On nodal filter theory of EQ-algebras

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Abstract. In this article, we mainly focus on a new kind of filter on EQ-algebras. At first, we introduce some new concepts of seminodes, nodes and nodal filters (n-filters, for short) on EQ-algebras and investigate the relationships among them and some other elements. Also, we investigate their lattice structures and obtain that the set $\mathcal{SN}(E)$ of all seminodes on an EQ-algebra is a Hertz-algebra and a Heyting-algebra under some conditions. Then, we discuss the properties of n-filters and show that there is a one-to-one correspondence between nodal principle filter and node element in an idempotent EQ-algebra. Furthermore, the relationships among it and other filters are presented. It is proved that each obstinate filter or each (positive) implicative filter is an n-filter under some conditions. At last, we introduce the algebraic structures and topological structures of the set of all n-filters on EQ-algebras and prove that $(NP(E), \tau)$ is a compact T_0 space. Moreover, we set up the connections from the set $NF(E)$ of all n-filters on an EQ-algebra to other algebraic structures, like BCK-algebras, Hertz algebras and so on.
Keywords: EQ-algebra, seminode, node, nodal filter, topological space.

1. Introduction

As we all know, logic is not only an important tool in mathematics and information science, but also a basic technology. Non-classical logic consists of fuzzy logic and multi-valued logic, they deal with uncertain information such as

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fuzziness and randomness. Therefore, all kinds of fuzzy logic algebras are widely introduced and studied, such as residuated lattices, BL-algebras, MV-algebras, which play a very important role in fuzzy logic algebra system. In [11], Goguen put forward a new point of view, which is that the algebraic structure of many-valued logic may be a residuated lattice satisfying some additional conditions. This view has been widely recognized by scholars at home and abroad. However, since the publication of Hájek's book [12] in 1998, fuzzy logic has been developed into different formal systems, and each one is based on a residuated lattice. With the passage of time, propositional logic and first-order logic have been widely developed. For this reason, in order to develop the higher-order fuzzy logic as a correspondence of the classical higher-order logic. Novák and De Bates [17] came out with a new algebra, which is called an EQ-algebra, for the first time. An EQ-algebra has three operations, which are fuzzy equality, multiplication and meet. By replacing the basic conjunction fuzzy equality with implication, EQ-algebras open up a new filed for another development of many-valued fuzzy logic and a possibility for developing a fuzzy logic with non commutative connection but only one implication. Since then, EQ-algebras have been widely concerned and many significant properties and conclusions have been proved [1], [10], [14][17], [21], [26].

Filter theory is of great significance to study the completeness of different logical systems and their matching logical algebras. Start with a logical viewpoint, we can use the filters to represent the provable formula sets in relevant reasoning systems. Also, the characters of filters is closely related to the structure properties of algebras. Hence, there are numerous researches on filter theory. In [17], Novák and De Bates introduced filters on EQ-algebra for the first time. In [10], M. El-Zekey and V. Novák proposed the concepts of (prime) prefilters on EQ-algebras. Moreover, their related properties were stated and proved. And then, in [14], implicative and positive implicative prefilters (filters) in EQ-algebra were proposed by Liu and Zhang and they also represented some related conclusions of them. Also, they discussed the properties of quotient algebras, which is induced by the positive implicative filters. Furthermore, they discussed the relationships between these two prefilters and concluded that in good IEQ-algebras positive implicative prefilters and implicative prefilters coincided.

Now, in this paper, we introduce a new kind of filter to EQ-algebras, which is said to be a nodal filter. Originally, Balbes and Horn [2] put forward the concept of nodes in a lattice. In [22], the definition of a nodal filter was introduced by Varlet in the (implicative) semilattice. Afterward, T. Khorami and B. Saeid [13] presented the concepts of nodes and nodal filters on BL-algebra and the congruence relations induced by nodal filters on BL-algebra is stated and proved. In [6], Bakhshi presented the concept of nodal filters in residuated lattices and obtained that the set of all nodal filters forms a Heyting algebra. Namdar and Borzooei [18] researched nodal filters theory in hoop algebras. Next, X. Xun and X.L. Xin [24] introduced it in equality algebras. Now, we introduce this concept to EQ-algebras, here is the outline of this paper: In the next Section, we recollect some basic definitions and properties of EQ-algebras. In Section 3, we introduce the concepts of seminodes and nodes on EQ-algebras and investigate the related properties of them. We obtain that the set $\mathcal{SN}(E)$ of all seminodes is a Hertz-algebra and a Heyting-algebra under some conditions. Moreover, we consider the relationships among seminodes, nodes and some other elements on an EQ-algebra. In Section 4, we present the notion of nodal filter (for short, n-filter) in an EQ-algebra and investigate their related properties. Furthermore, we discuss the relationships between nodal filters and node elements, as well as their relationships with other filters. In Section 5, we study the algebraic structures of $NF(E)$ and topological structures of $NP(E)$ on EQ-algebras.

2. Preliminaries

In this section, we present some basic concepts and conclusions relevant to EQ-algebras.

Definition 2.1 ([17]). *An algebra $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$ is said to be an EQ-algebra, if for all $x, y, p, q \in E$, it satisfies the following axioms:*

$(EQ1)$ $< E, \wedge, 1 >$ *is a commutative idempotent monoid.*

$(EQ2)$ $< E, \otimes, 1 >$ *is a monoid and $\otimes$ is isotone w.r.t. "$\leq$", where $x \leq y$ is defined as $x \wedge y = x$.*

$(EQ3)$ $x \sim x = 1$.

$(EQ4)$ $((x \wedge y) \sim p) \otimes (q \sim x) \leq p \sim (q \wedge y)$.

$(EQ5)$ $(x \sim y) \otimes (p \sim q) \leq (x \sim p) \sim (y \sim q)$.

$(EQ6)$ $(x \wedge y \wedge p) \sim x \leq (x \wedge y) \sim x$.

$(EQ7)$ $x \otimes y \leq x \sim y$.

An EQ-algebra $\mathcal{E}$ is bounded if there exists an element $0 \in E$ such that $0 \leq x$, for all $x \in E$. And we define the unary operation: $x' = x \to 0$, for all $x \in E$. If $x^2 = x$, for all $x \in E$, then $\mathcal{E}$ is called an idempotent EQ-algebra. For any $x \in E$, x is called:

(1) dense if $x' = 0$.

(2) atom if x is the minimal element in $E \setminus \{0\}$.

(3) co-atom if x is the maximal element in $E \setminus \{1\}$.

(4) involutive if $x'' = x$.

Definition 2.2 ([17]). *Let $\mathcal{E}$ be an EQ-algebra and $x, y, z \in E$. Then, it is called*

(1) *good if* $x \sim 1 = x$ *for each* $x \in E$.

(2) *prelinear if* 1 *is the unique upper bound of the set* $\{x \to y, y \to x\}$ *in* E, *for all* $x, y \in E$.

(3) *residuated if for each* $x, y, z \in E$, $(x \odot y) \wedge z = (x \odot y)$ *if and only if* $x \wedge ((y \wedge z) \sim y) = x$.

(4) *lattice-ordered if it has a lattice reduct.*

(5) *distributively lattice-ordered if the lattice reduct is distributive.*

Proposition 2.3 ([9, 10, 17])**.** *Let* $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ *be an EQ-algebra, and let* $x \to y := (x \wedge y) \sim x$ *and* $\bar{x} = x \sim 1$. *Then, for all* $x, y, w \in E$ *the following properties hold:*

(1) $x \otimes y \leq x, y$, $x \otimes y \leq x \wedge y$.

(2) $x \sim y \leq x \to y$, $x \sim y = y \sim x$.

(3) $x \leq \bar{x} \leq y \to x$, $\bar{1} = 1$.

(4) $x \to y \leq (w \to x) \to (w \to y)$, $x \to y \leq (y \to w) \to (x \to w)$.

(5) $x \to x \wedge y = x \to y$.

(6) *if* $x \leq y$, *then* $x \sim y = y \to x$, $w \to x \leq w \to y$ *and* $y \to w \leq x \to w$.

(7) $x \to y \leq (x \wedge w) \to (y \wedge w)$, $w \to (x \wedge y) \leq (w \to x) \wedge (w \to y)$.

(8) *if* $x \vee y$ *exists, then* $(x \vee y) \to w = (x \to w) \wedge (y \to w)$.

Proposition 2.4 ([9])**.** *Let* $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ *be an EQ-algebra. Then,* $\mathcal{E}$ *is residuated iff* $\mathcal{E}$ *is good and* $x \to y \leq (x \otimes z) \to (y \otimes z)$, *for all* $x, y, z \in E$.

Definition 2.5 ([17])**.** *Let* $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ *be an EQ-algebra. Then, a subset* H *of* E *is called a prefilter provided that, for all* $x, y, z \in E$, *the following conditions hold:*

$(F1)$ $1 \in H$.

$(F2)$ *If* $x, y \in H$, *then* $x \otimes y \in H$.

$(F3)$ *If* $x, x \to y \in H$, *then* $y \in H$.

A prefilter H *is called a filter provided that, for all* $x, y, z \in E$, *the following condition holds:*

$(F4)$ *If* $x \to y \in H$, *then* $(x \otimes z) \to (y \otimes z) \in H$.

The set of all filters of $\mathcal{E}$ is denoted by $\mathcal{F}(E)$.

Theorem 2.6 ([16])**.** *Let $\mathcal{E}$ be an EQ-algebra.*

(1) *For any $\emptyset \neq X \subseteq E$, the prefilter generated by X is written as $\langle X\rangle = \{x \in E \mid x_1 \to (x_2 \to (x_3 \to \cdots (x_n \to x)\cdots)) = 1$ for some $x_i \in X$ and $n \geq 1\}$. If $X = \{a\}$, then the prefilter $\langle a\rangle$ generated by $\{a\}$ is called a principal prefilter.*

(2) *If $\mathcal{E}$ is residuated, then $\langle X\rangle$ is a filter.*

(3) *$\langle x\rangle \cap \langle y\rangle = \langle x \vee y\rangle$, for all $x, y \in E$, where $\langle x\rangle$ denotes the principal prefilter generated by x.*

Definition 2.7 ([14])**.** *Let H be a filter of an EQ-algebra. Then:*

(1) *H is called an implicative filter if $z \to ((x \to y) \to x) \in H$ and $z \in H$ imply $x \in H$ for any $x, y, z \in E$.*

(2) *H is called a positive implicative filter if $x \to (y \to z) \in H$ and $x \to y \in H$, then $x \to z \in H$ for any $x, y, z \in E$.*

(3) *H is called an obstinate filter of $\mathcal{E}$ if, for all $x, y \in E$, $x, y \notin H$ implies $x \to y \in H$ and $y \to x \in H$.*

For any filter H of an EQ-algebra and $x, y \in E$, we define a relation $\approx_H$ on $\mathcal{E}$ as follows:

$$x \approx_H y \text{ iff } x \sim y \in H$$

In [17], we know that $\approx_H$ is a congruence relation on E. Define the factor algebra $\mathcal{E}/H = (E/H, \wedge, \odot, \sim_H, 1)$ as follows: $E/H = \{[x] \mid x \in E\}$, the operation $\wedge$ is defined by $[x] \wedge [y] = [x \wedge y]$, and similarly for the other operations. The ordering in $\mathcal{E}/H$ is defined by:

$$[x] \leq [y] \text{ iff } [x] \wedge [y] = [x] \text{ iff } x \wedge y \approx_H x \text{ iff } x \wedge y \sim x = x \to y \in H$$

Definition 2.8 ([4])**.** *An algebra $(E, \wedge, \to, 1)$ of type $(2, 2, 0)$ is called a Hertz-algebra provided that, for all $x, y, w \in E$, the following axioms hold:*

$(HE1)$ $x \to x = 1$.

$(HE2)$ $y \wedge (x \to y) = y$.

$(HE3)$ $x \wedge (x \to y) = x \wedge y$.

$(HE4)$ $x \to (y \wedge w) = (x \to y) \wedge (x \to w)$.

Definition 2.9 ([15])**.** *A BCK-algebra $(A, \to, 1)$ is an algebra of type $(2, 0)$, which satisfies the following conditions for any $x, y, w \in E$:*

$(B1)$ $(y \to w) \to ((w \to x) \to (y \to x)) = 1$.

$(B2)$ $y \to ((y \to x) \to x) = 1$.

$(B3)$ $x \to x = 1$.

$(B4)$ $x \to 1 = 1$.

$(B5)$ *If* $x \to y = 1$, $y \to x = 1$, *then* $x = y$.

Definition 2.10 ([8]). *An algebra* $(H, \to, 1)$ *of type* $(2, 0)$ *is said to be a Hilbert algebra, if for all* $x, y, w \in E$, *we have:*

$(HL1)$ $x \to (y \to x) = 1$.

$(HL2)$ $(x \to (y \to w)) \to ((x \to y) \to (x \to w)) = 1$.

$(HL3)$ *If* $x \to y = y \to x = 1$, *then* $x = y$.

Definition 2.11 ([3]). *If* $(E, \vee, \wedge, 1)$ *is a lattice, which satisfies* $x \leq y \to z$ *iff* $x \wedge y \leq z$ *for any* $x, y, z \in E$, *then the algebra* $(E, \vee, \wedge, \to, 1)$ *is said to be a Heyting-algebra.*

Definition 2.12 ([19, 20]). *If* $(L, \vee, \wedge, 0, 1)$ *is a distributive lattice satisfying* $0' = 1$, $1' = 0$, *and* $(x \wedge y)'' = x'' \wedge y''$, $(x \vee y)' = x' \wedge y'$ *and* $x''' = x'$ *hold for any* $x, y \in L$. *Then, the algebra* $(L, \vee, \wedge, ', 0, 1)$ *of type* $(2, 2, 1, 0, 0)$ *is said to be a semi-De Morgan algebra.*

3. Seminodes and nodes on EQ-algebras

In this section, we present the concepts of seminodes and nodes on EQ-algebras and study their related properties. Moreover, we consider the relationships among seminodes, nodes and some other elements on an EQ-algebra.

Definition 3.1. *Let* $\mathcal{E}$ *be an EQ-algebra and* $x \in E$. *Then,* x *is called a:*

(1) *seminode, if the set* $\{x \to y, y \to x\}$ *has a unique upper bound* 1, *for all* $y \in E$;

(2) *node, if either* $x \leq y$ *or* $y \leq x$ *for any* $y \in E$.

Let us denote the set of all seminodes of an EQ-algebra by $\mathcal{SN}(E)$ and the set of all nodes of an EQ-algebra by $\mathcal{ND}(E)$. Since $1 \in \mathcal{SN}(E)$ and $1 \in \mathcal{ND}(E)$, it readily follows that $\mathcal{SN}(E)$ and $\mathcal{ND}(E)$ are nonempty.

Example 3.2 ([5]). (1) Assume that $E = \{0, u, v, w, 1\}$ with $0 < u < v < w < 1$. Then, one can check that $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra, where the two operations $\otimes$ and $\sim$ are given by:

$\otimes$	0	u	v	w	1
0	0	0	0	0	0
u	0	0	0	0	u
v	0	0	0	0	v
w	0	0	u	u	w
1	0	u	v	w	1

$\sim$	0	u	v	w	1
0	1	w	v	v	0
u	w	1	w	w	u
v	v	w	1	w	v
w	v	w	w	1	w
1	0	u	v	w	1

Obviously, $\mathcal{SN}(E) = \mathcal{ND}(E) = \{0, u, v, w, 1\}$. But the element u is not a co-atom and dense element, w is not a dense element and a atom and v is not a dense element. Moreover, the involutive elements are $\{0, v, 1\}$.

(2) Suppose that $E = \{0, u, v, p, q, 1\}$ with $0 < u < v < p, q < 1$. Then, $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra, where the operations $\otimes$ and $\sim$ are given by the next tables:

$\otimes$	0	u	v	p	q	1
0	0	0	0	0	0	0
u	0	0	0	u	0	u
v	0	0	v	v	v	v
p	0	u	v	p	v	p
q	0	0	v	v	q	q
1	0	u	v	p	q	1

$\sim$	0	u	v	p	q	1
0	1	q	u	0	0	0
u	q	1	u	u	u	u
v	u	u	1	q	p	v
p	0	u	q	1	v	p
q	0	u	p	v	1	q
1	0	u	v	p	q	1

One can check that $\mathcal{SN}(E) = \{0, u, v, p, q, 1\}$ and $\mathcal{ND}(E) = \{0, u, v, 1\}$. Although p and q are not node elements, they are dense elements and co-atoms. In addition, the involutive elements are $\{0, 1\}$.

(3) Let $E = \{0, u, v, p, q, 1\}$ satisfies $0 < u, v < p < q < 1$. Then, $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra with respect to the following operations $\otimes$ and $\sim$:

$\otimes$	0	u	v	p	q	1
0	0	0	0	0	0	0
u	0	0	0	0	u	u
v	0	0	0	0	v	v
p	0	0	0	0	p	p
q	0	u	v	p	q	q
1	0	u	v	p	q	1

$\sim$	0	u	v	p	q	1
0	1	p	p	p	0	0
u	p	1	p	p	u	u
v	p	p	1	p	v	v
p	p	p	p	1	p	p
q	0	u	v	p	1	1
1	0	u	v	p	1	1

It is apparent that $\mathcal{SN}(E) = \{0, u, v, p, q, 1\}$ and $\mathcal{ND}(E) = \{0, p, q, 1\}$. Although u and v are atoms, they are not dense elements and nodes. Moreover, 0 and p are involutive elements, but they are not atoms and dense elements.

According to the above example, we see immediately that seminodes and nodes are different from dense elements, (co-)atoms and involutive elements in an EQ-algebra. In addition, they have the following properties:

Remark 3.3. Suppose $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra.

(1) If E is a chain, then each element of E is a node.

(2) If E has at most one node u, then $u = 1$. Therefore, it is neither an atom nor a co-atom.

(3) Each node of E is a seminode of E. But the converse is not true. In fact, by definitions of nodes and seminodes, we can easily check that each node is a seminode. Also, by Example 3.2 (3), we know that u and v are seminodes, but not nodes. Therefore, we conclude that a seminode element is more general than a node.

In general EQ-algebra, we can only obtain that $(q_1 \wedge q_2) \to q_3 \geq (q_1 \to q_3) \vee (q_2 \to q_3)$ and $q_1 \to (q_2 \wedge q_3) \leq (q_1 \to q_2) \wedge (q_1 \to q_3)$ hold. But when we define q_1, q_2, q_3 in the set $\mathcal{SN}(E)$, we shall prove the equations hold.

Proposition 3.4. *Let $\mathcal{E}$ be a lattice-ordered EQ-algebra. Then, the following hold, for all $q_1, q_2 \in \mathcal{SN}(E)$ and $q_3 \in E$:*

(1) $(q_1 \wedge q_2) \to q_3 = (q_1 \to q_3) \vee (q_2 \to q_3)$.

(2) $q_1 \to (q_2 \wedge q_3) = (q_1 \to q_2) \wedge (q_1 \to q_3)$.

Proof. (1) From the Proposition 2.3 (5) and (4), we get $q_1 \to q_2 = q_1 \to (q_1 \wedge q_2) \leq ((q_1 \wedge q_2) \to q_3) \to (q_1 \to q_3) \leq ((q_1 \wedge q_2) \to q_3) \to ((q_1 \to q_3) \vee (q_2 \to q_3))$. Similarly, we obtain that $q_2 \to q_1 \leq ((q_1 \wedge q_2) \to q_3) \to ((q_1 \to q_3) \vee (q_2 \to q_3))$. Since $q_1 \in \mathcal{SN}(E)$, it implies that $(q_1 \to q_2) \vee (q_2 \to q_1) = 1$, and so $((q_1 \wedge q_2) \to q_3) \to ((q_1 \to q_3) \vee (q_2 \to q_3)) = 1$. Thus, we obtain $((q_1 \wedge q_2) \to q_3) \leq ((q_1 \to q_3) \vee (q_2 \to q_3))$. In addition, because $q_1 \wedge q_2 \leq q_1, q_2$, we have $q_1 \to q_3$, $q_2 \to q_3 \leq (q_1 \wedge q_2) \to q_3$. Thus, it readily follows that $(q_1 \to q_3) \vee (q_2 \to q_3) \leq (q_1 \wedge q_2) \to q_3$. Therefore, we see immediately that $(q_1 \wedge q_2) \to q_3 = (q_1 \to q_3) \vee (q_2 \to q_3)$.

(2) By Proposition 2.3 (5) and (4), we obtain $q_2 \to q_3 = q_2 \to (q_2 \wedge q_3) \leq (q_1 \to q_2) \to (q_1 \to (q_2 \wedge q_3)) \leq ((q_1 \to q_2) \wedge (q_1 \to q_3)) \to (q_1 \to (q_2 \wedge q_3))$. Analogously, $q_3 \to q_2 \leq ((q_1 \to q_2) \wedge (q_1 \to q_3)) \to (q_1 \to (q_2 \wedge q_3))$ holds. Since $q_2 \in \mathcal{SN}(E)$, we obtain $(q_2 \to q_3) \vee (q_3 \to q_2) = 1$, and then $((q_1 \to q_2) \wedge (q_1 \to q_3)) \to (q_1 \to (q_2 \wedge q_3)) = 1$. Thus, it follows that $(q_1 \to q_2) \wedge (q_1 \to q_3) \leq q_1 \to (q_2 \wedge q_3)$. In addition, since $q_2 \wedge q_3 \leq q_2, q_3$, it readily implies $q_1 \to (q_2 \wedge q_3) \leq q_1 \to q_2, q_1 \to q_3$, and so $q_1 \to (q_2 \wedge q_3) \leq (q_1 \to q_2) \wedge (q_1 \to q_3)$. Therefore, it readily follows $q_1 \to (q_2 \wedge q_3) = (q_1 \to q_2) \wedge (q_1 \to q_3)$. □

Theorem 3.5. *Let $\mathcal{E}$ be a lattice-latticed EQ-algebra. Then, the following conclusions hold:*

(1) *Denote $BL(E) = \{u \in E \mid u \vee m = 1,\ u \wedge m = 0$ for some $m \in E\}$. Then, $\mathcal{ND}(E) \cap BL(E) = \{0, 1\}$.*

(2) *If $\mathcal{E}$ is distributive, then $(\mathcal{SN}(E), \wedge, \vee)$ is a distributive lattice.*

(3) *$(\mathcal{ND}(E), \vee, \wedge)$ is a distributive lattice, too.*

Proof. (1) It is clear that $\{0,1\} \subseteq \mathcal{ND}(E) \cap BL(E)$. Conversely, for any $u \in \mathcal{ND}(E) \cap BL(E)$, we have $u \in \mathcal{ND}(E)$ and $u \in BL(E)$. From $u \in \mathcal{ND}(E)$, we know that either $u \leq m$ or $m \leq u$ for any $m \in E$. Moreover, it follows from $u \in BL(E)$ that $u \vee m = 1$ and $u \wedge m = 0$ for some $m \in E$, which implies that $u \vee m = m$, $u \wedge m = u$ or $u \vee m = u$, $u \wedge m = m$. Hence, $u = 0$ or $u = 1$, and so $u \in \{0,1\}$. Therefore, we obtain $\mathcal{ND}(E) \cap BL(E) = \{0,1\}$.

(2) Firstly, we prove $((u \wedge m) \to w) \vee (w \to (u \wedge m)) = 1$ for any $w \in E$ and $u, m \in \mathcal{SN}(E)$. In fact, by Proposition 3.4, we have $((u \wedge m) \to w) \vee (w \to (u \wedge m)) = ((u \to w) \vee (m \to w)) \vee ((w \to u) \wedge (w \to m)) = [(u \to w) \vee (m \to w) \vee (w \to u)] \wedge [(u \to w) \vee (m \to w) \vee (w \to m)] \geq [(u \to w) \vee (w \to u)] \wedge [(m \to w) \vee (w \to m)] = 1$. Thus, it readily follows that $u \wedge m \in \mathcal{SN}(E)$.

Now, we shall prove that $((u \vee m) \to w) \vee (w \to (u \vee m)) = 1$ for any $w \in E$. Indeed, by Proposition 2.3 (8), we obtain $((u \vee m) \to w) \vee (w \to (u \vee m)) = ((u \to w) \wedge (m \to w)) \vee (w \to (u \vee m)) = ((u \to w) \vee (w \to (u \vee m))) \wedge ((m \to w) \vee (w \to (u \vee m))) \geq [(u \to w) \vee (w \to u)] \wedge [(m \to w) \vee (w \to m)] = 1$ Therefore, we get that $u \vee m \in \mathcal{SN}(E)$, and so $(\mathcal{SN}(E), \wedge, \vee)$ is a distributive lattice.

(3) Let $u, m \in \mathcal{ND}(E)$. It suffices to show that $u \vee m$, $u \wedge m \in \mathcal{ND}(E)$. Assume that $w \in E$. If $w \leq u, m$, then $w \leq u \wedge m$. And, if $u \leq w \leq m$ or $m \leq w \leq u$, then $u \wedge m \leq u \leq w$ or $u \wedge m \leq m \leq w$. Thus $u \wedge m \in \mathcal{ND}(E)$. Analogously, $u \vee m \in \mathcal{ND}(E)$ also holds. Therefore, $(\mathcal{ND}(E), \vee, \wedge, 0, 1)$ is a lattice. By definition of $\mathcal{ND}(E)$, we see immediately that it is a distributive lattice. □

Theorem 3.6. *Let $\mathcal{E}$ be an EQ-algebra. If for any $x, y \in \mathcal{SN}(E)$, $x \wedge (x \to y) = x \wedge y$ holds and $\mathcal{SN}(E)$ is closed with the operator $\to$. Then, $(\mathcal{SN}(E), \wedge, \vee, \to, 1)$ is a Hertz-algebra and a Heyting-algebra.*

Proof. Firstly, we prove that it is a Hertz-algebra. Obviously, (HE1) holds. By Proposition 2.3 (3), we know that (HE2) holds. By hypothesis, the (HE3) is valid. Moreover, from Proposition 3.4 (2), it implies that (HE4) holds. Hence, $(\mathcal{SN}(E), \wedge, \vee, \to, 1)$ is a Hertz-algebra.

Now, we show that it is a Heyting-algebra. For any $x, y, w \in \mathcal{SN}(E)$, if $x \leq y \to w$, then $x \wedge y \leq y \wedge (y \to w) = y \wedge w \leq w$, i.e. $x \wedge y \leq w$. Conversely, if $x \wedge y \leq w$, then it follows that $x \leq y \to x = 1 \wedge (y \to x) = (y \to y) \wedge (y \to x) = y \to (y \wedge x) \leq y \to w$ by Proposition 2.3 (3) and Proposition 3.4 (2). Therefore, the conclusion holds. □

4. Nodal filters on EQ-algebras

In this section, we introduce the notion of an nodal filter on EQ-algebras and give the equivalent characterization of it. Furthermore, the relationships between nodal filters and node elements, as well as between nodal filters and other filters are discussed.

Definition 4.1. *Let H be a filter of an EQ-algebra. If H is a node in poset $(\mathcal{F}(E), \subseteq)$, then it is said to be an nodal filter (for short, n-filter).*

Let us denote the set of all n-filters of $\mathcal{E}$ by $NF(E)$ in the sequel.

Example 4.2 ([16])**.** Let $E = \{0, u, v, p, q, 1\}$ such that $0 < u, v < p < 1$, $0 < v < q < 1$. Then, $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra, where the operations $\otimes$ and $\sim$ are given by the following tables:

$\otimes$	0	u	v	p	q	1
0	0	0	0	0	0	0
u	0	u	0	u	0	u
v	0	0	0	0	v	v
p	0	u	0	u	v	p
q	0	0	v	v	q	q
1	0	u	v	p	q	1

$\sim$	0	u	v	p	q	1
0	1	q	p	v	u	0
u	q	1	v	p	0	u
v	p	v	1	q	p	v
p	v	p	q	1	v	p
q	u	0	p	v	1	q
1	0	u	v	p	q	1

It is easy for us to check that $\mathcal{F}(E) = \{\{1\}, \{q, 1\}, \{u, p, 1\}, \{u, v, p, q, 1\}, E\}$, but $\mathcal{NF}(E) = \{\{1\}, \{u, v, p, q, 1\}, E\}$.

Example 4.3 ([7])**.** Suppose that $E = \{0, u, v, p, q, 1\}$ with $0 < u < v, p < q < 1$. Then, we can verify that $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra, where the operations $\otimes$ and $\sim$ are given by the next tables:

$\otimes$	0	u	v	p	q	1
0	0	0	0	0	0	0
u	0	0	0	0	0	u
v	0	0	0	0	0	v
p	0	0	0	0	0	p
q	0	0	0	0	q	q
1	0	u	v	p	q	1

$\sim$	0	u	v	p	q	1
0	1	1	u	u	u	u
u	1	1	u	u	u	u
v	u	u	1	p	p	p
p	u	u	p	1	p	p
q	u	u	p	p	1	q
1	u	u	p	p	q	1

Obviously, $\mathcal{F}(E) = \{\{1\}, \{q, 1\}, \{v, q, 1\}, \{u, p, q, 1\}, \{u, v, p, q, 1\}, E\}$, but $NF(E) = \{\{1\}, \{q, 1\}, \{u, v, p, q, 1\}, E\}$.

From the above Examples, we see immediately that $n-$filters are distinct from filters of EQ-algebras.

Theorem 4.4. *Let H be a filter of an idempotent and good EQ-algebra. Then, H is an n-filter if and only if $u \in H$ and $v \notin H$ imply $v < u$ for any $u, v \in E$.*

Proof. $(\Rightarrow)$ Assume that $u \in H$ and $v \notin H$ for any $u, v \in E$. Then, it follows from H is an n-filter that $\langle u \rangle \subseteq H$ and $H \subseteq \langle v \rangle$, which implies $u \in \langle v \rangle$. Hence, $v^n \leq u$ for some $n \in N$. Moreover, by assumption, we get $v = v^n$. If $v = u$, then $v \in H$, which is a contradiction. Hence, it readily follows that $v < u$.

$(\Leftarrow)$ Suppose that $v < u$, for all $u \in H$ and $v \notin H$. If there exists a filter J such that H and J are incomparable. Then, $u \in H \setminus J$ and $v \in J \setminus H$ for some

$u, v \in E$. Now, since J is a filter and $v < u$, it implies that $u \in J$, which is impossible. Hence, either $H \subseteq J$ or $J \subseteq H$ for any filter J of $\mathcal{E}$. Therefore, we obtain that H is an n-filter. □

Corollary 4.5. *If $\mathcal{E}$ is linearly ordered, then each filter is an n-filter.*

Proof. For any filter H such that $u \in H$ and $v \notin H$. Since $u \in \mathcal{ND}(E)$, we get $v < u$. Indeed, if $u \leq v$, then $v \in H$ as H is a filter. Hence, by the Theorem above, we obtain that H is an n-filter. □

Proposition 4.6. *Let H be a filter of a good EQ-algebra. If $u \in H$ is a node, then H is an n-filter. Especially, the filter $\langle u \rangle$ generated by u is also an n-filter.*

Proof. Assume H is a filter of $\mathcal{E}$ and $v \notin H$. If $u \in \mathcal{ND}(E)$, then either $u \leq v$ or $v \leq u$. If $u \leq v$, then $v \in H$, which is a contradiction. Thus, it readily follows that $v < u$. By Theorem 4.4, we obtain that H is an n-filter. □

Remark 4.7. In Example 4.2, we obtain that $\{u, v, p, q, 1\}$ is an n-filter of $\mathcal{E}$, but $v \notin \mathcal{ND}(E)$, which implies that the converse of Proposition 4.6 may not hold, in general.

Proposition 4.8. *Let $\mathcal{E}$ be an idempotent and good EQ-algebra.*

(1) *If $\langle u \rangle \in NF(E)$, then $u \in \mathcal{ND}(E)$.*

(2) *If E has n node elements, then it has at least n n-filters.*

Proof. (1) For any $v \in E$, then either $v \in \langle u \rangle$ or $v \notin \langle u \rangle$. If $v \notin \langle u \rangle$, then we obtain that $v < u$ by Theorem 4.4. If $v \in \langle u \rangle$, then $u^n = u \leq v$ for some $n \in N$. Hence, u is a node element.

(2) Let $u \in \mathcal{ND}(E)$. Then, it follows that $\langle u \rangle$ is a nodal filter by Proposition 4.6. Now, assume u and v are two nodes of E. If $\langle u \rangle = \langle v \rangle$, then $u \in \langle v \rangle$ and $v \in \langle u \rangle$. Since $u^2 = u$ and $v^2 = v$, we obtain $u \geq v$ and $v \geq u$, which implies that $u = v$. Therefore, we see immediately that it has at least n n-filters. □

Combining Proposition 4.6 and Proposition 4.8, we know that there is a one-to-one correspondence between nodal principle filters and node elements in an idempotent EQ-algebra.

Proposition 4.9. *Suppose that H is an n-filter of a residuated EQ-algebra. Then, for any $u \in \mathcal{ND}(E)$, $H(u) = \langle H \cup \{u\} \rangle$ is an n-filter.*

Proof. If $u \in H$, then $H(u) = H$. Thus, it readily implies that $H(u)$ is an n-filter of $\mathcal{E}$. By the above Proposition, we obtain that $\langle u \rangle$ is an n-filter. Now, suppose that $J \in \mathcal{F}(E)$ and $J \nsubseteq H(u)$. Note that if $J \subseteq H$ or $J \subseteq \langle u \rangle$, then $J \subseteq H(u)$, which is contradiction. Hence, we get H, $\langle u \rangle \subseteq J$. If $v \in H(u)$, then $u \to_n v \in H \subseteq J$ for some $n \in N$. Thus, we know that $v \in J$ as J is a filter. Hence, $H(u) \subseteq J$, which readily follows that $H(u)$ is an n-filter. □

Example 4.10. In Example 4.2, we know that $H = \{1\}$ is an n-filter. And, one can check that $p \notin \mathcal{ND}(E)$ and $H(p) = \{u, p, 1\} \notin NF(E)$. Moreover, $J = \{q, 1\} \notin NF(E)$ but $J(v) = H(v) = E \in NF(E)$. That is to say, the converse of Proposition 4.9 may not hold, in general.

Proposition 4.11. *Assume that E_1 and E_2 are two idempotent and good EQ-algebras and $g : E_1 \to E_2$ is a homomorphism.*

(1) *If g is injective and $H \in NF(E_2)$, then $g^{-1}(H) = \{a \in E_1 \mid g(a) \in H\} \in NF(E_1)$.*

(2) *If g is surjective and $H \in NF(E_1)$, then $g(H) \in NF(E_2)$.*

Proof. (1) Firstly, we show that $g^{-1}(H)$ is a filter. Since $g(1_{E_1}) = 1_{E_2} \in H$, we get $1_{E_1} \in g^{-1}(H)$, i.e. (F1) holds. For any $a, b \in g^{-1}(H)$, it implies that $g(a), g(b) \in H$. And, because $H \in NF(E_2)$, we obtain $g(a \otimes b) = g(a) \otimes g(b) \in H$, which implies $(a \otimes b) \in g^{-1}(H)$, i.e. (F2) holds. For any $a, b \in E_1$, assume $a, a \to b \in g^{-1}(H)$. Then, $g(a), g(a \to b) \in H$, i.e. $g(a), g(a) \to g(b) \in H$. Thus, $g(b) \in H$ and so $b \in g^{-1}(H)$, i.e. (F3) holds. Let $a \to b \in g^{-1}(H)$. Then, $g(a) \to g(b) = g(a \to b) \in H$, which readily follows that $(g(a) \otimes g(c)) \to (g(b) \otimes g(c)) \in H$, where $c \in E_1$ and $g(c) \in H$, i.e. $g((a \otimes c) \to (b \otimes c)) \in H$. Hence $(a \otimes c) \to (b \otimes c) \in g^{-1}(H)$, i.e. (F4) holds. Therefore, we see immediately that $g^{-1}(H)$ is a filter.

Now, we shall prove that $g^{-1}(H)$ is an n-filter. Let $a \in g^{-1}(H)$ and $b \notin g^{-1}(H)$. Then, $g(a) \in H$ and $g(b) \notin H$. Since H is an n-filter and $a^2 = a$ holds, for all $a \in E_1$, we have $g(b) < g(a)$ by Theorem 4.4, which implies that $g(b \to a) = g(b) \to g(a) = 1_{E_2}$. Moreover, since $g(1_{E_1}) = 1_{E_2}$ and g is injective, we obtain that $b \to a = 1_{E_1}$ and so $b \leq a$. If $b = a$, then $g(b) = g(a)$, which generates a contradiction, and so $b < a$. Now, by Theorem 4.4, we see immediately that $g^{-1}(H)$ is an n-filter.

(2) Analogously, we show that $g(H)$ is a filter firstly. Since $1_{E_2} = g(1_{E_1}) \in g(H)$, it implies that (F1) holds. Let $a, b \in g(H)$. Since g is surjective, there exist $a_1, b_1 \in H$ such that $g(a_1) = a$, $g(b_1) = b$. Hence $a \otimes b = g(a_1) \otimes g(b_1) = g(a_1 \otimes b_1) \in g(H)$, i.e. (F2) holds. Now, let a, $a \to b \in g(H)$, i.e. $g(a_1)$, $g(a_1) \to g(b_1) = g(a_1 \to b_1) \in g(H)$. Thus, we get a_1, $a_1 \to b_1 \in H$, and so $b_1 \in H$. Hence, we obtain that $b = g(b_1) \in g(H)$, i.e. (F3) holds. Moreover, let $a \to b \in g(H)$. Then, $g(a_1) \to g(b_1) = g(a_1 \to b_1) \in g(H)$, i.e. $a_1 \to b_1 \in H$. Hence, $(a_1 \otimes c_1) \to (b_1 \otimes c_1) \in H$, where $c_1 \in E_1$, and so $(g(a_1) \otimes g(c_1)) \to (g(b_1) \otimes g(c_1)) = g((a_1 \otimes c_1) \to (b_1 \otimes c_1)) \in H$, i.e. (F4) holds. Therefore, we see immediately that $g(H)$ is a filter.

Now, we prove $g(H)$ is an n-filter. Let $a \in g(H)$ and $b \notin g(H)$. Since g is surjective, there exists $a_1 \in H$ such that $g(a_1) = a$. But there is no $b_1 \in H$ such that $g(b_1) = b$. Moreover, because $b_1 \notin H$, then we get $b_1 < a_1$ and so $b_1 \to a_1 = 1$. Thus, it implies $g(b_1) \to g(a_1) = 1$, i.e. $g(b_1) \leq g(a_1)$. If $g(b_1) = g(a_1)$, i.e. $a = b$, which is a contradiction. Hence, $g(b_1) < g(a_1)$, i.e.

$b < a$. Therefore, we see immediately that $g(H)$ is an n-filter by Theorem 4.4. □

In what follows, we will prove the relationships among n-filters, (positive) implicative filters, prime filters and obstinate filters, in genaral. Futhermore, we discuss the relationships among them.

Definition 4.12 ([10]). *Let H be a proper filter of an EQ-algebra. Then, H is called prime if $x \to y \in H$ or $y \to x \in H$ for any $x, y \in E$.*

Example 4.13. (1) In Example 4.2, we obtain that $H_1 = \{1\}$ is an n-filter. Now, since $p \vee q = 1 \in \{1\}$, but $p, q \notin \{1\}$, we obtain that it is not a prime filter. Moreover, $H_2 = \{u, p, 1\} \notin NF(E)$, but it is a implicative filter and a prime filter. Furthermore, $H_3 = \{q, 1\}$ is a obstinate filter, but $H_3 \notin NF(E)$.

(2) In Example 4.3, although $H_3 = \{1\} \in NF(E)$, it is not a positive implicative filter as $p \to (1 \to v) = 1 \in \{1\}$, $p \to 1 = 1 \in \{1\}$, but $p \to v = u \in \{1\}$. Also, $H_2 = \{u, p, q, 1\}$ is a positive implicative and obstinate filter, but it is not an n-filter.

Lemma 4.14. *Let H be a filter of a prelinear and lattice-orderd EQ-algebra. Then, H is a prime filter iff for any $x, y \in E$, $x \vee y \in H$ implies $x \in H$ or $y \in H$.*

Proof. ($\Rightarrow$) Let $x \to y \in H$ and $x \vee y \in H$. Since $(x \vee y) \leq (x \to y) \to y$, we have $(x \to y) \to y \in H$, and so $y \in H$. As to another case, we can immediately obtain that $x \in H$.

($\Leftarrow$) Let $x, y \in E$. Since $(x \to y) \vee (y \to x) = 1 \in H$, we have $x \to y \in H$ or $y \to x \in H$ by assumption. Therefore, it readily follows that H is a prime filter. □

Proposition 4.15. *Each non principal n-filter H is a prime filter of a prelinear EQ-algebra.*

Proof. Suppose there are $x, y \in E$ satisfying $x \vee y \in H$ but $x \notin H$, $y \notin H$. Then, we know that $\langle x \vee y \rangle \subseteq H$, $\langle x \rangle \nsubseteq H$ and $\langle y \rangle \nsubseteq H$. And, by the fact that H is a nodal filter, it follows that $H \subseteq \langle x \rangle$ and $H \subseteq \langle y \rangle$. Thus, by Theorem 2.6 (3), we obtain $H \subseteq \langle x \rangle \cap \langle y \rangle = \langle x \vee y \rangle$. For this reason, we get that $H = \langle x \vee y \rangle$, which is a contradiction. Hence, we obtain that $x \in H$ or $y \in H$, and so H is a prime filter. □

Proposition 4.16. *Let H be an obstinate filter of a bounded EQ-algebra. If $(x \otimes y') \leq y$ for any $x, y \in E$, then H is an n-filter.*

Proof. Assume H is not an n-filter. Then, we get $J \nsubseteq H$ and $H \nsubseteq J$ for some $J \in \mathcal{F}(E)$. Thus, there are $u, v \in E$ such that $u \in H/J$ and $v \in J/H$. It follows from H is an obstinate filter that $v' = v \to 0 \in H$, and so $u \otimes v' \in H$. Moreover, since $(u \otimes v') \leq v$, we get $v \in H$, which generates a contradiction. Hence, we see immediately that H is an n-filter. □

Proposition 4.17. *Suppose H is an implicative filter of a good EQ-algebra. If d is a dense element for any $d \in E$, then H is an n-filter.*

Proof. Suppose H is not an n-filter. Firstly, we show that $d'' \to d \in H$ for any $d \in E$. Since $d' \to 0 \leq d' \to d$, we have $d'' \to (d' \to d) = 1 \in H$. And, because $d \leq d'' \to d$, we get $d' \to d \leq (d'' \to d)' \to d$. Thus $d'' \to (d' \to d) \leq d'' \to [(d'' \to d)' \to d] \in H$, which implies that $1 \to [(d'' \to d)' \to (d'' \to d)] = (d'' \to d)' \to (d'' \to d) = d'' \to [(d'' \to d)' \to d] \in H$. By definition of an implicative filter, we know that $d'' \to d \in H$.

Assume H is not an n-filter of $\mathcal{E}$. Then, $J \nsubseteq H$ and $H \nsubseteq J$ for some $J \in \mathcal{F}(E)$. Thus, $v \in J/H$ for some $v \in E$. By the conclusion above, we obtain that $v'' \to v \in H$. Since v is a dense element, we have $v'' \to v = v \in H$, which generates a contradiction. Hence, we see immediately that H is an n-filter. □

Proposition 4.18. *Assume H is a positive implicative filter of a residuated EQ-algebra. If $y \to (x \odot y) = x \to y$ holds for any $x, y \in E$, then H is an n-filter.*

Proof. Assume that H is not an n-filter. Firstly, we shall prove that for any $x \in E$, $x \to x^2 \in H$. Since $x \to (x \to x^2) = x^2 \to x^2 = 1 \in H$ and $x \to x = 1 \in H$. Then, by definition of a positive implicative filter, we get $x \to x^2 \in H$. If H is not an n-filter, then there is $J \in \mathcal{F}(E)$ satisfying $J \nsubseteq H$ and $H \nsubseteq J$. Moreover, assume $x \in H/J$ and $y \in J/H$. By the conclusion above, it follows that $y \to y^2 \in H$. Then, $x \otimes (y \to y^2) \in H$. And, because $x \otimes (y \to y^2) \leq y \to (x \otimes y^2) \leq y \to (x \otimes y) = x \to y$, we have $x \to y \in H$, and so $y \in H$, which is a contradiction. Hence, we obtain that H is an n-filter. □

Proposition 4.19. *Let H be a non principal n-filter of an EQ-algebra $\mathcal{E}$. Then, $(E/H, \wedge, \odot, \sim_H, 1)$ is linearly ordered.*

Proof. Let x/H, $y/H \in E/H$ and $x/H \nleq y/H$. Then, we can obtain that $x \to y \notin H$. Moreover, because H is a non principal n-filter, then from Theorem 4.15 that we get H is a prime filter. Hence, it readily follows that $y \to x \in H$, and so $[y] \leq [x]$. Thus, we see immediately that E/H is a chain. □

Lemma 4.20 ([9])**.** *Assume θ is a congruence relation on a separated EQ-algebra. Then, $F = [1]_\theta = \{a \in E \mid a\theta 1\}$ is a filter.*

Theorem 4.21. *Assume $\mathcal{E}$ is an EQ-algebra. Then, $[1]_\theta$ is an n-filter iff θ is a node of $Con(E)$, where $Con(E)$ denotes the set of all congruence relation of E.*

Proof. Note that the mapping $\theta \mapsto F_\theta$ of $Con(E)$ on to $NF(E)$ is an isomorphism and F_θ is an n-filter iff it is a node of $NF(E)$. □

5. The structures of the set of all nodal filters on EQ-algebras

In this section, we study the algebraic properties $NF(E)$ and topological properties of $NP(E)$ on EQ-algebras.

Let $O, J \in NF(E)$. Define five operations as follows:

$$O \sqcap J := O \cap J, O \sqcup J := \langle O \cup J \rangle, O \to J := \{a \in E \mid O \cap \langle a \rangle \subseteq J\},$$
$$O \otimes J := \{o \otimes j \mid o \in O, j \in J\}, O' := O \to \{1\}.$$

Proposition 5.1. *Let $\mathcal{E}$ be an EQ-algebra. Then, for any $O, J \in NF(E)$, the following properties hold:*

(1) *$O \sqcap J$, $O \sqcup J \in NF(E)$.*

(2) *$O \to J \in NF(E)$.*

(3) *$O \otimes J \in NF(E)$ and $O \otimes J = O \cup J$.*

Proof. (1) For any $K \in \mathcal{F}(E)$. If $O, J \subseteq K$, then $O \sqcup J = \langle O \cup J \rangle \subseteq K$. And, if $K \subseteq O, J$, we have $K \subseteq O \subseteq \langle O \cup J \rangle = O \sqcup J$. Now, if $O \subseteq K \subseteq J$ or $J \subseteq K \subseteq O$, we obtain that $K \subseteq \langle O \cup J \rangle = O \sqcup J$. Thus, it readily follows that $O \sqcup J \in NF(E)$. Analogously, we can prove that $O \sqcap J \in NF(E)$ hold.

(2) If $O = J$, we can get that $O \to J = E \in NF(E)$. Now, if $O \neq J$. Suppose that $O \subseteq J$. Then, $O \cap \langle a \rangle \subseteq O \subseteq J$ for any $a \in E$, which implies that $O \to J = E$. If $J \subseteq O$, we shall prove that $O \to J = J$. In fact, for any $a \in O \to J$, if $a \in J$, then $O \to J \subseteq J$. And, if $a \notin J$ and $a \in O$, we get $\langle a \rangle \subseteq O$. Thus, $\langle a \rangle = O \cap \langle a \rangle \subseteq J$, which is a contradiction. Suppose that $a \notin J$ and $a \notin O$. Then, we have $O \subseteq \langle a \rangle$, which means $O = O \cap \langle a \rangle \subseteq J$. Moreover, because $J \subseteq O$, we get that $O = J$, which is a contradiction. Hence, $O \to J \subseteq J$. Conversely, for any $a \in J$, we can easily get $\langle a \rangle \subseteq J$, which implies $O \cap \langle a \rangle \subseteq \langle a \rangle \subseteq J$, that is $a \in O \to J$. Hence $J \subseteq O \to J$, and so $O \to J = J$.

(3) If $O \subseteq J$, then $O \otimes J = \{o \otimes j \mid o \in O, j \in J\} = J \in NF(E)$. Similarly, if $J \subseteq O$, then $O \otimes J = O \in NF(E)$. In any cases, $O \otimes J = O$ or J holds. Thus, we see immediately that $O \otimes J = O \cup J$. □

Remark 5.2. In particular, we know that $H' := H \to \{1\} \in NF(E)$ for any $H \in NF(E)$.

Proposition 5.3. *Let $\mathcal{E}$ be an EQ-algebra. Then, for any $O, J, K \in NF(E)$, the following properties hold:*

(1) *$E \to O = O$, $O \to O = E$, $O \to E = E$, $\{1\} \to O = E$.*
(2) *$O' = \{1\}$, $O'' = E$, for $O \neq \{1\}$.*
(3) *$O \to J' = J \to O'$ for $O, J \neq \{1\}$.*
(4) *$O \subseteq J$ implies $J \to K \subseteq O \to K$, $K \to O \subseteq K \to J$.*
(5) *$O \subseteq J$ iff $O \to J = E$.*
(6) *$O \subseteq J \to O$ and $O, J \subseteq O \otimes (O \to J)$.*
(7) *$O \otimes (J \otimes K) = (O \otimes J) \otimes K$.*

Proof. (1) By definition, we have $E \to O = \{a \in E \mid E \cap \langle a \rangle \subseteq O\} = \{a \in E \mid \langle a \rangle \subseteq O\} = O$. Similarly, we can prove other equations hold.

(2) By definition, it readily implies $O' = O \to \{1\} = \{a \in E \mid O \cap \langle a \rangle \subseteq \{1\}\}$. Now, let $a \in O'$ and $a \neq 1$. If $a \in O$, then $O \cap \langle a \rangle = \langle a \rangle \nsubseteq \{1\}$, which is a contradiction. Thus $a = 1$, and so $O' = O \to \{1\} = \{1\}$. Furthermore, by (1), we see immediately that $O'' = O' \to \{1\} = \{1\} \to \{1\} = E$.

(3) By (2), we get that $O' = J' = \{1\}$. Then, $O \to J' = O \to \{1\} = O' = \{1\}$. Similarly, we can obtain $J \to O' = \{1\}$. Hence, we obtain that $O \to J' = J \to O'$.

(4) For any $a \in J \to K$, we get $J \cap \langle a \rangle \subseteq K$. And, since $O \subseteq J$, it readily follows that $O \cap \langle a \rangle \subseteq J \cap \langle a \rangle \subseteq K$. Thus $a \in O \to K$. That is $J \to K \subseteq O \to K$. Analogously, we can obtain that $K \to O \subseteq K \to J$.

(5) By definition, we know that $O \subseteq J$ iff $\langle a \rangle \cap O \subseteq J$ holds for any $a \in E$ iff $O \to J = E$.

(6) By the proof of Proposition 5.1, we obtain that if $J \subseteq O$, then $O \otimes (O \to J) = O \otimes J = O$ and $J \to O = E$. And, if $O \subseteq J$, then $O \otimes (O \to J) = E$ and $J \to O = J$. Therefore, in any case, we have $O \subseteq J \to O$ and $O, J \subseteq O \otimes (O \to J)$.

(7) The proof is clear. □

Proposition 5.4. *Let $\mathcal{E}$ be an EQ-algebra. Then, $(NF(E), \sqcup, \sqcap)$ is a bounded distributive lattice.*

Proof. By Proposition 5.1 (1), we know that $(NF(E), \sqcup, \sqcap)$ is a lattice. Next we shall show that $O \cap \langle J \cup K \rangle = \langle \langle O \cap J \rangle \cup \langle O \cap K \rangle \rangle$ holds for any $O, J, K \in NF(E)$. Let us consider the following six cases:

Case 1. Assume $O \subseteq J \subseteq K$. Then, $O \cap \langle J \cup K \rangle = O \cap K = O = \langle O \cup O \rangle = \langle \langle O \cap J \rangle \cup \langle O \cap K \rangle \rangle$.

Case 2. Assume $O \subseteq K \subseteq J$. Then, $O \cap \langle J \cup K \rangle = O \cap J = O = \langle O \cup O \rangle = \langle \langle O \cap J \rangle \cup \langle O \cap K \rangle \rangle$.

Case 3. Assume $K \subseteq O \subseteq J$. Then, $O \cap \langle J \cup K \rangle = O \cap J = O = \langle O \cup K \rangle = \langle \langle O \cap J \rangle \cup \langle O \cap K \rangle \rangle$.

Case 4. Assume $K \subseteq J \subseteq O$. Then, $O \cap \langle J \cup K \rangle = O \cap J = J = \langle J \cup K \rangle = \langle \langle O \cap J \rangle \cup \langle O \cap K \rangle \rangle$.

Case 5. Assume $J \subseteq K \subseteq O$. Then, $O \cap \langle J \cup K \rangle = O \cap K = K = \langle J \cup K \rangle = \langle \langle O \cap J \rangle \cup \langle O \cap K \rangle \rangle$.

Case 6. Assume $J \subseteq O \subseteq K$. Then, $O \cap \langle J \cup K \rangle = O \cap K = O = \langle J \cup O \rangle = \langle \langle O \cap J \rangle \cup \langle O \cap K \rangle \rangle$.

Hence, we obtain that $(NF(E), \sqcup, \sqcap)$ is a bounded distributive lattice. □

Theorem 5.5. *Assume that $\mathcal{E}$ is an EQ-algebra. Then, $(NF(E), \sqcap, \to, E)$ is a Hertz-algebra.*

Proof. It is apparent that (HE1) is valid. By Proposition 5.3 (6), we know that (HE2) holds. For (HE3), if $O \subseteq J$, then $O \sqcap (O \to J) = O \sqcap E = O = O \sqcap J$. And, if $J \subseteq O$, then $O \sqcap (O \to J) = O \sqcap J$. Hence, it implies that (HE3) holds. Now, we prove that (HE4) is valid and we consider the following scenarios:

Case 1. Suppose that $O \subseteq J \subseteq K$. Then, $O \to (J \sqcap K) = O \to J = E = E \sqcap E = (O \to J) \sqcap (O \to K)$.

Case 2. If $O \subseteq K \subseteq J$, it follows that $O \to (J \sqcap K) = O \to K = E = E \sqcap E = (O \to J) \sqcap (O \to K)$.

Case 3. If $J \subseteq O \subseteq K$, we conclude that $O \to (J \sqcap K) = O \to J = J = J \sqcap E = (O \to J) \sqcap (O \to K)$.

Case 4. Suppose $J \subseteq K \subseteq O$, we obtain that $O \to (J \sqcap K) = O \to J = J = J \sqcap K = (O \to J) \sqcap (O \to K)$.

Case 5. If $K \subseteq O \subseteq J$, it implies that $O \to (J \sqcap K) = O \to K = K = E \sqcap K = (O \to J) \sqcap (O \to K)$.

Case 6. If $K \subseteq J \subseteq O$, we have $O \to (J \sqcap K) = O \to K = K = J \sqcap K = (O \to J) \sqcap (O \to K)$.

Hence, $(HE4)$ holds. Therefore, we obtain that $(NF(E), \sqcap, \to, E)$ is a Hertz-algebra. □

Theorem 5.6. *Let $\mathcal{E}$ be an EQ-algebra. Then, the following properties hold:*

(1) $(NF(E), \otimes, \{1\})$ *is a commutative monoid.*

(2) $(NF(E), \to, E)$ *is a Hilbert algebra.*

(3) $(NF(E), \sqcup, \sqcap, \to, E)$ *is a Heyting algebra.*

(4) $(NF(E), \to, E)$ *is a BCK-algebra.*

Proof. (1) If $O \subseteq J$, then $O \otimes J = J = J \otimes O$. And, if $J \subseteq O$, we get $O \otimes J = O = J \otimes H$. Moreover, because $O \otimes \{1\} = O = \{1\} \otimes O$, we see immediately that $(NF(E), \otimes, \{1\})$ is a commutative monoid.

(2) Firstly, we show that (HL1) is valid. If $O \subseteq J$, then we obtain $O \to (J \to O) = O \to O = E$ by Proposition 5.1 and Proposition 5.3 (1). Similarly, if $J \subseteq O$, it follows that $O \to (J \to O) = O \to E = E$. Hence, we conclude that (HL1) holds.

Next, we shall prove that (HL2). If $O \subseteq J \subseteq K$, then $[O \to (J \to K)] \to [(O \to J) \to (O \to K)] = (O \to E) \to (E \to E) = E \to E = E$. And, if $O \subseteq K \subseteq J$, then $[O \to (J \to K)] \to [(O \to J) \to (O \to K)] = (O \to K) \to (E \to E) = E$. Moreover, if $K \subseteq O \subseteq J$ or $K \subseteq J \subseteq O$ or $J \subseteq K \subseteq O$ or $J \subseteq O \subseteq K$, we can prove it in a similar way. Thus, we obtain that (HL2) holds.

Finally, by Proposition 5.3 (5), we can easily check that (HL3) holds. Therefore, $(NF(E), \to, E)$ is a Hilbert algebra.

(3) By Proposition 5.4, we know that $(NF(E), \sqcup, \sqcap)$ is a bounded distributive lattice. Now, for any $O, J, K \in NF(E)$, we shall prove that $O \cap K \subseteq J$ iff $K \subseteq O \rightarrow J$. Let us take the following six cases into account:

Case 1. If $O \subseteq J \subseteq K$, then $O \cap K = O \subseteq J$ iff $K \subseteq E = O \rightarrow J$.

Case 2. If $O \subseteq K \subseteq J$, then $O \cap K = O \subseteq J$ iff $K \subseteq E = O \rightarrow J$.

Case 3. If $K \subseteq O \subseteq J$, then $O \cap K = K \subseteq J$ iff $K \subseteq E = O \rightarrow J$.

Case 4. If $K \subseteq J \subseteq O$, then $O \cap K = K \subseteq J$ iff $K \subseteq E = O \rightarrow J$.

Case 5. If $J \subseteq O \subseteq K$, then $O \cap K = O \nsubseteq J$ iff $K \nsubseteq J = O \rightarrow J$.

Case 6. If $J \subseteq K \subseteq O$, then $O \cap K = K \nsubseteq J$ iff $K \nsubseteq J = O \rightarrow J$.

Hence, we obtain that $(NF(E), \sqcup, \sqcap, \rightarrow, E)$ is a Heyting algebra.

(4) Firstly, we show that (B1) holds. Let us consider the following six scenarios:

Case 1. Assume $O \subseteq J \subseteq K$. Then, $(J \rightarrow K) \rightarrow [(K \rightarrow O) \rightarrow (J \rightarrow O)] = E \rightarrow (O \rightarrow O) = E \rightarrow E = E$.

Case 2. If $O \subseteq K \subseteq J$, then $(J \rightarrow K) \rightarrow [(K \rightarrow O) \rightarrow (J \rightarrow O)] = K \rightarrow (O \rightarrow O) = K \rightarrow E = E$.

Case 3. If $K \subseteq O \subseteq J$, then $(J \rightarrow K) \rightarrow [(K \rightarrow O) \rightarrow (J \rightarrow O)] = K \rightarrow (E \rightarrow O) = K \rightarrow O = E$.

Case 4. Suppose $K \subseteq J \subseteq O$, then $(J \rightarrow K) \rightarrow [(K \rightarrow O) \rightarrow (J \rightarrow O)] = K \rightarrow (E \rightarrow E) = K \rightarrow E = E$.

Case 5. If $J \subseteq K \subseteq O$, then $(J \rightarrow K) \rightarrow [(K \rightarrow O) \rightarrow (J \rightarrow O)] = E \rightarrow (E \rightarrow E) = E$.

Case 6. If $J \subseteq O \subseteq K$, then $(J \rightarrow K) \rightarrow [(K \rightarrow O) \rightarrow (J \rightarrow O)] = E \rightarrow (O \rightarrow E) = E$.

Hence, we obtain that $(B1)$ holds.

As for (B2), if $O \subseteq J$, then it implies that $J \rightarrow ((J \rightarrow O) \rightarrow O) = J \rightarrow (O \rightarrow O) = J \rightarrow E = E$ by Proposition 5.3 (1). Similarly, if $J \subseteq O$, we can get that $J \rightarrow ((J \rightarrow O) \rightarrow O) = J \rightarrow (E \rightarrow O) = J \rightarrow O = E$. Hence, we conclude that (B2) holds. Moreover, from Proposition 5.3 (1) and (5), we can easily check that (B3), (B4) and (B5) hold. Therefore, we obtain that $(NF(E), \rightarrow, E)$ is a BCK-algebra. □

Theorem 5.7. *Suppose that $\mathcal{E}$ is an EQ-algebra. If for any $\{1\} \neq O, J \in NF(E)$, $O \cap J \neq \{1\}$, then $(NF(E), \sqcup, \sqcap, ', \{1\}, E)$ is a semi-De Morgan algebra.*

Proof. Similar to above, it follows that it is a bounded distributive lattice by Theorem 5.4. Now, for any $O, J \in NF(E)$, we shall show that $(O \sqcup J)' = O' \sqcap J'$, $(O \sqcap J)'' = O'' \sqcap J''$ and $O' = O'''$. If $O = J = \{1\}$, since $O' = E$ and $J' = E$, we get $(O \sqcup J)' = \langle O \cup J \rangle \rightarrow \{1\} = \{1\} \rightarrow \{1\} = E = E \cap E = O' \cap J' = O' \sqcap J'$,

$(O \sqcap J)'' = ((O \sqcap J)')' = E' = \{1\} = \{1\} \cap \{1\} = O'' \cap J'' = O'' \sqcap J''$ and $O''' = E = O'$. Now, assume $O = \{1\}$ and $J \neq \{1\}$. Because $O' = E$, it follows that $(O \sqcup J)' = \langle O \cup J \rangle \to \{1\} = J \to \{1\} = J' = J' \cap E = J' \cap O' = J' \sqcap O'$ and $(O \sqcap J)'' = O'' = \{1\} = \{1\} \cap J'' = O'' \cap J'' = O'' \sqcap J''$ and $O' = E = O'''$ by Proposition 5.3 (2). Finally, assume $O \neq \{1\}$ and $J \neq \{1\}$. Since $O' = J' = \{1\}$, we obtain $(O \sqcup J)' = \langle O \cup J \rangle \to \{1\} = \{1\} = \{1\} \cap \{1\} = O' \cap J' = O' \sqcap J'$, $(O \sqcap J)'' = ((O \sqcap J) \to \{1\})' = \{1\}' = E = E \cap E = O'' \cap J'' = O'' \sqcap J''$ and $O''' = \{1\}'' = E' = \{1\} = O'$. Hence, the conclusion holds. □

In the following, some topological properties of $NF(E)$ will be stated and proved. By Proposition 4.15, we know that each non principal nodal filter is prime. Let us call this kind of filter nodal prime filter and denote the set of all nodal prime filters by $NP(E)$.

Proposition 5.8. *Suppose H is a prime filter of an EQ-algebra.*

(1) *If H_1 is a proper filter with $H \subseteq H_1$, then H_1 is a prime filter.*

(2) *If $\{H_i \mid i \in I\} \subseteq \mathcal{F}(E)$ satisfying $H \subseteq \bigcap_{i \in I} H_i$, then $\{H_i \mid i \in I\}$ is a chain.*

Proof. (1) It follows from H is a prime that either $a \to b \in H \subseteq H_1$ or $b \to a \in H \subseteq H_1$ for any $a, b \in E$. Thus, we obtain that H_1 is a prime filter.

(2) Let $H_1, H_2 \in \{H_i \mid i \in I\}$. When $H_1 = E$ or $H_2 = E$, the proof is obvious. Now, let $H_1 \neq E$, $H_2 \neq E$ and $H_1 \nsubseteq H_2$, $H_2 \nsubseteq H_1$. Then, $u \in H_1 \setminus H_2$ and $v \in H_2 \setminus H_1$ for some $u, v \in E$. Since $H \subseteq \bigcap_{i \in I} H_i \subseteq H_1 \cap H_2$, we know that $H_1 \cap H_2$ is prime. Moreover, since $\langle u \rangle \in H_1$ and $\langle v \rangle \in H_2$, it follows that $\langle u \rangle \cap \langle v \rangle \subseteq H_1 \cap H_2$, and so $u \in \langle u \rangle \subseteq H_1 \cap H_2$ or $v \in \langle v \rangle \subseteq H_1 \cap H_2$, which generates a contradiction. Therefore, $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$, it turns out that $\{H_i \mid i \in I\}$ is a chain. □

Theorem 5.9. *Let H be a filter of an EQ-algebra and $\emptyset \neq I \subseteq E$ with $I \cap H = \emptyset$. Then, there is a prime filter J satisfying $H \subseteq J$ and $I \cap J = \emptyset$.*

Proof. Denote $\Gamma = \{K \in \mathcal{N}(F) \mid H \subseteq K \text{ and } I \cap K = \emptyset\}$. It follows from $H \in \Gamma$ that Γ is non-empty. Assume $\{K_i \mid i \in I\} \subseteq \Gamma$ is a chain. Then, $J = \bigcup_{i \in I} K_i$ is a maximal element in Γ by Zorn's Lemma, and so we shall show that J is a filter. Obviously, $1 \in J$. For any $u \in J$ and $u \leq v$, we get $u \in K_{i_1}$ for some $i_1 \in I$. And, since K_{i_1} is a filter, we obtain that $v \in K_{i_1} \subseteq J$. Suppose that $x, y \in J$. Then, there are $i, j \in I$ such that $x \in K_i, y \in K_j$. If $K_i \subseteq K_j$, then we get $x \otimes y \in K_i \subseteq J$. Otherwise, we obtain that $x \otimes y \in K_j \subseteq J$. Now, for any $u \to v \in J$, there exists $i_2 \in I$ such that $u \to v \in K_{i_2}$. Thus, it follows from K_{i_2} is a filter that $u \odot w \to v \odot w \in K_{i_2} \subseteq J$ for any $w \in E$. Hence, we obtain that J is a filter. By Proposition 5.8, we know that J is a prime filter. Therefore, we see immediately that J is what we want. □

Corollary 5.10. *Let H be a filter of an EQ-algebra and $x \notin H$. Then, there is a prime filter J satisfying $H \subseteq J$ and $x \notin J$.*

For any $A \subseteq E$, denote $T(A) = \{H \in NP(E) \mid A \nsubseteq H\}$. Next, we will present the properties of $T(A)$ and the topology space induced by it.

Proposition 5.11. *Let $\mathcal{E}$ be an EQ-algebra. Then, for any $M, N \subseteq E$, the following properties hold:*

(1) *If $M \subseteq N$, then $T(M) \subseteq T(N)$.*

(2) *$T(\{0\}) = NP(E)$, $T(\emptyset) = \emptyset$.*

(3) *If $\langle M \rangle = E$, then $T(M) = NP(E)$.*

(4) *$T(M) = T(\langle M \rangle)$.*

(5) *$T(M) = T(N)$ iff $\langle M \rangle = \langle N \rangle$.*

(6) *$T(M) \cap T(N) = T(\langle M \rangle \cap \langle N \rangle)$.*

(7) *Let $\{M_i \mid i \in I\} \subseteq E$. Then, $T(\bigcup_{i\in I} M_i) = \bigcup_{i\in I} T(M_i)$.*

Proof. (1) For any $H \in T(M)$, we get $M \nsubseteq H$. And, by assumption, it follows that $N \nsubseteq H$, which means $H \in T(N)$. Thus, we obtain that $T(M) \subseteq T(N)$.

(2) Let $H \in NP(E)$. Since H is a prime filter, it implies that H is proper, which means $0 \in H$, that is $\{0\} \subseteq H$. Thus, we obtain that $H \in T(\{0\})$, and it readily follows that $T(\{0\}) = NP(E)$. Obviously, $T(\emptyset) = \emptyset$ holds.

(3) If $\langle M \rangle = E$, we know that E is the smallest filter containing M by definition. Then, for any $H \in NP(E)$, it readily follows that $M \nsubseteq H$. Thus $H \in T(M)$ holds, and then $NP(E) \subseteq T(M)$. Hence, we obtain that $T(M) = NP(E)$.

(4) Since $M \subseteq \langle M \rangle$, we get $T(M) \subseteq T(\langle M \rangle)$ by (1). Conversely, let $H \in T(\langle M \rangle)$. Then, $\langle M \rangle \nsubseteq H$. If $M \subseteq H$, it follows from the definition of $\langle M \rangle$ that $\langle M \rangle \subseteq H$, which generates a contradiction. Hence, $M \nsubseteq H$, and so $H \in T(M)$. Therefore, we see immediately that $T(M) = T(\langle M \rangle)$.

(5) Assume $\langle M \rangle = \langle N \rangle$. Then, we get $T(\langle M \rangle) = T(\langle N \rangle)$, and so $T(M) = T(N)$ by (4). Conversely, let $T(M) = T(N)$. If $\langle M \rangle \neq \langle N \rangle$, then we obtain that there is a prime filter H satisfying $\langle M \rangle \subseteq H$ and $\langle N \rangle \nsubseteq H$ by Proposition 5.9. Thus, $H \notin T(M)$ and $H \in T(N)$, which contradict to $T(M) = T(N)$. Therefore, $\langle M \rangle = \langle N \rangle$ holds.

(6) By (4), it suffices to show that $T(\langle M \rangle) \cap T(\langle N \rangle) = T(\langle M \rangle \cap \langle N \rangle)$. Obviously, $\langle M \rangle \cap \langle N \rangle \subseteq \langle M \rangle, \langle N \rangle$, which implies that $T(\langle M \rangle \cap \langle N \rangle) \subseteq T(\langle M \rangle)$, $T(\langle N \rangle)$, and so $T(\langle M \rangle \cap \langle N \rangle) \subseteq T(\langle M \rangle) \cap T(\langle N \rangle)$. Conversely, for any $H \in T(\langle M \rangle) \cap T(\langle N \rangle)$, we obtain that $\langle M \rangle \nsubseteq H$ and $\langle N \rangle \nsubseteq H$. Hence, there are $a \in \langle M \rangle$ and $b \in \langle N \rangle$ satisfying $a \notin H$ and $b \notin H$. Now, we show that $\langle M \rangle \cap \langle N \rangle \nsubseteq H$. Otherwise, it follows from $a \vee b \in \langle M \rangle \cap \langle N \rangle$ that $a \vee b \in H$.

By the fact that H is prime, we obtain that $a \in H$ or $b \in H$, which generates a contradiction. Hence, it follows that $\langle M\rangle \cap \langle N\rangle \nsubseteq H$, and so $H \in T(\langle M\rangle \cap \langle N\rangle)$.

(7) Since $M_i \subseteq \bigcup_{i\in I} M_i$ for any $i \in I$, we get $T(M_i) \subseteq T(\bigcup_{i\in I} M_i)$ for any $i \in I$, that is $\bigcup_{i\in I} T(M_i) \subseteq T(\bigcup_{i\in I} M_i)$. Conversely, assume $H \in T(\bigcup_{i\in I} M_i)$, we have $\bigcup_{i\in I} M_i \nsubseteq H$ by definition. Hence, there is M_{i_1} satisfying $H \in T(M_{i_1})$, and so $M_{i_1} \nsubseteq H$. It follows that $\bigcup_{i\in I} M_{i_1} \nsubseteq H$ and $H \in T(\bigcup_{i\in I} M_i)$. Hence, we obtain that $T(\bigcup_{i\in I} M_i) = \bigcup_{i\in I} T(M_i)$. □

Proposition 5.12. *Let H, J be two filters of an EQ-algebra. Then, the equations $T(H \sqcup J) = T(H) \cup T(J)$ and $T(H \cap J) = T(H) \cap T(J)$ hold.*

Proof. Let $K \in T(H) \cup T(J)$. Then, $H \nsubseteq K$ or $J \nsubseteq K$. Now, because $H, J \subseteq H \sqcup J$, we get $H \sqcup J \nsubseteq K$, that is $K \in T(H \sqcup J)$. Conversely, for any $K \in T(H \sqcup J)$, it readily implies that $H \sqcup J \nsubseteq K$. Assume that $H \subseteq K$ and $J \subseteq K$. Then, $H \sqcup J \subseteq K$, which is a contradiction. Thus, we get $H \nsubseteq K$ or $J \nsubseteq K$, it follows that $K \in T(H)$ or $K \in T(J)$, that is $K \in T(H) \cup T(J)$. Hence, $T(H \sqcup J) = T(H) \cup T(J)$ holds.

Now, we prove that $T(H \cap J) = T(H) \cap T(J)$ holds. Obviously, $T(H \cap J) \subseteq T(H) \cap T(J)$ is valid. Conversely, for any $K \in T(H) \cap T(J)$, it implies that $H \nsubseteq K$ and $J \nsubseteq K$. Thus, $u \in H$ and $u \notin K$ for some $u \in E$. If $K \nsubseteq T(H \cap J)$, we get $H \cap J \subseteq K$, and then $u \vee v \in H \cap J \subseteq K$ for some $v \in J$. Moreover, since K is prime and $u \notin K$, it follows that $v \in K$, and so $J \subseteq K$, which is a contradiction. Hence, $K \in T(H \cap J)$, which implies $T(H \cap J) = T(H) \cap T(J)$. □

Especially, if $A = \{u\}$, then we denote $T(u) = \{H \in NP(E) \mid u \notin H\}$. Analogously, we have the following properties:

Proposition 5.13. *Assume $\mathcal{E}$ is an EQ-algebra. Then, for any $x, y \subseteq E$, the following properties hold:*

(1) *If $x \leq y$, then $T(y) \leq T(x)$.*

(2) *$T(0) = NP(E)$, $T(1) = \emptyset$.*

(3) *If $\langle x\rangle = E$, then $T(x) = NP(E)$.*

(4) *$T(x) = T(\langle x\rangle)$.*

Proposition 5.14. *Let $\mathcal{E}$ be an EQ-algebra. Then, for any $x, y \subseteq E$, the following properties hold:*

(1) *$\bigcup_{x\in E} T(x) = NP(E)$.*

(2) *If $x \vee y$ exists, then $T(x) \cap T(y) = T(x \vee y)$.*

(3) *$T(x) \cup T(y) = T(x \wedge y) = T(x \otimes y)$.*

Proof. (1) It follows from Proposition 5.11 (2).

(2) Let $H \in T(x) \cap T(y)$. Then, we have $H \in T(x)$ and $H \in T(y)$, which implies $x \notin H$, $y \notin H$. If $x \vee y \in H$, then by the fact that H is prime, we get $x \in H$ or $y \in H$, which generates a contradiction. Thus, we get $x \vee y \notin H$, which means $H \in T(x \vee y)$. Hence, it follows that $T(x) \cap T(y) \subseteq T(x \vee y)$. Conversely, for any $H \in T(x \vee y)$, it implies that $x \vee y \notin H$. If $x \in H$ or $y \in H$, then we get $x \vee y \in H$ by $x, y \leq x \vee y$, which generates a contradiction. Hence, it follows that $x \notin H$ and $y \notin H$, that is $H \in T(x)$ and $H \in T(y)$, and so $H \in T(x) \cap T(y)$. Therefore, we obtain that $T(x) \cap T(y) = T(x \vee y)$.

(3) For any $H \in T(x) \cup T(y)$, it implies that $H \in T(x)$ or $H \in T(y)$, which means $x \notin H$ or $y \notin H$. Now, since H is a filter, we get $x \wedge y \notin H$, that is $H \in T(x \wedge y)$, and so $T(x) \cup T(y) \subseteq T(x \wedge y)$. Conversely, for any $H \in T(x \wedge y)$, we have $x \wedge y \notin H$. If $x, y \in H$, then $x \otimes y \in H$, and so $x \wedge y \in H$, which generates a contradiction. Hence, $x \notin H$ or $y \notin H$, that is $H \in T(x) \cup T(y)$. Therefore, $T(x) \cup T(y) = T(x \wedge y)$. Analogously, $T(x) \cup T(y) = T(x \otimes y)$ also holds. □

Let $\mathcal{E}$ be an EQ-algebra and $\tau = \{T(M) \mid M \subseteq E\}$. Then, by the above Proposition, we have:

(1) $\emptyset$, $NP(E) \in \tau$.

(2) If $T(M)$, $T(N) \in \tau$, then $T(M) \cap T(N) \in \tau$.

(3) If $\{T(M_i) \mid i \in I\} \subseteq \tau$, then $\bigcup_{i \in I} T(M_i) \in \tau$.

Hence, τ is a topology on $NP(E)$ and $(NP(E), \tau)$ is a topological space of nodal prime filters.

Proposition 5.15. *Assume that $\mathcal{E}$ is an EQ-algebra. Then, $\{T(m) \mid m \in E\}$ is a topological base of $(NP(E), \tau)$.*

Proof. Let $T(M) \in \tau$. Then, we get $T(M) = T(\bigcup_{i \in I} m_i) = \bigcup_{i \in I} T(m_i)$, that is to say each element in τ can be expressed by the union of elements in subset of $\{T(m) \mid m \in E\}$. Hence, $\{T(m) \mid m \in E\}$ is a topological base of $(NP(E), \tau)$. □

Proposition 5.16. *Suppose that $\mathcal{E}$ is an EQ-algebra. Then, $(NP(E), \tau)$ is a compact T_0 space.*

Proof. Firstly, we show that $T(u)$ is compact set in $(NP(E), \tau)$ for any $u \in E$. By definition of compact, we shall prove that each open covering of $T(u)$ has a finite open covering. Assume $T(u) = \bigcup_{i \in I} T(u_i) = T(\bigcup_{i \in I} u_i)$. Then, from Proposition 5.11 (5), we obtain that $\langle u \rangle = \langle \bigcup_{i \in I} u_i \rangle$, and so $u \in \langle \bigcup_{i \in I} u_i \rangle$. Hence, there are finite $u_{i_1}, u_{i_2}, \cdots, u_{i_n}$ satisfying $u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n} \leq u$, which implies $T(u) \leq T(u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n}) = T(u_{i_1}) \cup T(u_{i_2}) \cup \cdots \cup T(u_{i_n}) \subseteq \bigcup_{i \in I} T(u_i) = T(u)$. Therefore, it follows that $(NP(E), \tau)$ is compact.

Next, we show that $(NP(E), \tau)$ is a T_0 space. Assume that $H, J \in NP(E)$ with $H \neq J$. Then, we get $H \nsubseteq J$ or $J \nsubseteq H$. If $H \nsubseteq J$, then there exists a such that $a \in H$ but $a \notin J$. Let $U = T(a)$. Then, it implies that $J \in U$ and $H \nsubseteq U$. If $J \nsubseteq H$, the proof is similar. Hence, the conclusion holds. □

6. Conclusion

In this article, we presented the definitions of seminodes, nodes and nodal filters in EQ-algebras and their related properties are stated and proved. At first, we exemplify that the seminodes and nodes are different with other specific elements and show that the set $\mathcal{ND}(E)$ is a distributive lattice and the set $\mathcal{SN}(E)$ is a Hertz-algebra and a Heyting-algebra under some conditions. Then, we introduced the concept of n-filters, we studied it with the help of node elements and obtained that there is a one-to-one correspondence between nodal principle filters and node elements in an idempotent EQ-algebra. Furthermore, the relationships among it and other filters were given. It was turned out that each obstinate filter and each (positive) implicative filter is an n-filter under some conditions. Finally, we investigated the algebraic structures of $NF(E)$ and topological structures of $NP(E)$ on EQ-algebras and set up the connections from the set $NF(E)$ of all nodal filters in an EQ-algebra $\mathcal{E}$ to other algebraic structures, like BCK$-$algebras, Hertz algebras and so on. In addition, we concluded that $(NP(E), \tau)$ is a compact T_0 space.

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