Multiset group and its generalization to (A, B)-multiset group

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Abstract. Multiset groups are multisets with its elements taken from a group and the characteristic function of the multiset satisfying certain conditions. Apart from the definition and examples of multiset groups, we try to explain some properties, that a multiset should satisfy in order to become a multiset group. From this point, we broaden the concept of multiset group to a new scenario, (A, B)- multiset group, where A and B are non negative real numbers. The multiplicity of the identity element e has its own importance in an (A, B)- multiset group. The count value of the elements depends largely on the values of A and B. We have also delved upon the peculiarities of an (A, B)- multiset group drawn from a cyclic group and defined and explored an (A, B)- multiset normal group and cosets of (A, B)- multiset group.
Keywords: multiset, characteristic function, root set, multiset group, multiset subgroup, level set, (A, B)- multiset group, (A, B)-multiset normal group

1. Introduction

The limitations of classical set theory is what led to the other forms of sets, such as fuzzy set or multiset. Many researchers contributed in the development of these generalized sets. Looking to the case of multisets (also, known as Bags), D. E. Knuth pointed out the essentialness of such a set ([1]). Chris Brink in his studies explained the relations and operations with multisets [2]. Later Wayne D. Blizard developed some of the fundamental structures in multiset background ([3]). C. S. Calude [4], N.J. Wildberger [5], D. Singh [6] are some of the persons who were put milestones in this journey. K.P. Girish and S.J. John [7] explores the relations and functions in multiset context.

The algebraic structures, group, ring, ideal etc. with fuzzy set context are being applied in subjects like computer science, physics and so on. Some of the

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research work in this area are done by Azriel Rosenfield [8], Sabu Sebastian and T. V. Ramakrishnan [9], and Yuying Li et all [10]. The structure with multiset base are yet to be used and implemented widely. Multiset groups (shortly mset groups) and some of its properties have been studied by the authors like A.M. Ibrahim and P.A. Ajegwa [11], Binod Chandra Tripathy [12], A.A. Johnson [13], P.A. Ejegwa [14], S.K. Nazmul [15], Tella [16]. Suma P. and Sunil J. John [17] extended this to ring and ideal structures.

This paper is an attempt to extend the properties of multiset group to a generalized form (A, B)- multiset group. Here, A and B are non negative real numbers with $A < B$. Section 3 is a discussion of multiset group and some of the properties of mset normal groups and cosets of mset groups. In section 4, these properties are analysed in (A, B)- mset group.

2. Preliminaries

In this section, we will be revisiting some of the fundamental properties of Multiset that have been developed by several researchers, which are necessary for this paper.

A *Multiset* (shortly *mset*) T drawn (or derived) from a set U is represented by a function $C_T : U \to N$, where N is the set of non negative integers. $C_T(u)$ represents the number of occurrences of the element u in the multiset T. The function C_T is known as *Characteristic function* or *Count Function* and $C_T(u)$ is the *Count value* of u in T (see, Girish and John (2009)).

Let T be an mset drawn from U, and let $\{u_1, u_2, \cdots, u_n\}$ be a subset of T, with u_1 appearing k_1 times, u_2 appearing k_2 times and so on. Then T is written as

$$T = \{k_1|u_1, k_2|u_2, \cdots, k_n|u_n\}.$$

The subset $S = \{u_1, u_2, \cdots, u_n\}$ of U is called the *Root Set* of T.

Operations of multisets:-

1. Let T_1 and T_2 be two msets drawn from a set U. T_1 is a *submultiset* of T_2, $(T_1 \subseteq T_2)$ if $C_{T_1}(u) \leq C_{T_2}(u)$ for all u in U.

2. Two msets T_1 and T_2 are *equal* if $T_1 \subseteq T_2$ and $T_2 \subseteq T_1$.

3. The *intersection* of T_1 and T_2 is a multiset, $T = T_1 \cap T_2$, with the count function $C_T(u) = \min\{C_{T_1}(u), C_{T_2}(u)\}$, for every $u \in U$.

4. The *union* of T_1 and T_2 is a multiset, $T = T_1 \cup T_2$, with the count function $C_T(u) = \max\{C_{T_1}(u), C_{T_2}(u)\}$, for every $u \in U$.

   More details in [7].

3. Multiset group

Consider the group (G, *) and a multiset T drawn from G. Then, T is said to be a *multiset group* or shortly *mset group* if the characteristic function satisfies the following properties:

(1) $C_T(g * h) \geq \min\{C_T(g), C_T(h) : g, h \in G\}$;

(2) $C_T(g) = C_T(g^{-1})$ for all $g \in G$ where g^{-1} is the inverse of g in G.

Let T be an mset group. A subset P of T is an mset subgroup, if P itself is an mset group on G ([15]).

Example 3.1. Let $G = \{1, -1, i, -i\}$. Then $(G, *)$ is a group, where $*$ is the usual multiplication of real numbers. Consider the multiset $T = \{5|1, 3|-1, 4|i, 4|-i\}$. Here T is a multiset group.

Theorem 3.1. *Let T be a multiset group derived from a group (G, *) and let S be the root set of T. Then S is a subgroup of G.*

Proof. Let $g, h \in S$. Then $C_T(g) > 0$ and $C_T(h) > 0$, $C_T(g * h^{-1}) \geq \min\{C_T(g), C_T(h^{-1})\} = \min\{C_T(g), C_T(h)\} > 0$ means that $g * h^{-1} \in S$. □

Proposition 3.1. *Consider a group (G, *) with identity element e and a multiset group T drawn from G. Then:*

(1) $C_T(e) \geq C_T(g), \forall g \in G$;

(2) $C_T(g^n) \geq C_T(g), \forall g \in G$, and all natural number n. Here, g^n means $g * g * \cdots n$ times.

Proof. (1) Since $e = g * g^{-1}, \forall g \in G$, $C_T(e) \geq \min\{C_T(g), C_T(g^{-1})\} = C_T(g)$;

(2) Applying mathematical induction on n. For $n = 1, C_T(g) = C_T(g)$, and hence the result is true. Assume the result is true for $n-1$ i.e., $C_T(g^{n-1}) \geq C_T(g)$.

Now, $C_T(g^n) = C_T(g^{n-1} * g) \geq \min\{C_T(g^{n-1}),\ C_T(g)\} = C_T(g)$, by induction hypothesis. □

Theorem 3.2. *If T is an mset derived from a group G, then T is an mset group if and only if $C_T(g * h^{-1}) \geq \min\{C_T(g), C_T(h)\}, \forall g, h \in G$.*

Proof. First assume that T is an mset group. Then

$$C_T(g * h^{-1}) \geq \min\{C_T(g),\ C_T(h^{-1})\} = \{\min\{C_T(h),\ C_T(h)\}.$$

Conversely, suppose $C_T(g * h^{-1}) \geq \min\{C_T(g),\ C_T(h)\}, \forall g, h \in G$.

Now, $C_T(e) = C_T(g * g^{-1}), \forall g \in G. \geq \min\{C_T(g),\ C_M(g)\}$, by assumption. So, $C_T(e) \geq C_T(g), \forall g \in G$. Now, $C_T(g^{-1}) = C_T(e*g^{-1}) \geq \min\{C_T(e),\ C_T(g)\} \geq C_T(g)$. Similarly, $C_T(g) = C_T(e * g) \geq \min\{C_T(e),\ C_T(g^{-1})\} \geq C_T(g^{-1})$. Hence, we get $C_T(g) = C_T(g^{-1}), \forall g \in G$, which is the second e condition of Mset group. Now, to show the first condition, take two arbitrary elements g and h from G.

$$\begin{aligned} C_T(g * h) &= C_T(g * (h^{-1})^{-1}) \geq \min\{C_T(g),\ C_T(h^{-1})\}, \text{ by assumption} \\ &= \min\{C_T(g),\ C_T(h)\}. \end{aligned}$$

□

Theorem 3.3. *Let $(G,\ *)$ be a group with identity e and T be an mset group derived from G. If $E = \{g \in G : C_T(g) = C_T(e)\}$, then E is a subgroup of G.*

Proof. Take g and h from E. Then, $C_T(g) = C_T(h) = C_T(e)$. $C_T(g * h^{-1}) \geq \min\{C_T(g),\ C_T(h)\}$, by Theorem 3.4 $= C_T(e)$. Therefore, $g * h^{-1} \in E$. Hence, E is a subgroup of G. □

Definition 3.1. *Let T be an mset drawn from a group G. The subset $\{g : C_T(g) \geq r\}$ of G is known as the* Level Set of T, *denoted by T_r. Here, r is a non negative real number.*

Theorem 3.4. *If T is an mset group drawn from a group $(G,\ *)$ having identity element e, then the level sets T_r are all subgroups of G.*

Proof. If $T_r = \phi$, then T_r is a subgroup.

If T_r is a singleton set, then $T_r = \{e\}$, which is also a subgroup of G. Otherwise, Let $g, h \in T_r$. Then, $C_T(g) \geq r$ and $C_T(h) \geq r$. Now, $C_T(g * h^{-1}) \geq \min\{C_T(g),\ C_T(h)\} \geq r$. So, T_r is a subgroups of G for all positive real number r. □

Theorem 3.5. *Let T be an mset group drawn from a group $(G,\ *)$ having identity element e. If $C_T(g * h^{-1}) = C_T(e)$, for some g and h in* G, *then $C_T(g) = C_T(h)$.*

Proof. $C_T(g) = C_T(g*e) = C_T(g*(h^{-1}*h)) = C_T((g*h^{-1})*h) \geq \min\{C_T(g*h^{-1}),\ C_T(h)\}, = C_T(h)$.

Similarly, starting from $C_T(h)$, we can show that $C_T(h) \geq C_T(g)$. □

Definition 3.2. *An mset group T drawn from a group G is said to be an* Mset Normal group, *if $C_T(g * h * g^{-1}) \geq C_T(h), \forall g, h$ in G.*

Proposition 3.2. *If T is an mset normal group, then $C_T(g * h) = C_M(h * g)$, for every g and h in G.*

Proof. Suppose T is an mset normal group derived from G. Then $C_T g * h * g^{-1}) \geq C_T(h), \forall g, h$ in G. Replacing h by $h*g$, $C_T(g*(h*g)*g^{-1}) \geq C_T(h*g)$.

By associativity $C_T(g * h) \geq C_T(h * g)$. Interchanging the role of g and h, $C_T(h * g) \geq C_T(g * h)$. □

Proposition 3.3. *Let T an mset group drawn from a group G. If T is an mset normal group, then T_r is a normal subgroup of G, for every $r > 0$.*

Proof. Take an mset normal group T derived from G and r a positive real number. Then $C_T(g * h * g^{-1}) \geq C_T(h), \forall g, h$ in G. Choose a $h \in T_r$. Then, $C_T(h) \geq r$. For any $g \in G$, $C_T(g * h * g^{-1}) \geq C_T(h) \geq r$, $g * h * g^{-1} \in T_r$. T_r is a normal subgroup of G. □

Theorem 3.6. *Let T be an mset group drawn from a cyclic group G with generator a. Then $C_T(g) \geq C_T(a), \forall x \in G$.*

Proof. Let $g \in G$. Then $g = a^n$ for some non negative integer n and $C_T(g) \geq C_T(a)$, by Proposition 3.3. □

Corollary 3.1. *Let T be an mset group drawn from a cyclic group G with generators a and b.Then $C_T(a) = C_T(b)$.*

Proof. since a is a generator, and $b \in G$, by above theorem $C_T(a) \leq C_T(b)$. By interchanging the roles of a and b, $C_T(b) \leq C_T(a)$. □

Corollary 3.2. *Let T be an mset group drawn from a group G of prime order. Then $C_T(g)$ are all equal for all $g \in G$ other than the identity element.*

Proof. Being prime order, G is cyclic and every element other than the identity element of G are generators. The proof is then straight forward from above theorem and corollary. □

Definition 3.3. *Let T be an mset group drawn from a group G and $g \in G$ such that $C_T(g) = 0$. The* Left Coset *gM is defined as $C_{gT}(x) = C_T(g * x)$, for $x \in G$.*

Similarly, the Right Coset *Tg is $C_{Tg}(x) = C_T(x * g)$, for $x \in G$.*

Proposition 3.4. *If T is an mset group drawn from G, and $g, h \in G$, then*

(a) $eT = Te = T$.

(b) $g(hT) = (g * h)T$

(c) $(Tg)h = T(g * h)$.

(d) $gT = hT \Leftrightarrow T = (g^{-1} * h)T \Leftrightarrow T = (h^{-1} * g)T$

(e) $Tg = Th \Leftrightarrow T = T(h * g^{-1}) \Leftrightarrow T = T(g * h^{-1})$.

Proposition 3.5. *Let T and R are two mset groups drawn from the same group G, and $g, h \in G$*

(a) $gT = hR \Leftrightarrow T = (g^{-1} * h)R \Leftrightarrow (h^{-1} * g)T = R$.

(b) $Tg = Rh \Leftrightarrow T = R(h * g^{-1}) \Leftrightarrow T(g * h^{-1}) = R$.

4. (A, B)-multiset group

Definition 4.1. *Let M be an mset drawn from a group G, and A, B are two real numbers with $0 \leq A < B$. Then M is called an (A, B)-* multiset group *if the characteristic function satisfies the following conditions.*

1. $\max\{C_M(x * y), A\} \geq \min\{C_M(x), C_M(y), B\}$;
2. $\max\{C_M(x^{-1}), A\} \geq \min\{C_M(x), B\}$,

for every x and y in G.

Notation 4.1. *An (A, B)- mset group is denoted by M_{AB}.*

Proposition 4.1. *If M is an mset group derived from a group G, then it is an (A, B)- mset group for every real number A and B with $0 \leq A < B$.*

Proof. M is an mset group means $C_M(x * y) \geq \min\{C_M(x), C_M(y)\}$, for every x and y in G. For $0 \leq A < B$,

$$\begin{aligned}\max\{C_M(x * y), A\} &\geq C_M(x * y)\\ &\geq \min\{C_M(x), C_M(y)\}\\ &\geq \min\{C_M(x), C_M(y), B\}\end{aligned}$$

M is an (A, B)- mset group. □

Proposition 4.2. *If an mset M derived from a group G is a $(0, N)$- mset group, where $N = \max\{C_M(x) : x \in G\}$, then it is an mset group.*

Proof. For any $x, y \in G, \max\{C_{0N}(x * y), 0\} \geq \min\{C_{M_{0N}}(x), C_{M_{0N}}(y), N\}$

$$C_{M_{0N}}(x * y) \geq \min\{C_{M_{0N}}(x), C_{M_{0N}}(y)\},$$

since $N \geq C_M(x)$ and $N \geq C_M(y)$.

Similarly, by the second condition of (A, B)- mset group

$$\begin{aligned}&\max\{C_{M_{0N}}(x^{-1}, 0)\} \geq \min\{C_{M_{0N}}(x), N\},\\ &C_{M_{0N}}(x^{-1}) \geq C_{M_{0N}}(x).\end{aligned}$$

Hence, the two conditions of mset group is satisfied by M_{0N}. □

Note 4.1. *If an mset drawn from a group G, is not an (A, B) mset group for all A and B with $0 \leq A < B$, then M need not be an mset group.*

Example 4.1. Consider the group $G = \{1, -1, i, -i\}$ with usual multiplication and the mset $M = \{3|1, 4| - 1\}$. Here, M is a (5,6)- mset group,because both th conditions of the definition of (A, B)-mset group is satisfied. But M is not a (1,5)- mset group. Taking $x = y = -1$, LHS of condition (1) of definition is $\max\{C_M(-1 * -1), A\} = \max\{3, 1\} = 3$.

RHS becomes $\min\{C_M(-1), C_M(-1), 5\} = \min\{4, 4, 5\} = 4$. We get LHS=3 and RHS=4, so that the first condition is not satisfied and hence not a $(1, 5)$-mset group. Note that M is not an mset group.

Example 4.2. Consider the group $G = \{1, -1, i, -i\}$ with usual multiplication and the mset $M = \{3|1, 3| - 1, 2|i, 2| - i\}$. Here, M is an (A, B) mset group for all A and B. M is an mset group also.

Definition 4.2. *Let M_{AB} be an (A, B) mset group drawn from a group G. The subset $\{x \in G : C_{M_{AB}}(x) \geq r\}$ of G is known as* level set *of M_{AB} and is denoted by M_r, where r is any positive number.*

The following theorem gives some of the properties of the count value of the identity element e in an (A, B)- mset group.

Theorem 4.1. *If G is a group with identity element e, and M_{AB}- is an (A, B)-mset group drawn from G, then:*

(a) $\max\{C_{M_{AB}}(e), A\} \geq \min\{C_{M_{AB}}(x), B\}, \forall x \in G$.

(b) *If $C_{M_{AB}}(x) \geq B$, for some $x \in G$, then $C_{M_{AB}}(e) \geq B$.*

(c) *If $C_{M_{AB}}(x) < B, \forall x \in G$, and $C_{M_{AB}}(x) > A$, for atleast one $x \in G$, then $C_{M_{AB}}(e) = \max\{C_{M_{AB}}(x) : x \in G\}$.*

(d) *If $C_{M_{AB}}(e) \leq A$, then $C_{M_{AB}}(x) \leq A, \forall x \in G$.*

(e) *If $A < C_{M_{AB}}(e) < B$, then $C_{M_{AB}}(x) \leq C_{M_{AB}}(e), \forall x \in G$.*

Proof. (a) In condition 1 of the definition of (A, B)- mset group, taking $y = x^{-1}$, we get

$$\begin{aligned} \max\{C_{M_{AB}}(x * x^{-1}), A\} &\geq \min\{C_{M_{AB}}(x), C_{M_{AB}}(x^{-1}), B\} \text{ i.e.} \\ \max\{C_{M_{AB}}(e), A\} &\geq \min\{C_{M_{AB}}(x), C_{M_{AB}}(x^{-1}), B\} \\ &\geq \min\{C_{M_{AB}}(x), B\}. \end{aligned}$$

(b) Suppose there is an $x_0 \in G$ with $C_{M_{AB}}(x_0) \geq B$. By part (a)

$$\max\{C_{M_{AB}}(e), A\} \geq \min\{C_{M_{AB}}(x_0), B\} = B,$$

since $C_{M_{AB}}(x_0) \geq B C_{M_{AB}}(e) \geq B$, because $A < B$.

(c) If $C_{M_{AB}}(x) < B, \forall x \in G$, $\min\{C_{M_{AB}}(x), B\} = C_{M_{AB}}(x), \forall x \in G$. So, by part (a),

$$\max\{C_{M_{AB}}(e), A\} \geq C_{M_{AB}}(x), \forall x \in G. \tag{1}$$

Suppose, there is an $x_0 \in G$ with

$$C_{M_{AB}}(x_0) \geq A.$$

For this particular x_0, (4.1) becomes $\max\{C_{M_{AB}}(e), A\} \geq C_{M_{AB}}(x_0)$, $C_{M_{AB}}(e) \geq C_{M_{AB}}(x_0)$. Since, x_0 is arbitrary, $C_{M_{AB}}(e) = \max\{C_{M_{AB}}(x) : x \in G\}$.

(d) If $C_{M_{AB}}(e) \leq A$, $\max\{C_{M_{AB}}(e), A\} = A$. Then, by part (a),

$$A \geq \min\{C_{M_{AB}}(x), B\}, \forall x \in G$$

$A \geq C_{M_{AB}}(x), \forall x \in G$, since $A < B$.

(e) If possible, let $C_{M_{AB}}(x_0) \geq B$ for some $x_0 \in G$. Then, by part (b), $C_{M_{AB}}(e) \geq B$, which is not the case. Therefore, $C_{M_{AB}}(x) \leq B, \forall x \in G$.

Since $C_{M_{AB}}(e) > A$, by part (c), $C_{M_{AB}}(e) = \max\{C_{M_{AB}}(x) : x \in G\}$. i.e. $C_{M_{AB}}(x) \leq C_{M_{AB}}(e), \forall x \in G$.

□

Corollary 4.1. *If $A < C_{M_{AB}}(e) < B$, then $C_{M_{AB}}(x) = C_{M_{AB}}(e), \forall x \in M_k$, where $k = C_{M_{AB}}(e)$*

Proof. For $x \in M_k,\ C_{M_{AB}}(x) \geq k$, $C_{M_{AB}}(x) \geq C_{M_{AB}}(e)$. By Theorem 4.9 (e),

$$C_{M_{AB}}(x) \leq C_{M_{AB}}(e), \forall x \in G.$$

Hence, for $x \in M_k,\ C_{M_{AB}}(x) = C_{M_{AB}}(e)$. □

Theorem 4.2. *Let M be an mset drawn from a group G. If M is an (A, B)-mset group, then the level set M_r is a subgroup of G for $A < r \leq B$.*

Proof. If $M_r = \phi$, then it is a subgroup trivially.

If M_r has exactly one element say x, then, by Theorem 4.9 (a), $x = e$, the identity element of G and is a subgroup of G.

Otherwise, take two element x and y from M_r, for a particular r. $C_{M_{AB}}(x) \geq r$ and $C_{M_{AB}}(y) \geq r$ and $A < r \leq B$, will give $\min\{C_{M_{AB}}(x), C_{M_{AB}}(y), B\} \geq r$.

By definition, $C_{M_{AB}}(x * y^{-1}) \geq r$.

$\implies x * y^{-1} \in M_r$, completes the proof. □

Corollary 4.2. *If $C_{M_{AB}}(x) \geq B$ and $C_{M_{AB}}(y) \geq B$ for $x \in G, y \in G$, then $C_{M_{AB}}(x * y) \geq B$.*

Proof. $x \in M_B$, $y \in M_B$ and M_B is a subgroup will imply $x * y \in M_B$. □

Example 4.3. In Example 4.7, $M_r = G$, if $r \leq 2$, $M_r = \{1, -1\}$, if $2 < r \leq 3$, and $M_r = \phi$, if $r > 3$.

In all cases, M_r is a subgroup of G.

Theorem 4.3. *If $A < C_{M_{AB}}(x) < B$, for $x \in G$, then $C_{M_{AB}}(x * y) = C_{M_{AB}}(x), \forall y \in G$ with $C_{M_{AB}}(y) > C_{M_{AB}}(x)$.*

Proof. By the definition of M_{AB} mset group

$$\max\{C_{M_{AB}}(x * y), A\} \geq \min\{C_{M_{AB}}(x), C_{M_{AB}}(y), B\} = C_{M_{AB}}(x),$$

since both B and $C_{M_{AB}}(y)$ are greater than $C_{M_{AB}}(x)$.

$$\therefore C_{M_{AB}}(x * y) \geq C_{M_{AB}}(x). \tag{2}$$

If $C_{M_{AB}}(x * y) > C_{M_{AB}}(x)$, let $r_0 = \min\{C_{M_{AB}}(x * y), C_{M_{AB}}(y), B\}$. Then $r_0 > C_{M_{AB}}(x)$. Also, $A < r_0 \leq B$ and hence M_{r_0} is a subgroup of G.

$$x * y \in M_{r_0}, y \in M_{r_0} \implies (x * y) * y^{-1} \in M_{r_0} \implies x \in M_{r_0},$$

i.e. $C_{M_{AB}}(x) \geq r_0 > C_{M_{AB}}(x)$, a contradiction and this completes the proof. □

Theorem 4.4. *If $C_{M_{AB}}(x) \leq A$ and $C_{M_{AB}}(y) > A$, for x, y in G, then $C_{M_{AB}}(x* y) \leq A$.*

Proof. If possible, let $C_{M_{AB}}(x*y) > A$. Take $r_0 = \min\{C_{M_{AB}}(x*y), C_{M_{AB}}(y), B\}$. Then $A < r_0 \leq B$ and hence M_{r_0} is a subgroup of G

$$x * y \in M_{r_0}, y \in M_{r_0} \implies (x * y) * y^{-1} \in M_{r_0} \implies x \in M_{r_0},$$

i.e. $C_{M_{AB}}(x) \geq r_0 > A$, a contradiction. □

Theorem 4.5. *If $A < C_{M_{AB}}(x) < B$, then $C_{M_{AB}}(x^n) \geq C_{M_{AB}}(x)$, for a positive integer n.*

Proof. By definition

$$\max\{C_M(x * x), A\} \geq \min\{C_M(x), C_M(x), B\},$$
$$\max\{C_{M_{AB}}(x^2), A\} \geq \min\{C_{M_{AB}}(x), B\},$$
$$C_{M_{AB}}(x^2) \geq C_{M_{AB}}(x),$$

since $A < C_{M_{AB}}(x) < B$. By the same argument $C_{M_{AB}}(x^3) \geq C_{M_{AB}}(x^2) \geq C_{M_{AB}}(x)$. Proceeding like this, $C_{M_{AB}}(x^n) \geq C_{M_{AB}}(x)$. □

Proposition 4.3. *If G is a group and M_{AB} is an (A, B)- mset group drawn from G, then*

(a) *If $C_{M_{AB}}(x) \leq A$, for some $x \in G$, then $C_{M_{AB}}(x^{-1}) \leq A$, for those x.*

(b) *If $A < C_{M_{AB}}(x) < B$, for some $x \in G$, then $C_{M_{AB}}(x) = C_{M_{AB}}(x^{-1})$.*

(c) *If $C_{M_{AB}}(x) \geq B$, for some $x \in G$, then $C_{M_{AB}}(x^{-1}) \geq B$.*

Proof. (a) Suppose $C_{M_{AB}}(x_0) \leq A$, for $x_0 \in G$. If possible, let $C_{M_{AB}}(x_0^{-1}) > A$. Let $r_0 = \min\{C_{M_{AB}}(x_0^{-1}), B\}$. Then $r_0 > A$, $x_0^{-1} \in (M_{AB})_{r_0}$ and being $(M_{AB})_{r_0}$ is a subgroup of G, $x_0 \in (M_{AB})_{r_0}$. Therefore, $C_{M_{AB}}(x_0) \geq r_0 > A$, a contraduction.

(b) choose x_0 from G such that

$$A < C_{M_{AB}}(x_0) < B. \tag{3}$$

By condition 2 of the definition of (A, B)-mset group,

$$\max\{C_{M_{AB}}(x_0^{-1}), A\} \geq \min\{C_{M_{AB}}(x_0), B\},$$

$$\max\{C_{M_{AB}}x_0^{-1}, A\} \geq C_{M_{AB}}(x_0), \text{ by (4.1)}$$

since, $A < C_{M_{AB}}(x_0)$,

$$C_{M_{AB}}(x_0^{-1}) \geq C_{M_{AB}}(x_0). \tag{4}$$

Again by applying condition 2 of the definition of (A, B)-mset group to the point (x_0^{-1})

$$\max\{C_{M_{AB}}(x_0), A\} \geq \min\{C_{M_{AB}}(x_0^{-1}), B\}.$$

In view of equation (4.4), this can be reduced to

$$C_{M_{AB}}(x_0) \geq C_{M_{AB}}(x_0^{-1}) \tag{5}$$

the required result is obtained from the equations (4.4) and (4.5).

(c) Choose an x_0 from G such that $C_{M_{AB}}(x_0) \geq B$.

Consider M_B. $x_0 \in M_B$. Since M_B is a subgroup of G, $x_0^{-1} \in M_B$, which gives $C_{M_{AB}}(x_0^{-1}) \geq B$. □

4.1 M_{AB} drawn from a cyclic group G

Theorem 4.6. *Let G be a cyclic group with generator a, and M_{AB} be an (A, B)-mset group drawn from G.*

If $A < C_{M_{AB}}(a) < B$, then $C_{M_{AB}}(x) \geq C_{M_{AB}}(a), \forall x \in G$.

Proof. By an above theorem, $C_{M_{AB}}(x) \leq C_{M_{AB}}(e), \forall x \in G$. So, $C_{M_{AB}}(a) \leq C_{M_{AB}}(e)$.

Now, for $x \neq e, x = a^n$, for some positive integer n. Again, by a previous theorem, $C_{M_{AB}}(a^n) \geq C_{M_{AB}}(a)$ i.e. $C_{M_{AB}}(x) \geq C_{M_{AB}}(a)$. □

Theorem 4.7. *Let G be a cyclic group with generator a, and M_{AB} be an (A, B)-mset group drawn from G. If $C_{M_{AB}}(a) \geq B$, then $G = M_B$.*

Proof. M_B is a subgroup of G. Now to show $G \subseteq M_B$.

Let $x \in G$. Then $x = a^n$ for a positive integer n. Given, $C_{M_{AB}}(a) \geq B \implies a \in M_B \implies a^n \in M_B \implies x \in M_B$. Hence, $G = M_B$. □

Theorem 4.8. *Let G be a cyclic group with two generators a and b and M_{AB} be an (A, B)- mset group drawn from G. If $A < C_{M_{AB}}(a) < B$, then $C_{M_{AB}}(a) = C_{M_{AB}}(b)$.*

Proof. By Theorem 4.18,

$$C_{M_{AB}}(b) \geq C_{M_{AB}}(a). \tag{6}$$

If possible, let $C_{M_{AB}}(b) \geq B$. Then, by Theorem 4.14, $G = M_B$ and so $a \in M_B$

$$\implies C_{M_{AB}}(a) \geq B,$$

a contradiction. Therefore,

$$C_{M_{AB}}(b) < B. \tag{7}$$

From (4.6) and (4.7), $A < C_{M_{AB}}(b) < B$. By Theorem 4.13

$$C_{M_{AB}}(a) \geq C_{M_{AB}}(b) \tag{8}$$

(4.6) and (4.8) together provides the requirement. □

Corollary 4.3. *If G is a cyclic group of prime order with generator a and identity element e, then $C_{M_{AB}}(x) = C_{M_{AB}}(a), \forall x \neq e$ of G.*

Proof. For a cyclic group of prime order, every element other than e, is a generator, and hence the result is obtained by above theorem. □

4.2 (A, B)- Mset normal group

Definition 4.3. *An (A, B)- mset group drawn from a group G is said to be an (A, B)- mset Normal group if* $\max\{C_{M_{AB}}(x * y * x^{-1}), A\} \geq \min\{C_{M_{AB}}(y), B\}$, *for every x and y in G.*

Proposition 4.4. *If an (A, B)- mset group is an (A, B) mset normal group, then* $\max\{C_{M_{AB}}(x * y), A\} \geq \min\{C_{M_{AB}}(y * x), B\}$, *for every x and y in G.*

Proof. Replacing y by $y * x$ in the definition of (A, B)- mset normal group, we get this proposition. □

Corollary 4.4. *For an abelian group G, M_{AB} is normal iff $A < C_{M_{AB}}(x) < B$ for all x in G.*

Proposition 4.5. *$If M_{AB}$ is an mset normal group drawn from a group G, then M_r is a normal subgroup of G, for $A < r \leq B$.*

Proof. Choose r such that $A < r \leq B$. If $M_r = \phi$, is a normal subgroup of G. If M_r is a singleton set, then $m_r = \{e\}$, again a subgroup of G.

On the other hand, if M_r contains more than one element. Take two arbitrary elenemts x and y from M_r. Then, $C_{M_{AB}}(x) \geq r$ and $C_{M_{AB}}(y) \geq r$. Therefore, $\min\{C_{M_{AB}}(y), B\} = r$. From the definition of (A, B)- mset normal group $\max\{C_{M_{AB}}(x * y * x^{-1}, A\} \geq r$.

$C_{M_{AB}}(x * y * x^{-1} \geq r$, since $A < r \leq B$.

$\implies x * y * x^{-1} \in M_r$, proving that M_r is a normal subgroup of G. □

Proposition 4.6. *M_{AB} is an (A, B)- mset normal group drawn from a group G, and x, y elements of G.*

(a) *If $C_{M_{AB}}(x) \geq B$, then $C_{M_{AB}}(y * x * y^{-1}) \geq B$.*

(b) *If $A < C_{M_{AB}}(x) < B$, then $C_{M_{AB}}(y * x * y^{-1}) = C_{M_{AB}}(x)$.*

(c) *If $C_{M_{AB}}(x * y) \leq A$, then $C_{M_{AB}}(y * x) \leq A$.*

(d) *if $A < C_{M_{AB}}(x * y) < B$, then $C_{M_{AB}}(x * y) = C_{M_{AB}}(y * x)$.*

(e) *If $C_{M_{AB}}(x * y) \geq B$, then $C_{M_{AB}}(y * x) \geq B$.*

Proof. The poof is straight forward from the definition of (A, B)- mset normal group. □

4.3 Cosets of (A, B)- mset group

Definition 4.4. *Let M_{AB} be an (A, B)- mset group drawn from a group G and let $g \in G$. The* left coset *gM_{AB} is defined as $C_{gM_{AB}}(x) = \min\{\max(C_{M_{AB}}(g^{-1} * x), A), B\}, \forall x \in G$. The* right coset *$M_{AB}g$ is $C_{M_{AB}g}(x) = \min\{\max(C_{M_{AB}}(x * g^{-1}), A), B\}, \forall x \in G$.*

Proposition 4.7. *If M_{AB} is an (A, B)- mset group drawn from a group G with identity element e, then $eM_{AB} = M_{AB}e$.*

Proof. By Definition,

$$\begin{aligned} C_{eM_{AB}}(x) &= \min\{\max(C_{M_{AB}}(e^{-1} * x), A), B\}, \forall x \in G \\ &= \min\{\max(C_{M_{AB}}(e * x), A), B\}, \forall x \in G \\ &= \min\{\max(C_{M_{AB}}(x), A), B\}, \forall x \in G \\ &= \min\{\max(C_{M_{AB}}(x * e), A), B\}, \forall x \in G \\ &= \min\{\max(C_{M_{AB}}(x * e^{-1}), A), B\}, \forall x \in G \\ &= C_{M_{AB}e}(x). \end{aligned}$$ □

Proposition 4.8. (a) *$C_{eM_{AB}}(x) = A$ if $C_{M_{AB}}(x) \leq A$.*

(b) *If $A < C_{M_{AB}}(x) < B$, then $C_{eM_{AB}}(x) = C_{M_{AB}}(x)$.*

(c) *$C_{eM_{AB}}(x) = B$ if $C_{M_{AB}}(x) \geq B$.*

Proof. The proof is obtained directly from the definition of left coset. □

Corollary 4.5. *$eM_{AB} = M_{AB}$ if $A \leq C_{M_{AB}}(x) \leq B, \forall x \in G$.*

Note 4.2. *Similar results hold for right cosets also.*

Proposition 4.9. (a) *If M_{AB}- is an (A, B) mset group, then both eM_{AB} and $M_{AB}e$ are (A, B)- mset groups.*

(b) *If M_{AB} is an (A, B)- mset normal group, then both eM_{AB} and $M_{AB}e$ are (A, B)- mset normal groups.*

Theorem 4.9. *If M_{AB} is an (A, B)- mset group drawn from a group G with identity element e. Suppose $C_{M_{AB}}(e) \geq B$. An element $a \neq e \in M_B$, if and only if $aM_{AB} = eM_{AB}$.*

Similar result hold for right cosets also.

Proof. Let $a \neq e \in M_B$. Then $a^{-1} \in M_B$.
Case 1. For $x \in G$ with $C_{M_{AB}}(x) \geq B$,

$$\begin{aligned} x \in M_B &\implies a^{-1} * x \in M_B \\ &\implies C_{M_{AB}}(a^{-1} * x) \geq B \\ &\implies C_{aM_{AB}}(x) = B, \end{aligned}$$

by definition of left coset. For the same x, $C_{eM_{AB}}(x) = \min\{\max(C_{M_{AB}}(x), A), B\}$ $= B$. So, $C_{aM_{AB}}(x) = C_{eM_{AB}}(x)$.
Case 2. For $x \in G$ with $A < C_{M_{AB}}(x) < B$,

$$\begin{aligned} C_{M_{AB}}(a^{-1} * x) &= C_{M_{AB}}(x), \text{ by Theorem } 4.7 \\ &= C_{M_{AB}}(e^{-1} * x) \\ \therefore C_{aM_{AB}}(x) &= C_{eM_{AB}}(x) \end{aligned}$$

Case 3 : For $x \in G$ with $C_{M_{AB}}(x) \leq A$,

$$\begin{aligned} C_{M_{AB}}(a^{-1} * x) &\leq A, \text{ by Theorem 4.8} \\ \therefore C_{aM_{AB}}(x) &= A \\ &= C_{eM_{AB}}(x). \end{aligned}$$

Hence, in all the three cases, $C_{aM_{AB}}(x) = C_{eM_{AB}}(x)$ and this completes one part of the proof.

Conversely, assume that $aM_{AB} = eM_{AB}$ for some $a \in G$. $C_{aM_{AB}}(x) = C_{eM_{AB}}(x), \forall x \in G$ i.e., $\min\{\max(C_{M_{AB}}(a^{-1}*x), A), B\} = \min\{\max(C_{M_{AB}}(e^{-1}* x), A), B\}, \forall x \in G$. Taking $x = a$,

$$\begin{aligned} \min\{\max(C_{M_{AB}}(a^{-1} * a), A), B\} &= \min\{\max(C_{M_{AB}}(e^{-1} * a), A), B\} \\ i.e. \min\{\max(C_{M_{AB}}(e), A), B\} &= \min\{\max(C_{M_{AB}}(a), A), B\} \\ &\implies B \\ &= \min\{\max(C_{M_{AB}}(e^{-1} * a), A), B\} \\ &\implies C_{M_{AB}}(a) \geq B \\ &\implies a \in M_B. \end{aligned}$$

□

Corollary 4.6. *Let M_{AB} is an (A, B)- mset group drawn from a group G with identity element e. If $a \in M_B$, then $aM_{AB} = M_{AB}a = eM_{AB} = M_{AB}e$.*

Proof. if $a \in M_B$, then by above theorem $aM_{AB} = eM_{AB}$ and $aM_{AB} = eM_{AB}$. But by Proposition 4.24, $eM_{AB} = M_{AB}e$. □

Corollary 4.7. *Let M_{AB} is an (A, B)- mset group drawn from a group G and let $a, b \in G$. $aM_B = bM_B$ if and only if $aM_{AB} = bM_{AB}$.*
Similarly for right cosets.

Proof.

$$\begin{aligned} & aM_B = bM_B \\ & \Leftrightarrow a^{-1}b \in M_B \\ & \Leftrightarrow (a^{-1}b)M_{AB} = eM_{AB} \\ & \Leftrightarrow bM_{AB} = aM_{AB}. \end{aligned}$$

□

Theorem 4.10. *Let M_{AB} is an (A, B)- mset group drawn from a group G with identity element e and suppose $A < C_{M_{AB}}(e) < B$. Then, for an element $a \in G$, $C_{M_{AB}}(a) = C_{M_{AB}}(e)$ if and only if $aM_{AB} = eM_{AB}$*

Proof. Assume first that $C_{M_{AB}}(a) = C_{M_{AB}}(e)$. Choose an $x \in G$.
Case 1. $C_{M_{AB}}(x) \leq A$. Then $C_{M_{AB}}(a^{-1} * x) \leq A$, by Theorem 4.8 and Proposition 4.10 (b). Hence, by definition of left coset and Proposition 4.25 $C_{aM_{AB}}(x) = A = C_{eM_{AB}}(x)$.
Case 2. $A < C_{M_{AB}}(x) < C_{M_{AB}}(e)$ then, $C_{M_{AB}}(a^{-1} * x) = C_{M_{AB}}(x) = C_{M_{AB}}(e^{-1} * x)$, by Theorem 4.7 and Proposition 4.10 (b) i.e. $C_{aM_{AB}}(x) = C_{eM_{AB}}(x)$.
Case 3. $C_{M_{AB}}(x) \geq C_{M_{AB}}(e)$. Let $C_{M_{AB}}(e) = m$. $C_{M_{AB}}(x) = m$, by Theorem 4.11 (e).

Here, $a \in M_m$, by assumption and M_m being a subgroup, $a^{-1} \in M_m$. Also, $x \in M_m \implies (a^{-1} * x) \in M_m \implies C_{M_{AB}}(a^{-1} * x) = m$.

$$\begin{aligned} \therefore C_{aM_{AB}}(x) &= \min\{\max(C_{M_{AB}}(a^{-1} * x), A), B\} \\ &= \min\{\max(m, A), B\} \\ &= \min\{\max(C_{M_{AB}}(e^{-1} * x), A), B\} \\ &= C_{eM_{AB}}(x). \end{aligned}$$

From the above three cases, $aM_{AB} = eM_{AB}$. Conversely, assume that $aM_{AB} = eM_{AB}$

$$\begin{aligned} C_{aM_{AB}}(x) &= C_{eM_{AB}}(x), \forall x \in G \\ C_{aM_{AB}}(a) &= C_{eM_{AB}}(a) \\ \min\{\max(C_{M_{AB}}(a^{-1} * a), A), B\} &= \min\{\max(C_{M_{AB}}(e^{-1} * a), A), B\} \\ \min\{\max(C_{M_{AB}}(e), A), B\} &= \min\{\max(C_{M_{AB}}(a), A), B\} \\ C_{M_{AB}}(a) &= C_{M_{AB}}(e). \end{aligned}$$

□

5. Conclusion and future work

We have broadened the group structure in multiset context to a new scenario , (A, B) multiset group. Here both A and B are non negative real numbers and the (A, B) multiset group depends on A, B and the count value of the elements. Hence, in practical situations, it will be more adequate to apply (A, B) multiset groups, rather than multiset groups, and in this way, we are providing a novel path for research.

References

[1] D. E. Knuth, *The art of computer programming*, Semi numerical Algorithms, Adison- Wesley, 2 (1981).

[2] Chris Brink, *Multisets and the algebra of relevance logic*, Non-Classical Logic, 5 (1988), 75-95.

[3] Wayne D. Blizard, *The developement of multiset theory*, Modern logic, 1 (1991), 319-352.

[4] C. S. Calude, G. Paun, G. Rozenberg, A. Salomaa, *Mathematics of Multisets*, Springer Verlag, (2001), 347-358.

[5] N.J. Wildberger, *A new look at multisets*, School of Mathematics UNSW Sydney 2053, (2003).

[6] D. Singh, A. M. Ibrahim, T. Yohanna, J. N. SINGH, *An overview of the application of multiset*, Novi Sad J. Math, 37 (2007), 73-92.

[7] K. P. Girish, Sunil Jacob John, *Relations and functions in multiset context*, Information sciences, 179 (2009), 758-768.

[8] Azriel Rosenfield, *Fuzzy groups*, Journal of Mathematical Analysis and Applications, 35 (1971), 512-517.

[9] Sabu Sebastian, T. V. Ramakrishnan, *Multi-fuzzy sets: an expansion of fuzzy sets*, Fuzzy Information and Engineering, 1 (2011), 35-43.

[10] Yuying Li, Xuzhu Wang, Liqiong Yang, *A study of* (λ, μ)*-fuzzy subgroups*, Hindawi Publishing Corporation, Jounal of Applied Mathematics, vol. 2013 Article ID 485768 (2013).

[11] A.M. Ibrahim, P.A. Ejegwa, *A survey on the concept of multigroups*, Journal of the Nigerian Association of Mathematical Physics, 38 (2016), 1-8.

[12] Binod Chandra Tripathy, Shyamal Debnath, Debjani Rakshit, *On multiset group*, Proyecciones Journal of Mathematics, 33 (2018), 479-489.

[13] Johnson Aderemi Awolola, Adeku Musa Ibrahim, *Some results on multigroups*, Quasigroups and Related Systems, 24 (2016), 169-177.

[14] P.A. Ejegwa, *Upper and lower cuts of multigroups*, Prajna International Journal of Mathematical Sciences and Applications, 1 (2017), 19-26.

[15] S. K. Nazmul, P. Majumdar, S.K. Samanta, *On multisets and multigroups*, Annals of Fuzzy Mathematics and Informatics, 3 (2013), 643-656.

[16] Y. Tella, S. Daniel, *A study of group theory in the context of multiset theory*, International Journal of Science and Technology, 8 (2013), 609-615.

[17] Suma P, Sunil Jacob John, *Handbook of research on emerging applications of fuzzy algebraic structures*, IGI Global., Chapter 5, Multiset Approach to Algebraic Structures, (2019), 78-90.

[18] I. Deli, *Convex and concave sets based on soft sets and fuzzy soft sets*, Journal of New Theory, 29 (2019), 101-110.

[19] F. Karaaslan, I. Deli, *On soft neutrosophic classical sets*, Plaestine Journal of Mathematics, 9/1 (2020), 312-326.

[20] I. Deli, M. A. Keles, *Distance measures on trapezoidal fuzzy multi-numbers and application to multi-criteria decision-making problems*, Soft Computing, https://doi.org/10.1007/s00500-021-05588-6DOI s00500-021-05588-6 (2021).

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