Sensitivity analysis of interest rate derivatives in a normal inverse Gaussian Lévy market

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Abstract. Abrupt happenings in financial markets contribute to jumps of different magnitudes that invariably affect interest rate derivatives. Many of the existing interest rate models do not capture jumps, leading to inaccurate prediction of option prices and sensitivity analysis in the markets. To incorporate jumps in interest rate derivatives, we extend the Vasicek model with a Brownian motion as an underlying process to a model driven by a normal inverse Gaussian process, which is a subordinated Lévy process, use the extended model to obtain an expression for the price of an interest rate derivative called a zero-coupon bond. We employ Malliavin calculus to compute the greeks *delta* and *vega* of the derived price, which are important risk quantifiers in the interest rate derivative markets driven by a normal inverse Gaussian process.
Keywords: interest rate derivatives, Lévy process, Malliavin calculus, normal inverse Gaussian process, Vasicek model.

1. Introduction

Investing in an interest rate derivative market requires a good understanding of how to minimize risks. This may be achieved by formulating a model which incorporates sudden or rare occurrences that may lead to jumps in a market. Such occurrences often arise from changes in monetary policy, inflation, natural disaster, abrupt information, economic recession, presence of a pandemic, etc.

In the literature, many models of interest rate derivatives do not consider jumps and heavy tails. The present paper bridges this gap by adopting a subordinated Lévy process called a *normal inverse Gaussian (NIG) process* to derive an extended Vasicek interest rate model and use the extended model to derive an expression for the price of an interest rate derivative called a *zero-coupon*

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bond and compute its sensitivity to some of its parameters using *Malliavin calculus*. These will assist an investor and risk manager to make the right decision and minimize risks in an NIG interest rate derivative market.

The NIG process was introduced by Barndorf-Nielsen [2] to generate good models for log-return process of prices and exchange rates [7]. Using the NIG process allows jumps and heavy tails to be captured. Examples of NIG markets include (i) volatile markets such as an electricity market, whose forward price has a return distribution with excess kurtosis and heavy tails [1]; and (ii) stock market prices [19]. Núñez [15] introduced the process as a replacement of the Gaussian assumption of underlying asset returns since it takes care of the heavy tails found in returns data series. Dhull and Kumar [9] emphasized the usefulness of the process in modelling various real-life time-series data. Lahcene [13] discussed an extension of the process in modelling and analyzing statistical data with emphasis on extensive sets of observations and some applications. Pintoux and Privault [18] discussed an interest rate derivative *zero-coupon bond price* using the Dothan model driven by a Wiener process while Yin et al. [22] emphasized that non-Gaussian Ornstein-Uhlenbeck process based on a negative/positive subordinated Lévy process fits and provides a better economic interpretation of the associated time series. Sabino [20] considered how to price energy derivatives for spot prices driven by a tempered stable Ornstein-Uhlenbeck process, while Hainaut [12] discussed an interest rate model driven by a mean reverting Lévy process with a sub-exponential memory of sample path achieved by considering an Ornstein–Uhlenbeck process in which the exponential decaying kernel is replaced by a Mittag–Leffler function. We adopt the Vasicek model since it has the property of mean-reversion and possibility of a negative interest rate. Research has shown that a good model should take care of negative interest rates that now occur in the current market environment as observed by Orlando et al. [16].

Bavouzet-Morel and Messaoud [3] discussed the Malliavin calculus for jump processes while Petrou [17] extended the theory of the calculus adding some tools for the computation of sensitivities. Bayazit and Nolder [4] applied the calculus to the sensitivities of an option whose underlying is driven by an exponential Lévy process. This work extends Bayazit and Nolder [4] to the sensitivity analysis of interest rate derivatives in a normal inverse Gaussian Lévy market.

In the next section, we discuss important mathematical tools to be employed in our results. In Section 3, we derive an extended Vasicek model driven by the NIG process and derive an equivalent expression for the zero-coupon bond price. In Section 4, we compute the greeks of the derived price using the Malliavin calculus, and discuss sensitivity analysis of the interest rate derivatives. In a previous publication [21], we derived expressions for certain greeks in a model involving the variance gamma process.

2. Foundational notion

In this section, we discuss important mathematical tools employed for the success of the paper.

2.1 The normal inverse Gaussian process

The inverse Gaussian process is a random process with infinite number of jumps for each finite period. The NIG process is a subordinated Lévy process.

Remark 2.1. 1. Let X be a random variable with an NIG distribution denoted $X \sim \text{NIG}(x;\alpha,\beta,\mu,\delta)$, then its probability density function is given by

$$f_{NIG}(x) = \frac{\alpha\delta\exp(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu))}{\pi\cdot\sqrt{\delta^2+(x-\mu)^2}}K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})$$

where $\alpha > 0$, $|\beta| < \alpha$, $\delta > 0$, and $K_1(x)$ is the modified Bessel function of the third kind with index λ given by

$$K_\lambda(x) = \frac{1}{2}\int_0^\infty t^{\lambda-1}\exp\Big(-\frac{1}{2}x\big(t+\frac{1}{t}\big)\Big)dt,\ x > 0.$$

2. The parameters α, β, δ and μ are for tail heaviness, symmetry, scale and location, respectively.

3. The characteristic function of the NIG process is given by

$$\phi_t(u) = \exp\big(-\delta t((\alpha^2-(\beta+iu)^2)^{\frac{1}{2}}-(\alpha^2-\beta^2)^{\frac{1}{2}})\big).$$

4. In what follows, we discuss the Malliavin calculus to be employed in the computation of greeks.

2.2 The Malliavin calculus for Lévy processes

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $X_i, i = 1,\dots,n$ be a sequence of random variables with piecewise differentiable probability density functions. Let $C^p(\mathbb{R}^n)$ where $p, n \geq 1$, be the space of p times continuously differentiable functions. The following basic definitions will be utilized in the sequel.

Definition 2.1. *Let $L^0(\Omega,\mathbb{R})$ be the linear space of all $\mathbb{R}$-valued random variables on $(\Omega,\mathcal{B},\mathbb{P})$. A map $F : (L^0(\Omega,\mathbb{R}))^n \to L^0(\Omega,\mathbb{R})$, $n \in \mathbb{N}$ is defined as* (n,p)-simple functional *of the n random variables if there exists an $\mathbb{R}$-valued function $\widehat{F} \in C^p(\mathbb{R}^n)$ where*

$$F(X_1,\dots,X_n)(\omega) = \widehat{F}(X_1(\omega),\dots,X_n(\omega)),\ \omega\in\Omega,\ X_1,\dots,X_n \in L^0(\Omega,\mathbb{R}).$$

An (n,p)-simple process *of length n is a sequence of random variables $U = (U_i)_{i\leq n}$ such that $U_i(\omega) = u_i(X_1(\omega),\dots,X_n(\omega))$ where $u_i \in C^p(\mathbb{R}^n)$, $X_1,\dots,X_n \in L^0(\Omega,\mathbb{R})$ and $\omega\in\Omega$.*

We write $S_{(n,p)}$ for the space of all (n,p)-simple functionals and $P_{(n,p)}$ for the space of all (n,p)-simple processes.

Definition 2.2. *Let* $F \in S_{(n,1)}$*, where* $F(X_1,\dots,X_n)(\omega)=\widehat{F}(X_1(\omega),\dots,X_n(\omega))$*,* $\omega\in\Omega$*,* $\widehat{F} \in C^1(\mathbb{R}^n)$*, and* $X_1,\dots,X_n \in L^0(\Omega,\mathbb{R})$*. Define the operator* $D : S_{(n,1)} \to (P_{(n,0)})^n$ *called the* Malliavin derivative operator *by* $DF = (D_iF)_{i\le n}$ *where*

$$D_iF(X_1,\dots,X_n)(\omega) = \left(\frac{\partial\widehat{F}}{\partial x_i}\right)(X_1(\omega),\dots,X_n(\omega)),$$

$$D_iF(X)(\omega) = \left(\frac{\partial\widehat{F}}{\partial x}\right)(X(\omega)), \quad \text{when } n=1. \tag{1}$$

Definition 2.3. *Let* $F=(F_1,\dots,F_d)$ *be a d-dimensional vector of simple functionals where* $F_i\in S_{(n,1)}$*. The matrix* $\mathcal{M} = (\mathcal{M}(F)_{i,j})$ *defined by*

$$\mathcal{M}(F)_{i,j} = \langle DF_i, DF_j\rangle_n = \sum_{m=1}^{n} D_mF_iD_mF_j$$

is called the Malliavin covariance matrix *of F [4]. This implies that if* $n=1$*,*

$$\mathcal{M}(F)_{i,j} = \langle DF_i, DF_j\rangle = DF_iDF_j. \tag{2}$$

Definition 2.4. *Define the operator* $\widetilde{\delta} : P_{(n,1)} \to S_{(n,0)}$ *called the* Skorohod integral operator *for a simple process* $U = (U_i)_{i=1,\dots,n} \in P_{(n,1)}$*,* $U_i(\omega) = u_i(X_1(\omega), \dots, X_n(\omega))$*,* $\omega \in \Omega$ *by*

$$\begin{aligned}\widetilde{\delta}(U)(X_1,\dots,X_n) &= \sum_{i=1}^{n}\widetilde{\delta}_i(U)(X_1,\dots,X_n)\\ &= -\sum_{i=1}^{n}[D_iu_i(X_1,\dots,X_n) + u_i(X_1,\dots,X_n)\varphi_i(\mathbf{x})],\end{aligned}$$

where $\varphi_i(\mathbf{x}) = \frac{\partial \ln f_X(\mathbf{x})}{\partial x_i} = \frac{f'_{X_i}(\mathbf{x})}{f_X(\mathbf{x})}, f_X(\mathbf{x}) \neq 0,\ 1 \le i \le n,\ \mathbf{x} = x_1,\dots,x_n$ *and* $f_X(x)$ *is the density function of the random variable X.*

Definition 2.5. *The* Ornstein-Uhlenbeck (O-U) operator $L : S_{(n,2)} \to S_{(n,0)}$ *is defined as*

$$(LF)(X_1,\dots,X_n) = -\sum_{i=1}^{n}[(\partial^2_{ii}\widehat{F})(X_1,\dots,X_n) + \varphi_i(\mathbf{x})(\partial_i\widehat{F})(X_1,\dots,X_n)],$$

where $F \in S_{(n,2)}$*,* $X_1,\dots,X_n \in L^0(\Omega,\mathbb{R})$ *and* $\varphi_i(\mathbf{x})$ *is given by Definition* 2.4*. For* $n=1$*,*

$$LF(X) = -[DD\widehat{F}(X) + \varphi(x)D\widehat{F}(X)] \tag{3}$$

where

$$\varphi(x) = \frac{\partial \ln f_X(x)}{\partial x} = \frac{f'_X(x)}{f_X(x)}, \text{ and } f_X(x) \neq 0. \tag{4}$$

2.2.1 Malliavin integration by parts theorem

To compute the greeks of the interest rate derivative, we need the integration by parts theorem of the Malliavin calculus stated below.

Proposition 2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space; $X_1, \dots, X_n$, a sequence of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and $P = (P_1, \dots, P_d) \in (S_{(n,2)})^d$, $Q \in S_{(n,1)}$. Let $\mathcal{M} = (\mathcal{M}_{ij}(P))_{1\le i\le n, 1\le j\le n}$ be an invertible Malliavin covariance matrix with inverse given by $\mathcal{M}(P)^{-1} = (\mathcal{M}(P)_{ij})^{-1}_{1\le i\le n,1\le j\le n}$. Suppose that $\mathbb{E}[\det \mathcal{M}(P)^{-1}]^p < \infty$, $p \ge 1$, and $\Phi : \mathbb{R}^d \to \mathbb{R}$ represents a smooth bounded function with bounded derivative. Then,*

$$\mathbb{E}[\partial_i \Phi(P)Q] = \mathbb{E}[\Phi(P)H_i(P,Q)] \text{ where } \mathbb{E}[H_i(P,Q)] < \infty, \; i = 1, 2, \dots, n;$$

and the Malliavin weight is given by

$$H_i(P,Q) = \sum_{j=1}^{n} Q\mathcal{M}(P)^{-1}_{ij} LP_j - \mathcal{M}(P)^{-1}_{ij}\langle DP_j, DQ\rangle - Q\langle DP_j, D\mathcal{M}(P)^{-1}_{ij}\rangle.$$

Remark 2.2. For $d = n = 1$, the Malliavin weight is given by

$$H(P,Q) = Q\mathcal{M}(P)^{-1}LP - \mathcal{M}(P)^{-1}\langle DP, DQ\rangle - Q\langle DP, D\mathcal{M}(P)^{-1}\rangle.$$

We proceed to the next section and derive our results.

3. The Short rate model under the NIG process

In this section, we extend the Vasicek short rate model to a market driven by the NIG process and derive an expression for the price of an interest rate derivative called a *zero-coupon bond.*

The Vasicek (1977) interest rate model satisfies the stochastic differential equation given by

$$dr_t = a(b - r_t)dt + \sigma dX_t \tag{5}$$

where $X_t = X(t)$, b, a and σ denote the Lévy process, long-term mean rate, speed of mean reversion and volatility of the interest rate, respectively.
Integrating equation (5) by using Itô's formula, we obtain

$$r_t = r_0 e^{-at} + b(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)} dX_s. \tag{6}$$

We adopt the NIG model given by $X_t = \mathbf{w}t + \beta\delta^2 IG_t + \delta W(IG_t)$ [11] where $\mathbf{w}$ is the cumulant generating function given by

$$\mathbf{w} = -\frac{1}{t}\ln(\phi_t(-i)) = \delta(\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2}).$$

The parameters α, β and δ control the behaviour of the tail, skewness and scale of the distribution, respectively. $IG_t = IG(t)$ denotes the inverse Gaussian process. We represent the standard Brownian motion $W(t)$ as the process $W(t)-W(s) = \sqrt{|\, t-s\,|}Z$, $t, s \geq 0$, where Z is a $N(0,1)$ Gaussian random variable. Then, $W(t) = \sqrt{t}Z$ and $\mathbb{E}(W(t)W(s)) = \min(t,s)$, $t, s \geq 0$. Thus,

$$\begin{aligned} X_t &= \mathbf{w}t + \delta\sqrt{IG(t)}Z + \beta\delta^2 IG(t), \\ &\Longrightarrow dX_t = \mathbf{w}dt + \delta\Delta\sqrt{IG(t)}Z + \beta\delta^2\Delta IG(t). \end{aligned} \tag{7}$$

Substituting equation (7) into (6) and evaluating, we have

$$\begin{aligned} r_t = r_0 e^{-at} + b(1-e^{-at}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-at}) + \sigma\delta\Big(\sum_{0\leq s\leq t}(\Delta\sqrt{IG(s)}Z \\ + \beta\delta\Delta IG(s))e^{-a(t-s)}\Big). \end{aligned} \tag{8}$$

We adopt the above expression (8) to derive an expression for the zero-coupon bond price driven by the NIG process.

3.1 Expression for a zero-coupon bond price with a Vasicek short rate model under the NIG process

The dynamics of the zero-coupon bond price under a risk neutral measure is given by

$$dP = r_t P dt + \sigma P dX_t. \tag{9}$$

Applying Itô's lemma to equation (9), we obtain

$$\begin{aligned} d\ln P = r_t dt + \sigma\mathbf{w}dt + \sigma(\delta\Delta\sqrt{IG(t)}Z + \beta\delta^2\Delta IG(t)) - \frac{1}{2}\sigma^2(\delta\Delta\sqrt{IG(t)}Z \\ + \beta\delta^2\Delta IG(t))^2. \end{aligned} \tag{10}$$

Integrating equation (10), we get

$$\begin{aligned} \ln P(t,T) = -\Bigg(\int_t^T r_u du + \sigma\mathbf{w}\int_t^T du \\ + \sigma\Bigg(\sum_{0\leq u\leq T}(\delta\Delta\sqrt{IG(u)}Z + \beta\delta^2\Delta IG(u)) \\ - \sum_{0\leq u\leq t}(\delta\Delta\sqrt{IG(u)}Z + \beta\delta^2\Delta IG(u))\Bigg) - \frac{1}{2}\sigma^2\Bigg(\sum_{0\leq u\leq T}(\delta\Delta\sqrt{IG(u)}Z \\ + \beta\delta^2\Delta IG(u))^2 - \sum_{0\leq u\leq t}(\delta\Delta\sqrt{IG(u)}Z + \beta\delta^2\Delta IG(u))^2\Bigg)\Bigg). \end{aligned} \tag{11}$$

By equation (8), it follows that

$$
\begin{aligned}
\int_t^T r_u du &= \frac{-r_0}{a}(e^{-aT}-e^{-at}) + b\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big) \\
&+ \frac{\sigma\mathbf{w}}{a}\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big) \\
&+ \sigma\delta\big(\sum_{0\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z+\beta\delta\Delta IG(s))e^{-a(u-s)}\big) \\
&- \sigma\delta\big(\sum_{0\le u\le t}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z+\beta\delta\Delta IG(s))e^{-a(u-s)}\big).
\end{aligned}
\tag{12}
$$

Substituting equation (12) into (11) and evaluating, we obtain the zero-coupon bond price driven by the NIG process as

$$
\begin{aligned}
P(t,T) &= \exp\Bigg(-\Bigg[\frac{-r_0}{a}(e^{-aT}-e^{-at}) + b\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big) \\
&+ \frac{\sigma\mathbf{w}}{a}\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big) + \sigma\delta\big(\sum_{0\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z \\
&+ \beta\delta\Delta IG(s))e^{-a(u-s)}\big) - \sigma\delta\big(\sum_{0\le u\le t}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z+\beta\delta\Delta IG(s)) \\
&\cdot e^{-a(u-s)}\big) + \sigma\mathbf{w}[T-t] + \sigma\delta\Bigg(\sum_{0\le u\le T}(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u)) \\
&- \sum_{0\le u\le t}(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u))\Bigg) - \frac{1}{2}\sigma^2\delta^2\Bigg(\sum_{0\le u\le T}(\Delta\sqrt{IG(u)}Z \\
&+ \beta\delta\Delta IG(u))^2 - \sum_{0\le u\le t}(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u))^2\Bigg)\Bigg]\Bigg).
\end{aligned}
\tag{13}
$$

Besides being a function of t and T, the expression on the right hand side of equation (13) also depends on $r_0, \beta, \delta, \sigma, \mathbf{w}$ and Z. Thus, in the sequel, we shall regard P as a function of $t, T, r_0, \beta, \delta, \sigma, \mathbf{w}$ and Z.

The price of the zero-coupon bond driven by the NIG Lévy process given by equation (13) can be written as

$$
\begin{aligned}
P(t,T) = \exp\Bigg(-\Bigg(&\frac{-r_0}{a}(e^{-aT}-e^{-at}) + b\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big) \\
&+ \frac{\sigma\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big] + \mathbf{w}\sigma[T-t] \\
&+ \sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z+\beta\delta\Delta IG(s)e^{-a(u-s)}\big) \\
&+ \sigma\delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u)\big) \\
&- \frac{\sigma^2\delta^2}{2}\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)^2\Bigg)\Bigg)\Bigg).
\end{aligned}
\tag{14}
$$

We state the necessary lemmas for the computation of the delta which measures the sensitivity of a bond option price driven by the NIG process to changes in the initial interest rate and vega which measures the sensitivity of the bond option price with respect to changes in the volatility of the short rate model.

Lemma 3.1. *Let P be the price of a zero-coupon bond driven by the NIG process. Then, the Malliavin derivative on P is given by*

$$DP = -\Big[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) - \sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P. \tag{15}$$

Proof. By equation (1) of Definition 2.2 and the zero-coupon price given by equation (14), we get the Malliavin derivative

$$DP = -\Big[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) - \frac{\sigma^2\delta^2}{2}\Big(2\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P.$$

Hence, the result follows. □

Lemma 3.2. *Let P be the price of the zero-coupon bond driven by the NIG process. Then, the Ornstein Uhlenbeck operator L on P is given by*

$$\begin{aligned} LP = -\Big[&\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big) + \Big(\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) \\ &+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) - \sigma^2\delta^2\big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\big)\Big)^2 \\ &-\varphi(z)\Big(\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\ &-\sigma^2\delta^2\big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\big)\Big)\Big]P,\ \varphi(z) = -z. \end{aligned} \tag{16}$$

Proof. By equation (15) of Lemma 3.1, it follows that

$$\begin{aligned} DDP = {}&\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)P + \Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) \\ &+ \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) - \sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\delta\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^2 P. \end{aligned}$$

By equations (3) and (4) of Definition 2.5, we obtain

$$LP = -[DDP + \varphi(z)DP]$$

where

$$\varphi(z) = \frac{\partial \ln f_{\mathcal{N}}(z)}{\partial z} = \frac{\partial \ln\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\right)}{\partial z} = -z.$$

Substituting DDP and equation (15) of Lemma 3.1 into LP yields the desired result. □

Lemma 3.3. *Let P be the price of the zero-coupon bond driven by the NIG process and $\mathcal{M}(P)$, its Malliavin covariance matrix. Then,*

$$\begin{aligned} M(P)^{-1} = \Bigg[&\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t} \left(\Delta\sqrt{IG(s)}e^{-a(u-s)}\right) + \sigma\delta \sum_{t\le u\le T} (\Delta\sqrt{IG(u)}) \\ &- \sigma^2\delta^2\Bigg(\sum_{t\le u\le T} (\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]^{-2} P^{-2}. \end{aligned} \tag{17}$$

Proof. By equation (2) of Definition 2.3, $\mathcal{M}(P) = \langle DP, DP\rangle$. Thus, by equation (15), it follows that

$$\begin{aligned} M(P) = \Bigg[&\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t} \left(\Delta\sqrt{IG(s)}e^{-a(u-s)}\right) + \sigma\delta \sum_{t\le u\le T} (\Delta\sqrt{IG(u)}) \\ &- \sigma^2\delta^2\Bigg(\sum_{t\le u\le T} (\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]^{2} P^{2}. \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{M}(P)^{-1} = \Bigg(\Bigg[&\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t} \left(\Delta\sqrt{IG(s)}e^{-a(u-s)}\right) + \sigma\delta \sum_{t\le u\le T} (\Delta\sqrt{IG(u)}) \\ &- \sigma^2\delta^2\Bigg(\sum_{t\le u\le T} (\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg] P\Bigg)^{-2} \end{aligned}$$

which gives equation (17). □

Lemma 3.4. *Let P be the price of the zero-coupon bond driven by the NIG process. Then, the Malliavin derivative of the inverse Malliavin covariance matrix of P is given by*

$$\begin{aligned} D\mathcal{M}(P)^{-1} = 2\Bigg[&\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t} \left(\Delta\sqrt{IG(s)}e^{-a(u-s)}\right) + \sigma\delta \sum_{t\le u\le T} (\Delta\sqrt{IG(u)}) \\ &- \sigma^2\delta^2\Big(\sum_{t\le u\le T} (\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Bigg]^{-3} P^{-2} \end{aligned} \tag{18}$$

$$\Big[\Big(\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$
$$-\sigma^2\delta^2\big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\big)\Big)^2$$
$$+\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\Big].$$

Proof. Applying Malliavin derivative to equation (17) gives

$$D\mathcal{M}(P)^{-1} = 2\Big[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$
$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^{-3}P^{-2}$$
$$\cdot\Big[\Big[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$
$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^2$$
$$+\sigma^2\delta^2\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big]$$

which yields the desired result. □

4. The greeks of the zero-coupon bond price driven by the NIG Lévy process

The greeks serve as risk quantifiers. They give insight on various dimensions of insecurity involved in grabbing a bond option's position. Investors and risk managers use the greeks to predict future price and hedge risks. Some of the greeks are delta, vega, gamma and Theta. We shall concentrate on the delta and vega.

Remark 4.1. The price of a call option, with P as the underlying is given by

$$\mathbb{V} = e^{-r_0T}\mathbb{E}[\Phi(P)]$$

where $\Phi(P) = \max(P-K, 0)$ is the payoff with strike price K.

A greek is computed using the formula

$$\frac{\partial\mathbb{V}}{\partial\varsigma} = \frac{\partial\big(e^{-r_0T}\mathbb{E}[\Phi(P)]\big)}{\partial\varsigma}$$

where ς represents a parameter of the bond price whose effect is to be determined.

4.1 Computation of *delta* for NIG-driven interest rate derivatives

The greek *delta* measures the sensitivity of the zero-coupon bond option price to changes in its initial interest rate. It helps investors and portfolio managers by indicating the extent to which the bond option's price will move when the initial interest rate increases by a unit currency. This is very important because movements in the underlying, that is, the initial interest rate can change the worth of their investment [8].

Let P be the zero-coupon bond price given by equation (14), $\Phi(P)$ be the payoff function and $Q = \frac{\partial P}{\partial r_0}$. Then, by Proposition 2.1,

$$\triangle_{NIG} = \frac{\partial}{\partial r_0}[e^{-r_0 T}\mathbb{E}(\Phi(P))] = -Te^{-r_0 T}\mathbb{E}(\Phi(P)) + e^{-r_0 T}\mathbb{E}[\Phi(P)H(P,Q)].$$

Next, we establish Lemmas 4.1 - 4.4 using Lemmas 3.1-3.4, to obtain the Malliavin weight $H(P,Q)$.

Lemma 4.1. *Let P be the zero-coupon bond price driven by the NIG process and $Q = \frac{\partial P}{\partial r_0}$. Then the following hold:*

$$Q = \frac{1}{a}(e^{-aT} - e^{-at})P \tag{19}$$

and

$$\begin{aligned} DQ = &-\frac{1}{a}(e^{-aT} - e^{-at})\Big(\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) \\ &+ \sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\ &- \sigma^2\delta^2\big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\big)\Big)P. \end{aligned} \tag{20}$$

Proof. Applying partial derivative to equation (14) yields equation (19). Moreover, the Malliavin derivative

$$DQ = \frac{1}{a}(e^{-aT} - e^{-at})DP.$$

Substituting DP from equation (15) into the above equation yields equation (20). □

Lemma 4.2. *Let P be the zero-coupon bond price driven by the NIG process and L, the Ornstein-Uhlenbeck operator. Then,*

$$\begin{aligned} &Q\mathcal{M}(P)^{-1}LP \\ &= -\frac{1}{a}(e^{-aT} - e^{-at})\Big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\widehat{K}^{-2} + 1 - \varphi(z)\widehat{K}^{-1}\Big], \end{aligned} \tag{21}$$

where $\varphi(z) = -z$ *and* $\widehat{K}$ *is given by*

$$\begin{aligned}\widehat{K} &= \sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\ &\quad - \sigma^2\delta^2\big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\big).\end{aligned} \tag{22}$$

Proof. The result follows from Lemmas 3.2, 3.3 and 4.1 by substituting Q from equation (19) of Lemma 4.1, $\mathcal{M}(P)^{-1}$ from equation (17) of Lemma 3.3, and LP from equation (16) of Lemma 3.2 into $Q\mathcal{M}(P)^{-1}LP$. □

Lemma 4.3. *Let* P *be the zero-coupon bond price driven by the NIG process. Then,*

$$\mathcal{M}(P)^{-1}\langle DP, DQ\rangle = \frac{1}{a}(e^{-aT} - e^{-at}). \tag{23}$$

$$\begin{aligned}&Q\langle DP, D\mathcal{M}(P)^{-1}\rangle \\ &= -2\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)\Big[\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widehat{K}^2} + 1\Big],\end{aligned} \tag{24}$$

where $\widehat{K}$ *is given by equation* (22).

Proof. The result in equation (23) follows from Lemmas 3.1, 3.3 and 4.1 by substituting $\mathcal{M}(P)^{-1}$ from equation (17) of Lemma 3.3, DP from equation (15) of Lemma 3.1 and DQ from equation (20) of Lemma 4.1 into $\mathcal{M}(P)^{-1}\langle DP, DQ\rangle$; while the result in equation (24) follows from Lemmas 3.1, 3.4 and 4.1 by substituting Q from equation (19) of Lemma 4.1, DP from equation (15) of Lemma 3.1 and $D\mathcal{M}(P)^{-1}$ from equation (18) of Lemma 3.4 into $Q\langle DP, D\mathcal{M}(P)^{-1}\rangle$. □

Lemma 4.4. *Let* P *be the zero-coupon bond price driven by the NIG process and its payoff function be given by* $\Phi(P) = \max(P(t,T) - K, 0)$. *Then,*

$$\begin{aligned}\mathbb{E}[\Phi(P)] &= \int_K^\infty\int_K^\infty (p(t,T,y,z) - K)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2} \\ &\quad\cdot\Bigg(\frac{t(y(u))^{-3/2}\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}} \\ &\quad\cdot\exp\big(-\frac{1}{2}\big(\frac{t^2}{y(u)} + (\delta\sqrt{\alpha^2-\beta^2})^2 y(u)\big)\big)\cdot \mathbf{1}_{y>0}\Bigg)dydz\end{aligned}$$

where K *is the strike price and from equation* (14), $p(t,T) = p(t,T,y,z)$ *is given by*

$$\begin{aligned}p(t,T) = \exp\Bigg(&-\Big[\frac{-r_0}{a}(e^{-aT} - e^{-at}) + b\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big) \\ &+ \frac{\sigma\mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big]\end{aligned} \tag{25}$$

$$+ \mathbf{w}\sigma[T-t] + \sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\sqrt{y(s)}e^{-a(u-s)}z + \beta\delta y(s)e^{-a(u-s)}\big)$$
$$+ \sigma\delta \sum_{t\le u\le T}\big(\sqrt{y(u)}z+\beta\delta y(u)\big) - \frac{\sigma^2\delta^2}{2}\sum_{t\le u\le T}(\beta\delta y(u)+\sqrt{y(u)}z)^2\Big]\Big).$$

Proof. Let $f_{\mathcal{N}(z;0,1)}$ and $f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2})$ be the probability density functions for a Gaussian random variable and an inverse Gaussian random variable, respectively. Then,

$$\begin{aligned}\mathbb{E}[\Phi(P)] &= \int_{\mathbb{R}}\int_{\mathbb{R}} \Phi(p)\cdot f_{\mathcal{N}(z;0,1)} f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2})\big)dzdy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \max(p(t,T)-K,0)\cdot f_{\mathcal{N}(z;0,1)} f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2})\big)dzdy.\end{aligned}$$

where K is a constant, $f_{\mathcal{N}(z;0,1)} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ and

$$\begin{aligned}&f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2}) \\ &= \frac{ty^{-3/2}\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\exp\bigg(-\frac{1}{2}\big(\frac{t^2}{y}+(\delta\sqrt{\alpha^2-\beta^2})^2 y\big)\bigg)\cdot \mathbf{1}_{y>0}.\end{aligned}$$

Substituting the expression for $f_{\mathcal{N}(z;0,1)}$ and $f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2})$ into $\mathbb{E}[\Phi(P)]$ gives the desired result. □

Lemma 4.5. *Let P be the zero-coupon bond price driven by the NIG process and let $\mathbb{E}[\Phi(P)H(P,Q)] = \mathbb{E}[\Phi(P)H(P,\frac{\partial P}{\partial r_0})]$. Then,*

$$\begin{aligned}&\mathbb{E}\bigg[\Phi(P)H\bigg(P,\frac{\partial P}{\partial r_0}\bigg)\bigg] \\ &= \int_K^\infty\int_K^\infty (p(t,T,y,z)-K)H\bigg(p,\frac{\partial p}{\partial r_0}\bigg)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\cdot t(y(u))^{-3/2} \\ &\cdot\bigg(\frac{\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\exp\big(-\frac{1}{2}\big(\frac{t^2}{y(u)}+(\delta\sqrt{\alpha^2-\beta^2})^2y(u)\big)\big)\cdot \mathbf{1}_{y>0}\bigg)dydz\end{aligned}$$

and the Malliavin weight for the delta satisfies

$$H\bigg(p,\frac{\partial p}{\partial r_0}\bigg) = \frac{1}{a}(e^{-aT}-e^{-at})\bigg(\sigma^2\delta^2\sum_{t\le u\le T}(\sqrt{y(u)})^2\widehat{K}^{*^{-2}} - z\widehat{K}^{*^{-1}}\bigg) \tag{26}$$

where $\widehat{K}^*$ is obtained from $\widehat{K}$ given by equation (22) as

$$\begin{aligned}\widehat{K}^* &= \sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\sqrt{y(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\sqrt{y(u)}) \\ &\quad - \sigma^2\delta^2\big(\sum_{t\le u\le T}(\beta\delta y(u)+\sqrt{y(u)}z)\sqrt{y(u)}\big).\end{aligned} \tag{27}$$

Proof. From Proposition 2.1, the Malliavin weight becomes

$$H(P,Q)=H\Big(P,\frac{\partial P}{\partial r_0}\Big)=Q\mathcal{M}(P)^{-1}LP-\mathcal{M}(P)^{-1}\langle DP,DQ\rangle-Q\langle DP,D\mathcal{M}(P)^{-1}\rangle.$$

Substituting equation (21) from Lemma 4.2 for $Q\mathcal{M}(P)^{-1}LP$, equations (23) and (24) from Lemma 4.3 for $\mathcal{M}(P)^{-1}\langle DP,DQ\rangle$ and $Q\langle DP,D\mathcal{M}(P)^{-1}\rangle$, respectively into $H(P,Q)$, we obtain the expression in (26) from

$$H(P,Q)=\frac{1}{a}(e^{-aT}-e^{-at})\big(\sigma^2\delta^2\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\widehat{K}^{-2}+\varphi(z)\widehat{K}^{-1}\big),$$

where $\varphi(z)=-z$ and $\widehat{K}$ is given by equation (22). Hence, the result follows. □

Theorem 4.1. *Let P be the zero-coupon bond price driven by the NIG process and $Q=\frac{\partial P}{\partial r_0}$, then*

$$\begin{aligned}\triangle_{NIG}&=e^{-r_0T}\Big(-T\int_K^\infty\int_K^\infty(p(t,T,y,z)-K)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\\
&\cdot\Big(\frac{t(y(u))^{-3/2}\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\\
&\cdot\exp\big(-\frac{1}{2}(\frac{t^2}{y(u)}+(\delta\sqrt{\alpha^2-\beta^2})^2y(u))\big)\cdot\mathbf{1}_{y>0}\Big)dydz\\
&+\int_K^\infty\int_K^\infty(p(t,T,y,z)-K)H\Big(p,\frac{\partial p}{\partial r_0}\Big)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\cdot t(y(u))^{-3/2}\\
&\cdot\Big(\frac{\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\exp\big(-\frac{1}{2}(\frac{t^2}{y(u)}+(\delta\sqrt{\alpha^2-\beta^2})^2y(u))\big)\cdot\mathbf{1}_{y>0}\Big)dydz\Big)\end{aligned}$$

where $H(p,\frac{\partial p}{\partial r_0})$ is given by Lemma 4.5.

Proof. The greek *delta* is given by

$$\triangle_{NIG}=\frac{\partial}{\partial r_0}e^{-r_0T}\mathbb{E}[\Phi(P)]=e^{-r_0T}(-T\mathbb{E}[\Phi(P)]+\mathbb{E}[\Phi(P)H(P,Q)]).$$

Substituting $\mathbb{E}[\Phi(P)]$ given by equation (25) of Lemma 4.4 and $\mathbb{E}[\Phi(P)H(P,Q)]$ given by Lemma 4.5 into $\triangle_{NIG}$, gives the desired result. □

4.2 Computation of *vega* for the NIG-driven interest rate derivative

The greek *vega* $\mathcal{V}$ measures the sensitivity of the zero-coupon bond option price with respect to changes in its volatility. High vega implies that the bond option's value is very sensitive to little shift in volatility [6]. It presents uncertainty in future prices for the underlying contract [5]. It is given by

$$\mathcal{V}=\frac{\partial}{\partial\sigma}e^{r_0T}\mathbb{E}[\Phi(P)]=e^{-r_0T}\mathbb{E}\Big[\Phi'(P)\frac{\partial P}{\partial\sigma}\Big]=e^{-r_0T}\mathbb{E}\Big[\Phi(P)H\Big(P,\frac{\partial P}{\partial\sigma}\Big)\Big].$$

Lemma 4.6. *Let P be the zero-coupon bond price driven by the NIG process and $Q_\sigma = \frac{\partial P}{\partial \sigma}$. Then,*

$$
\begin{aligned}
Q_\sigma = -\Bigg[& \frac{\mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] \\
& + \mathbf{w}[T-t] + \delta \sum_{t \le u \le T} \sum_{0 \le s \le t} \big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z \\
& + \beta\delta\Delta IG(s)e^{-a(u-s)}\big) + \delta \sum_{t \le u \le T} \big(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)\big) \\
& - \sigma\delta^2 \sum_{t \le u \le T} (\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)^2\Bigg]P,
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
DQ_\sigma = -\Bigg[& \delta \sum_{t \le u \le T} \sum_{0 \le s \le t} (\Delta\sqrt{IG(s)}e^{-a(u-s)}) + \delta \sum_{t \le u \le T} (\Delta\sqrt{IG(u)}) \\
& - 2\sigma\delta^2\Bigg(\sum_{t \le u \le T} (\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]P + \widetilde{\Lambda}\widehat{K}P,
\end{aligned} \tag{29}
$$

where $\widehat{K}$ is given by equation (22) and

$$
\begin{aligned}
\widetilde{\Lambda} = & \frac{\mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] + \mathbf{w}[T-t] \\
& + \delta \sum_{t \le u \le T} \sum_{0 \le s \le t} \big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z \\
& + \beta\delta\Delta IG(s)e^{-a(u-s)}\big) + \delta \sum_{t \le u \le T} \big(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)\big) \\
& - \sigma\delta^2\Bigg(\sum_{t \le u \le T} (\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)^2\Bigg).
\end{aligned} \tag{30}
$$

Proof. Applying partial derivative to equation (14) yields equation (28). Hence, the Malliavin derivative

$$
\begin{aligned}
DQ_\sigma = -\Bigg[& \delta \sum_{t \le u \le T} \sum_{0 \le s \le t} (\Delta\sqrt{IG(s)}e^{-a(u-s)}) + \delta \sum_{t \le u \le T} (\Delta\sqrt{IG(u)}) \\
& - \sigma\delta^2\Bigg(2 \sum_{t \le u \le T} (\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]P \\
& + \Bigg(- \Bigg[\frac{\mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] + \mathbf{w}[T-t] \\
& + \delta \sum_{t \le u \le T} \sum_{0 \le s \le t} (\Delta\sqrt{IG(s)}e^{-a(u-s)}Z
\end{aligned}
$$

$$
\begin{aligned}
&+ \beta\delta\Delta IG(s)e^{-a(u-s)}\big) + \delta \sum_{t\le u\le T} \big(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)\big) \\
&- \sigma\delta^2\Big(\sum_{t\le u\le T} (\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)^2\Big)\Big] \\
&\cdot\Big(- \Big[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t} \big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) \\
&+ \sigma\delta \sum_{t\le u\le T} (\Delta\sqrt{IG(u)}) - \sigma^2\delta^2\Big(\sum_{t\le u\le T} (\beta\delta\Delta IG(u) \\
&+ \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]\Big)\Big)P
\end{aligned}
$$

which yields equation (29). □

Lemma 4.7. *Let P be the zero-coupon bond price driven by the NIG process. The following holds concerning the sensitivity with respect to σ:*

$$
\begin{aligned}
&Q_\sigma \mathcal{M}(P)^{-1}LP \\
&= \widetilde{\Lambda}\Big[\sigma^2\delta^2\big(\sum_{t\le u\le T} (\Delta\sqrt{IG(u)})^2\big)\widehat{K}^{-2} + 1 - \varphi(z)\widehat{K}^{-1}\Big],\ \varphi(z) = -z,
\end{aligned}
\tag{31}
$$

where $\widetilde{\Lambda}$ and $\widehat{K}$ are given by equations (30) *and* (22)*, respectively.*

Proof. The result follows from Lemmas 3.2, 3.3 and 4.6. Substituting equation (28) of Lemma 4.6 for Q_σ, equation (17) of Lemma 3.3 for $\mathcal{M}(P)^{-1}$, and equation (16) of Lemma 3.2 for LP into $Q_\sigma \mathcal{M}(P)^{-1}LP$ yields the expression in equation (31). □

Lemma 4.8. *Let P be the zero-coupon bond price driven by the NIG process. Then,*

$$
\begin{aligned}
&\mathcal{M}(P)^{-1}\langle DP, DQ_\sigma\rangle \\
&= \widehat{K}^{-1}\Big[\delta \sum_{t\le u\le T}\sum_{0\le s\le t} (\Delta\sqrt{IG(s)}e^{-a(u-s)}) + \delta \sum_{t\le u\le T} (\Delta\sqrt{IG(u)}) \\
&- 2\sigma\delta^2\Big(\sum_{t\le u\le T} (\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big] - \widetilde{\Lambda},
\end{aligned}
\tag{32}
$$

where $\widetilde{\Lambda}$ and $\widehat{K}$ are given by equations (30) *and* (22)*, respectively.*

Proof. The result follows from Lemmas 3.1, 3.3 and 4.6 by substituting equation (17) of Lemma 3.3 for $\mathcal{M}(P)^{-1}$, equation (15) of Lemma 3.1 for DP and equation (29) of Lemma 4.6 for DQ_σ into $\mathcal{M}(P)^{-1}\langle DP, DQ_\sigma\rangle$. □

Lemma 4.9. *Let P denote the zero-coupon bond price driven by the NIG process. Then, the following holds:*

$$Q_\sigma\langle DP, D\mathcal{M}(P)^{-1}\rangle = 2\widetilde{\Lambda}\Big[1+\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\widehat{K}^{-2}\Big], \tag{33}$$

where $\widetilde{\Lambda}$ and $\widehat{K}$ are given by equations (30) and (22), respectively.

Proof. The result follows from Lemmas 3.1, 3.4 and 4.6 by substituting equation (28) of Lemma 4.6 for Q_σ, equation (15) of Lemma 3.1 for DP and equation (18) of Lemma 3.4 for $D\mathcal{M}(P)^{-1}$ into $Q_\sigma\langle DP, D\mathcal{M}(P)^{-1}\rangle$. $\square$

Lemma 4.10. *Let P be the zero-coupon bond price driven by the NIG process. Then, the Malliavin weight for the greek* vega *is given by*

$$\begin{aligned}
H\Big(p,\frac{\partial p}{\partial\sigma}\Big) &= z\widetilde{\Lambda}\widehat{K}^{*^{-1}} - \widetilde{\Lambda}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\sqrt{y(u)})^2\big)\widehat{K}^{-2}\\
&- \widehat{K}^{-1}\Big[\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\sqrt{y(s)}e^{-a(u-s)})\\
&+\delta\sum_{t\le u\le T}(\sqrt{y(u)}) - 2\sigma\delta^2\Big(\sum_{t\le u\le T}(\beta\delta y(u)+\sqrt{y(u)}z)\sqrt{y(u)}\Big)\Big],
\end{aligned} \tag{34}$$

where $\widehat{K}^*$ is given by equation (27) and

$$\begin{aligned}
\widetilde{\Lambda}^* &= \frac{\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\mathbf{w}[T-t]\\
&+\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\sqrt{y(s)}e^{-a(u-s)}z\\
&+\beta\delta y(s)e^{-a(u-s)}\big)+\delta\sum_{t\le u\le T}\big(\sqrt{y(u)}z+\beta\delta y(u)\big)\\
&-\sigma\delta^2\Big(\sum_{t\le u\le T}(\beta\delta y(u)+\sqrt{y(u)}z)^2\Big).
\end{aligned} \tag{35}$$

Proof. The Malliavin weight $H(P,Q_\sigma)$ for the sensitivity with respect to volatility, is obtained by substituting equation (31) of Lemma 4.6 for $Q_\sigma\mathcal{M}(P)^{-1}LP$, equation (32) of Lemma 4.7 for $\mathcal{M}(P)^{-1}\langle DP, DQ_\sigma\rangle$ and equation (33) of Lemma 4.8 for $Q_\sigma\langle DP, D\mathcal{M}(P)^{-1}\rangle$ into $H(P,Q_\sigma)$. Thus,

$$\begin{aligned}
H(P,Q_\sigma) &= H\Big(P,\frac{\partial P}{\partial\sigma}\Big)\\
&= Q_\sigma\mathcal{M}(P)^{-1}LP - \mathcal{M}(P)^{-1}\langle DP, DQ_\sigma\rangle - Q_\sigma\langle DP, D\mathcal{M}(P)^{-1}\rangle
\end{aligned}$$

$$
\begin{aligned}
&= -\widetilde{\Lambda}\varphi(z)\widehat{K}^{-1} - \widetilde{\Lambda}\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)\widehat{K}^{-2} \\
&- \widehat{K}^{-1}\Bigg[\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{IG(s)}e^{-a(u-s)} \\
&+ \delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) - 2\sigma\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) \\
&+ \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg],
\end{aligned}
$$

where $\varphi(z) = -z$; $\widetilde{\Lambda}$ and $\widehat{K}$ are given by equations (30) and (22), respectively. Hence, the result follows. □

Theorem 4.2. *Let P be the zero-coupon bond price driven by the NIG process. Then, the greek* vega *is given by*

$$
\begin{aligned}
\mathcal{V} = &\int_K^\infty\int_K^\infty (p(t,T,y,z) - K)H\Big(p, \frac{\partial p}{\partial \sigma}\Big)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\cdot t(y(u))^{-3/2} \\
&\cdot\Bigg(\frac{\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2}\big(\frac{t^2}{y(u)} + (\delta\sqrt{\alpha^2-\beta^2})^2 y(u)\big)\Big)\cdot \boldsymbol{1}_{y>0}\Bigg)dydz,
\end{aligned}
$$

where the Malliavin weight $H(p, \frac{\partial p}{\partial \sigma})$ is given by equation (34) *of Lemma* 4.10.

Proof. Recall that $\mathcal{V} = e^{-r_0T}\mathbb{E}\Big[\Phi(P)H\Big(P, \frac{\partial P}{\partial \sigma}\Big)\Big]$. Thus,

$$
\begin{aligned}
\mathcal{V} = &\int_{\mathbb{R}}\int_{\mathbb{R}} \max(p(t,T,y,z) - K, 0)H\Big(p, \frac{\partial p}{\partial \sigma}\Big)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\cdot t(y(u))^{-3/2} \\
&\cdot\Bigg(\frac{\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2}\big(\frac{t^2}{y(u)} + (\delta\sqrt{\alpha^2-\beta^2})^2 y(u)\big)\Big)\cdot \mathbf{1}_{y>0}\Bigg)dydz.
\end{aligned}
$$

Hence, the result follows. □

5. Discussion and conclusion

In this paper, we have extended the work of Bavouzet-Morel & Messaoud [3] and Bayazit & Nolder [4] to the sensitivity analysis of an interest rate derivative market driven by a subordinated Lévy process. The Vasicek interest rate model was extended by considering the normal inverse Gaussian subordinated Lévy process. This was used to derive an expression for the price of a zero-coupon bond. The new model is important for transactions in a Lévy market situation where the prices of financial derivatives may experience jumps of different sizes. The greeks, namely: *delta* $\triangle_{NIG}$ and *vega* $\mathcal{V}$ were computed using the Malliavin integration by parts formula. The greeks assist an investor or decision maker to evaluate certain risks and predict the possibility of making money in a particular investment. Vega is important since an increase in volatility will increase the bond option price while a decrease in volatility will lead to a decrease in the bond option value. It helps investors to quantify the risk in the interest rate derivative Lévy market as the volatility changes. An investor or portfolio manager requires an adequate understanding of these greeks in order to predict future worth of a bond option so as to minimize risks. The work provided a better modelling of the interest rate derivative and understanding of sensitivities in a market driven by a normal inverse Gaussian process.

Appendix

Itô formula for semi-martingale [7]

Let $Y = (Y_t)_{0\leq t\leq T}$ be a semi-martingale and $f : [0,T]\times\mathbb{R} \to \mathbb{R}$, a $C^{1,2}$ function, then

$$\begin{aligned} f(t,Y_t) &= f(0,Y_0) + \int_0^t \frac{\partial f}{\partial s}(s,Y_s)ds + \int_0^t \frac{\partial f}{\partial y}(s,Y_{s_-})dY_s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial y^2}(s,Y_s)d[Y,Y]^c_s + \sum_{0\leq s\leq t,\Delta Y_s\neq 0} [f(s,Y_s) - f(s,Y_{s_-}) \\ &- \Delta Y_s \frac{\partial f}{\partial y}(s,Y_{s_-})], \end{aligned}$$

where $[Y,Y]^c_s$ is the continuous part of the quadratic variation of Y and $\Delta Y_s = Y_s - Y_{s_-}$.

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