumura in [18, 19] (see, also Cho [3, 4]). Nonexistence of the real hypersurfaces with a Killing type structure tensor field was proved by Cho in [5]. Some other results on the Lie derivative of the structure tensor field along ξ can also be found in [8, 11, 14, 15]. In 2013, Kaimakamis and Panagiotidou in [6, pp. 2091] proposed that it would be an interesting question for studying the Lie recurrency of the structure tensor field. In the present paper, we study the Lie derivative of the structure tensor field for real hypersurfaces in nonflat complex space forms $M^n(c)$, $c \neq 0$, and solved the question posed in [6].

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M^n(c)$ and N be a unit normal vector field of M. We denote by $\overline{\nabla}$ the Levi-Civita connection of the metric $\overline{g}$ of $M^n(c)$ and J the complex structure. Let g and ∇ be the induced metric from the ambient space and the Levi-Civita connection of the metric g, respectively. Then, the Gauss and Weingarten formulas are given respectively as the following:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \ \overline{\nabla}_X N = -AX, \tag{1}$$

for any $X, Y \in \mathfrak{X}(M)$, where A denotes the shape operator of M in $M^n(c)$. For any vector field $X \in \mathfrak{X}(M)$, we put

$$JX = \phi X + \eta(X)N, \ JN = -\xi. \tag{2}$$

We can define on M an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \tag{3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi), \tag{4}$$

for any $X, Y \in \mathfrak{X}(M)$. If the structure vector field ξ is *principal*, that is, $A\xi = \alpha\xi$ at each point, where $\alpha = \eta(A\xi)$, then M is called a Hopf hypersurface and α is called Hopf principal curvature.

Moreover, applying the parallelism of the complex structure (i.e., $\overline{\nabla}J = 0$) of $M^n(c)$ and using (1), (2) we have

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{5}$$

$$\nabla_X\xi = \phi AX, \tag{6}$$

for any $X, Y \in \mathfrak{X}(M)$.

3. Non-existence results

We denote by $\mathcal{L}$ the Lie derivative of a real hypersurface in a nonflat complex space form $M^n(c)$, $c \neq 0$, $n \geq 2$.

Definition 3.1. *The structure tensor field of a real hypersurface is called Lie Killing if*

$$(\mathcal{L}_X\phi)Y + (\mathcal{L}_Y\phi)X = 0, \tag{7}$$

for any vector fields X, Y.

Obviously, the above condition (7) is a generalization of the Lie parallelism of the structure tensor field, i.e., $\mathcal{L}_X\phi = 0$, for any $X \in \mathfrak{X}(M)$.

Theorem 3.1. *There exist no real hypersurfaces in nonflat complex space forms such that the structure tensor field is of Lie Killing type.*

Proof. By applying (5), we have

$$(\mathcal{L}_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi - \nabla_{\phi Y}X + \phi\nabla_Y X, \tag{8}$$

for any vector fields X, Y. Now suppose that the structure tensor field of a real hypersurface M is Lie Killing. From (7) and (8) we get

$$\eta(Y)AX - 2g(AX, Y)\xi - \nabla_{\phi Y}X + \phi\nabla_Y X + \eta(X)AY - \nabla_{\phi X}Y + \phi\nabla_X Y = 0, \tag{9}$$

for any vector fields X, Y. Taking the inner product of (9) with ξ, we obtain

$$\eta(Y)\eta(AX) - 2g(AX, Y) - \eta(\nabla_{\phi Y}X) + \eta(X)\eta(AY) - \eta(\nabla_{\phi X}Y) = 0, \tag{10}$$

for any vector fields X, Y. In (10), selecting $Y = \xi$ we obtain

$$A\xi = \eta(A\xi)\xi.$$

This means that M is a Hopf hypersurface. In (10), selecting $X, Y \in \ker\eta$, with the help of (3), (6) and $A\xi = \eta(A\xi)\xi := \alpha\xi$, we get

$$AX - \phi A\phi X = 0 \ (\Leftrightarrow A\phi X + \phi AX = 0), \tag{11}$$

for any $X \in \ker\eta$. On the other hand, recall that, for any Hopf hypersurfaces, we have (see, [17, Lemma 2.2]):

$$A\phi A - \frac{\alpha}{2}(A\phi + \phi A) - \frac{c}{4}\phi = 0. \tag{12}$$

Substituting $A\phi X + \phi AX = 0$ (for any $X \in \ker\eta$) into equality (12), then we obtain $A\phi AX = \frac{c}{4}\phi X$, for any $X \in \ker\eta$. Now, let X be a unit eigenvector field of A with eigenfunction λ orthogonal to ξ, then ϕX is also a unit eigenvector field of A with eigenfunction $c/(4\lambda)$. Notice that λ is nowhere vanishing. Otherwise we shall arrive at a contradiction (i.e., $c = 0$) according to $A\phi AX = \frac{c}{4}\phi X$, for any $X \in \ker\eta$. Therefore, with the aid of the second equality in (11), the inner product of $A\phi AX = \frac{c}{4}\phi X$ with ϕX gives

$$\frac{c}{4} = g(A\phi AX, \phi X) = g(\phi AX, A\phi X) = -|A\phi X|^2 = -\frac{c^2}{16\lambda^2}.$$

In view of the above equality, one sees that this situation occurs only for a real hypersurface in the complex hyperbolic space, and the two (distinct) principal curvatures λ and ν of the shape operator on the holomorphic distribution $\ker\eta$ are

$$\lambda = \frac{\sqrt{-c}}{2} \text{ and } \nu = -\frac{\sqrt{-c}}{2}. \tag{13}$$

Recall that the Hopf principal curvature for any Hopf hypersurface is a constant (see, [17, Theorem 2.1]). Thus, M is a Hopf hypersurface in $\mathbb{C}H^n(c)$ with constant principal curvatures. According to [1], M is locally congruent to a

- type (A_2) hypersurface whose two principal curvatures on holomorphic distribution $\ker\eta$ are $\frac{\sqrt{-c}}{2}\tanh(\frac{\sqrt{-c}}{2}r)$ and $\frac{\sqrt{-c}}{2}\coth(\frac{\sqrt{-c}}{2}r)$; or a
- type (B) hypersurface whose two principal curvatures on holomorphic distribution $\ker\eta$ are $\frac{\sqrt{-c}}{2}\tanh(\frac{\sqrt{-c}}{2}r)$ and $\frac{\sqrt{-c}}{2}\coth(\frac{\sqrt{-c}}{2}r)$.

Notice that the summation of the two principal curvatures of M on holomorphic distribution $\ker\eta$ in (13) vanishes, but by the above table this is impossible for type (A_2) or (B) hypersurfaces in $\mathbb{C}H^n(c)$. □

Corollary 3.1. *There are no real hypersurfaces in nonflat complex space forms with Lie parallel structure tensor field.*

Definition 3.2. *The structure tensor field of a real hypersurface is called Lie Codazzi if*

$$(\mathcal{L}_X\phi)Y = (\mathcal{L}_Y\phi)X, \tag{14}$$

for any vector fields X, Y.

Obviously, the above condition (14) is also a generalization of Lie parallelism of the structure tensor field.

Theorem 3.2. *There exist no real hypersurfaces in nonflat complex space forms such that the structure tensor field is of Lie Codazzi type.*

Proof. If the structure tensor field ϕ of a real hypersurface M in nonflat complex space forms is Lie Codazzi, from (8) and (14) we get

$$\eta(Y)AX - \nabla_{\phi Y}X + \phi\nabla_Y X = \eta(X)AY - \nabla_{\phi X}Y + \phi\nabla_X Y, \tag{15}$$

for any vector fields X, Y. Taking the inner product of (15) with ξ gives

$$\eta(Y)\eta(AX) - \eta(\nabla_{\phi Y}X) = \eta(X)\eta(AY) - \eta(\nabla_{\phi X}Y),$$

for any vector fields X, Y. In the above equality, replacing Y by ξ gives

$$A\xi = \eta(A\xi)\xi.$$

This means that M is a Hopf hypersurface. We may write $A\xi = \eta(A\xi)\xi := \alpha\xi$, and replacing Y by ξ in (15), we obtain

$$2AX + \phi\nabla_\xi X = 2\alpha\eta(X)\xi - \phi A\phi X, \tag{16}$$

for any vector field X. With the aid of $A\xi = \alpha\xi$, the operation of ϕ on (16) gives

$$2\phi AX - \nabla_\xi X + \eta(\nabla_\xi X)\xi = A\phi X,$$

for any vector field X. On the other hand, with the aid of $A\xi = \alpha\xi$, replacing X by ϕX in the above equality we have

$$2\phi A\phi X - \nabla_\xi \phi X = -AX + \alpha\eta(X)\xi,$$

for any vector field X. Thus, adding the above equality to (16), with the aid of (5), we get

$$AX + \phi A\phi X = \alpha\eta(X)\xi,$$

for any vector field X, where we have applied $\nabla_\xi\phi = 0$ which is obtained from (5) and $A\xi = \alpha\xi$. With the aid of $A\xi = \alpha\xi$, the operation of ϕ on the above equality gives

$$A\phi = \phi A. \tag{17}$$

In general, the above relation implies that M is a type (A) hypersurface. However, in our case, there are no real hypersurfaces satisfying the above relation. In fact, with the aid of (5), using (17) in (16) we get

$$AX = -\phi\nabla_\xi X + \alpha\eta(X)\xi, \tag{18}$$

for any vector field X. The operation of ϕ on (18) gives

$$\phi AX = \nabla_\xi X - \eta(\nabla_\xi X)\xi.$$

With the aid of (17) and $A\xi = \alpha\xi$, the operation of A on (18) gives

$$A^2X = -\nabla_\xi X + \eta(\nabla_\xi X)\xi + \alpha^2\eta(X)\xi,$$

for any vector field X. Eliminating $\nabla_\xi X$, according to the above relation and the previous one we get

$$A^2X + \phi AX = \alpha^2\eta(X)\xi,$$

for any vector field X. From the above equality, we conclude that all principal curvatures of the shape operator on $\ker\eta$ are zero. For any Hopf hypersurfaces, if $AU = \lambda U$ and $A\phi U = \nu\phi U$ for certain $U \in \ker\eta$, from [17, Corollary 2.3] we have

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \tag{19}$$

As all principal curvatures are zero on $\ker\eta$, applying this in (19) implies $c = 0$, a contradiction. □

Definition 3.3. *The structure tensor field of a real hypersurface is called Lie recurrent if*

$$(\mathcal{L}_X\phi)Y = \omega(X)\phi Y, \tag{20}$$

for any vector fields X, Y, and certain one-form ω.

Obviously, the above condition (20) is also a generalization of Lie parallelism of the structure tensor field.

Theorem 3.3. *There exist no real hypersurfaces in nonflat complex space forms such that the structure tensor field is Lie recurrent.*

Proof. If the structure tensor field ϕ of a real hypersurface M in nonflat complex space forms is Lie recurrent, from (8) and (20) we get

$$\eta(Y)AX - g(AX, Y)\xi - \nabla_{\phi Y}X + \phi\nabla_Y X = \omega(X)\phi Y, \tag{21}$$

for any vector fields X, Y. Taking the inner product of (21) with ξ gives

$$\eta(Y)\eta(AX) - g(AX, Y) - \eta(\nabla_{\phi Y}X) = 0, \tag{22}$$

for any vector fields X, Y. In (22), replacing X by ξ we see $A\xi = \eta(A\xi)\xi$, and hence M is Hopf. In (21), with the aid of $A\xi = \alpha\xi$, replacing X by ξ we obtain

$$-\phi A\phi Y - AY + \alpha\eta(Y)\xi = \omega(\xi)\phi Y, \tag{23}$$

for any vector field Y. With the aid of $A\xi = \alpha\xi$, the operation of ϕ on (23) gives

$$A\phi Y - \phi AY = \omega(\xi)\phi^2 Y.$$

In (23), with the aid of $A\xi = \alpha\xi$, replacing Y by ϕY gives

$$\phi AY - A\phi Y = \omega(\xi)\phi^2 Y.$$

Subtracting the last equality from the previous one gives $A\phi = \phi A$. Making use of this, with the aid of $A\xi = \alpha\xi$, selecting $X \in \ker\eta$ in (22), we obtain

$$AX = 0,$$

for any $X \in \ker\eta$. As seen in proof of Theorem 3.2, this is impossible because of (19). □

Remark 3.1. Corollary 3.1 is also a direct corollary of Theorems 3.2 and 3.3.

Remark 3.2. It has been proved in [6, Main Theorem] that there exist no real hypersurfaces in $M^n(c)$, $c \neq 0$, $n \geq 2$, whose structure Jacobi operator l is of Lie recurrent type, i.e., $\mathcal{L}_X l = \omega(X)l$, for any vector field X and certain one-form ω. This conclusion is still valid when the structure Jacobi operator l is replaced by the shape operator (see, [2, Theorem 8.116]) or the structure tensor field ϕ (see, Theorem 3.3).

We remark that it was proposed in [6] that how about if we weaken condition (20) to Lie $\mathcal{D}$-recurrent? Before closing this paper, we also answer this question and obtain again a nonexistence theorem. Next we denote by $\mathcal{D}$ the holomorphic distribution $\ker\eta$.

Definition 3.4. *The structure tensor field of a real hypersurface is called Lie $\mathcal{D}$-recurrent if*

$$(\mathcal{L}_X\phi)Y = \omega(X)\phi Y, \tag{24}$$

for any vector field Y and $X \in \mathcal{D}$, and certain one-form ω.

Obviously, condition (24) is much weaker than Lie parallelism (i.e., $\mathcal{L}_X\phi = 0$). Next, we extend Theorem 3.3 to the following form.

Theorem 3.4. *There exist no real hypersurfaces in nonflat complex space forms such that the structure tensor field is Lie $\mathcal{D}$-recurrent.*

Proof. By Definition 3.4, equalities (21) and (22) are still valid, for any vector field Y and $X \in \mathcal{D}$. Considering $X \in \mathcal{D}$ and $Y = \xi$ in (21), we get

$$AX - \eta(AX)\xi + \phi\nabla_\xi X = 0. \tag{25}$$

Replacing X by ϕX in (25) gives

$$A\phi X - \eta(A\phi X)\xi + \phi\nabla_\xi \phi X = 0,$$

which is operated by ϕ yielding

$$\phi A\phi X - \nabla_\xi \phi X + \eta(\nabla_\xi \phi X)\xi = 0.$$

Notice that from (6) and (4) we have $\eta(\nabla_\xi \phi X) + \eta(AX) = 0$, for any $X \in \mathcal{D}$, which is substituted into the above equality giving

$$\phi A\phi X - \nabla_\xi \phi X - \eta(AX)\xi = 0,$$

for any $X \in \mathcal{D}$. Adding this to (25) gives

$$(\nabla_\xi \phi)X = \phi A\phi X + AX - 2\eta(AX)\xi, \tag{26}$$

for any $X \in \mathcal{D}$. Comparing (26) with (5) we obtain

$$\phi A\phi X + AX - \eta(AX)\xi = 0, \tag{27}$$

for any $X \in \mathcal{D}$. On the other hand, considering $X \in \mathcal{D}$ in (22), with the aid of (6), we get

$$\phi A\phi X - AX + \eta(AX)\xi = 0.$$

Consequently, eliminating $\phi A\phi X$, from the above equality and (27) we obtain $AX = \eta(AX)\xi$, for any $X \in \mathcal{D}$. This implies that $g(AX, Y) = 0$, for any vector fields $X, Y \in \mathcal{D}$, and now the hypersurface is a ruled one (see, [2, 10, 17]). On a ruled hypersurface, there exists a unit vector field $U \in \mathcal{D}$ such that

$$A\xi = \alpha\xi + \beta U,\ AU = \beta\xi,\ AX = 0, \tag{28}$$

for any $X \in \{\xi, U\}^\perp$, where β is a non-vanishing function. Moreover, according to [7, pp. 404] (see, also, [10, 20]) we have

$$\nabla_X U = \begin{cases} \frac{1}{\beta}(\beta^2 - \frac{c}{4})\phi X, & X = U, \\ 0, & X = \phi U, \end{cases} \tag{29}$$

and

$$\mathrm{d}\beta(X) = \begin{cases} 0, & X = U, \\ \beta^2 + \frac{c}{4}, & X = \phi U. \end{cases} \tag{30}$$

In (21), considering $X = Y = U$, with the aid of (28), we obtain from (29) that

$$\beta^2 - \frac{c}{4} = 0 \text{ and } \omega(U) = 0.$$

The first equality implies that β is a constant, and hence according to (30) we obtain $\beta^2 + \frac{c}{4} = 0$, which is compared with the above equality implying $c = 0$, a contradiction. □

Remark 3.3. By Theorem 3.4, the structure tensor field of a real hypersurface in nonflat complex space forms cannot be Lie $\mathcal{D}$-parallel, but it can be Lie Reeb-parallel (i.e., $\mathcal{L}_\xi \phi = 0$). In fact, it has been proved in [13, Theorem A] that the structure tensor field of a real hypersurface is Lie Reeb-parallel if and only if the hypersurface is of type (A) (see, also, [12, 16]).

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