Improvements of Hölder's inequality via Schur convexity of functions

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Abstract. In this paper, we study the Schur convexity of some functions associated with Hölder's inequality, the results obtained are then used to establish the refined versions of Hölder's inequality under certain specified conditions. At the end of the paper, applications to inequalities for special means are given.

Keywords: Hölder inequality, Schur convexity, majorization, special means.

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1. Introduction and main results

The discrete Hölder inequality states that if $a_k \geq 0,\ b_k \geq 0, k = 1, 2, \ldots, n$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{k=1}^{n} a_k b_k \leq \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^{n} b_k^q\Big)^{\frac{1}{q}}. \tag{1}$$

Correspondingly, the integral version of Hölder's inequality can be formulated as

$$\int_a^b f(x)g(x)dx \leq \Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}} \Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}}, \tag{2}$$

where $f(x)$ and $g(x)$ are nonnegative integrable on $[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Hölder's inequality is one of the most important foundational inequality in analysis, it also plays a key role in dealing with various problems of pure and applied mathematics, see [1] for background information on Hölder's inequality. In the past more than 100 years, this classical inequality has been paid considerable attention, there have been a large number of literature focusing on its improvements, extensions and applications. For example, some refinements and generalizations of Hölder's inequality were established by Yang in the references [2] and [3], respectively. A sharpened version was given by Hu [4]. A complementary version of sharpening Hölder's inequality related to the work [4] was provided by Wu [5]. A generalization of the result of Hu [4] was obtained by Wu [6]. A further generalization and refinement of Hölder's inequality was proposed by Qiang and Hu in [7]. For more results regarding different improvements of Hölder's inequality can be found in monograph [8] and references therein.

In recent years, application of Schur convexity and majorization properties to establish and improve various inequalities has been a hot research topic. For details about the applications of Schur convexity of functions, we refer the reader to the references [9-13].

In this paper, we provide a novel method to study the improvements and variants of Hölder's inequality. More specifically, we will construct some functions associated with Hölder's inequality, and then we use Schur convexity of these functions to derive the refined versions of Hölder's inequality under certain specified conditions.

We denote the n-dimensional real vector by $\boldsymbol{V} = (v_1, v_2, \ldots, v_n)$, and let

$$\mathbb{R}^n = \{(v_1, v_2, \ldots, v_n) : v_i \in \mathbb{R}, i = 1, 2, \ldots, n\},$$

$$\mathbb{R}^n_+ = \{(v_1, v_2, \ldots, v_n) : v_i \geq 0, i = 1, 2, \ldots, n\}.$$

Our main results are as follows:

Theorem 1.1. *Let* $\boldsymbol{a} = (a_1, a_2, \dots, a_n), \boldsymbol{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n_+$, *and let* p, q *be two non-zero real numbers*

$$H_1(\boldsymbol{a}) = \Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}}.$$

If $p \geq 1$, *then for fixed* $\boldsymbol{b}$, $H_1(\boldsymbol{a})$ *is Schur-convex on* $\mathbb{R}^n_+$. *If* $p \leq 1$, *then for fixed* $\boldsymbol{b}$, $H_1(\boldsymbol{a})$ *is Schur-concave on* $\mathbb{R}^n_+$.

Theorem 1.2. *Let* $\boldsymbol{a} = (a_1, a_2, \dots, a_n), \boldsymbol{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n_+$, *and let* p, q *be two non-zero real numbers,* $A_n(\boldsymbol{a}) = \frac{1}{n}\sum_{k=1}^n a_k$,

$$H_2(\boldsymbol{b}) = n^{\frac{1}{p}} \Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} A_n(\boldsymbol{a}).$$

If $q \geq 1$, *then for fixed* $\boldsymbol{a}$, $H_2(\boldsymbol{b})$ *is Schur-convex on* $\mathbb{R}^n_+$. *If* $q \leq 1$, *then for fixed* $\boldsymbol{a}$, $H_2(\boldsymbol{b})$ *is Schur-concave on* $\mathbb{R}^n_+$.

Theorem 1.3. *Let* $f(x), g(x)$ *be two nonnegative and continuous functions on* I, *let* $\int_a^b f(x)g(x)dx \neq 0$, $\int_a^b (f(x))^p dx \neq 0$, $\int_a^b (g(x))^q dx \neq 0$, *for any* $a, b \in I$ $(a \neq b)$, $p, q \in \mathbb{R}$, *and let*

$$H_3(a,b) = \begin{cases} \Big(\frac{\int_a^b (g(x))^q dx}{\int_a^b f(x)g(x)dx}\Big)^p \Big(\frac{\int_a^b (f(x))^p dx}{\int_a^b f(x)g(x)dx}\Big)^q, & a \neq b, \\ [f(a)g(a)]^{pq-p-q}, \quad a = b. \end{cases} \tag{3}$$

Then, $H_3(a,b)$ *is Schur-convex (Schur-concave) on* I^2 *if and only if*

$$\frac{q(f^p(b) + f^p(a))}{\int_a^b f^p(x)dx} + \frac{p(g^q(b) + g^q(a))}{\int_a^b g^q(x)dx} \geq (\leq) \frac{(f(b)g(b) + f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx}. \tag{4}$$

2. Preliminaries

In this section, we introduce some essential definitions and lemmas.

Definition 2.1 ([14]). *Let* $\boldsymbol{U} = (u_1, u_2, \dots, u_n)$ *and* $\boldsymbol{V} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$.

(*i*) *The vector* $\boldsymbol{U}$ *is said to be majorized by the vector* $\boldsymbol{V}$, *symbolized as* $\boldsymbol{U} \prec \boldsymbol{V}$, *if* $\sum_{i=1}^{\ell} u_{[i]} \leq \sum_{i=1}^{\ell} v_{[i]}$ *for* $\ell = 1, 2, \dots, n-1$ *and* $\sum_{i=1}^n u_i = \sum_{i=1}^n v_i$, *where* $u_{[1]} \geq u_{[2]} \cdots \geq u_{[n]}$ *and* $v_{[1]} \geq v_{[2]} \cdots \geq v_{[n]}$ *are rearrangements of* $\boldsymbol{U}$ *and* $\boldsymbol{V}$ *in a descending order.*

(*ii*) *Let* $\Omega \subset \mathbb{R}^n$, Ψ: $\Omega \to \mathbb{R}$ *is said to be Schur-convex function on* Ω *if* $\boldsymbol{U} \prec \boldsymbol{V}$ *on* Ω *implies* $\Psi(\boldsymbol{U}) \leq \Psi(\boldsymbol{V})$, *while* Ψ *is said to be Schur-concave function on* Ω *if and only if* $-\Psi$ *is Schur-convex function.*

Lemma 2.1 ([14])**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a convex set and has a nonempty interior set Ω°, suppose also that $\Psi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω°. Then Ψ is the Schur-convex (or Schur-concave) function, if and only if it is symmetric on Ω and*

$$(v_1 - v_2)\left(\frac{\partial \Psi}{\partial v_1} - \frac{\partial \Psi}{\partial v_2}\right) \geq 0 \ (or \leq 0)$$

holds, for any $\boldsymbol{V} = (v_1, v_2, \ldots, v_n) \in \Omega^\circ$.

Lemma 2.2 ([15], Chebyshev inequality)**.** *Let $a_k \geq 0$, $b_k \geq 0, k = 1, 2, \ldots, n$.*

(i) If $\{a_k\}$ and $\{b_k\}$ $(k = 1, 2, \ldots, n)$ have opposite monotonicity, then

$$\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k \geq n \sum_{k=1}^{n} a_k b_k \tag{5}$$

(ii) If $\{a_k\}$ and $\{b_k\}$ $(k = 1, 2, \ldots, n)$ have same monotonicity, then

$$\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k \leq n \sum_{k=1}^{n} a_k b_k \tag{6}$$

Lemma 2.3 ([15], Hermite-Hadamard inequality)**.** *If $f(x)$ is a convex function on $[a, b]$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{7}$$

If $f(x)$ is a concave function on $[a, b]$, then inequality (7) is reversed.

Lemma 2.4 ([16])**.** *If $a \leq b$, $u(t) = tb + (1-t)a$, $v(t) = ta + (1-t)b$, $0 \leq t_1 \leq t_2 \leq \frac{1}{2}$ or $\frac{1}{2} \leq t_2 \leq t_1 \leq 1$, then*

$$\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (a, b) \tag{8}$$

Lemma 2.5 ([16])**.** *Let $\boldsymbol{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n_+$, $A_n(\boldsymbol{a}) = \frac{1}{n}\sum_{i=1}^{n} a_i$. Then*

$$\mathbf{u} = \Big(\underbrace{A_n(\boldsymbol{a}), A_n(\boldsymbol{a}), \cdots, A_n(\boldsymbol{a})}_{n}\Big) \prec (a_1, a_2, \cdots, a_n) = \boldsymbol{a}. \tag{9}$$

3. Proof of main results

Proof of Theorem 1.1. It is obvious that $H_1(\boldsymbol{a})$ is symmetric about $a_1, a_2, \ldots, a_n$ on $\mathbb{R}^n_+$, without loss of generality, we may assume that $a_1 \geq a_2$.

Differentiating $H_1(\boldsymbol{a})$ with respect to a_1 and a_2 respectively, we obtain

$$\frac{\partial H_1}{\partial a_1} = \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}-1}\Big(\sum_{k=1}^{n} b_k^q\Big)^{\frac{1}{q}} a_1^{p-1}, \quad \frac{\partial H_1}{\partial a_2} = \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}-1}\Big(\sum_{k=1}^{n} b_k^q\Big)^{\frac{1}{q}} a_2^{p-1}.$$

Hence, we have

$$\Delta_1 := (a_1 - a_2)\Big(\frac{\partial H_1}{\partial a_1} - \frac{\partial H_1}{\partial a_2}\Big) = (a_1 - a_2)\Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}-1}\Big(\sum_{k=1}^{n} b_k^q\Big)^{\frac{1}{q}}(a_1^{p-1} - a_2^{p-1}).$$

It is easy to see that $\Delta_1 \geq 0$ for $p \geq 1$, and $\Delta_1 \leq 0$ for $p \leq 1$. By Lemma 2.1, it follows that $H_1(\boldsymbol{a})$ is Schur-convex on $\mathbb{R}_+^n$ for $p \geq 1$, and $H_1(\boldsymbol{a})$ is Schur-concave on $\mathbb{R}_+^n$ for $p \leq 1$. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Using the same arguments as that described in the proof of Theorem 1.1, we can easily carry out the proof of Theorem 1.2. □

Proof of Theorem 1.3. Note that

$$H_3(a,b) = \Big(\frac{\int_a^b (g(x))^q dx}{\int_a^b f(x)g(x)dx}\Big)^p\Big(\frac{\int_a^b (f(x))^p dx}{\int_a^b f(x)g(x)dx}\Big)^q = \frac{\big(\int_a^b f^p(x)dx\big)^q\big(\int_a^b g^q(x)dx\big)^p}{\big(\int_a^b f(x)g(x)dx\big)^{p+q}}.$$

Since $H_3(a,b)$ is symmetric about a, b on $\mathbb{R}_+^2$, we may assume that $b \geq a$. Differentiating $H_3(\boldsymbol{a})$ with respect to b and a respectively gives

$$\begin{aligned}
\frac{\partial H_3}{\partial b} =& \frac{q\big(\int_a^b f^p(x)dx\big)^{q-1} f^p(b)\big(\int_a^b g^q(x)dx\big)^p\big(\int_a^b f(x)g(x)dx\big)^{p+q}}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}}\\
&+ \frac{p\big(\int_a^b g^q(x)dx\big)^{p-1} g^q(b)\big(\int_a^b f^p(x)dx\big)^q\big(\int_a^b f(x)g(x)dx\big)^{p+q}}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}}\\
&- \frac{(p+q)\big(\int_a^b f(x)g(x)dx\big)^{p+q-1} f(b)g(b)\big(\int_a^b f^p(x)dx\big)^q\big(\int_a^b g^q(x)dx\big)^p}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}},\\
\frac{\partial H_3}{\partial a} =& -\frac{q\big(\int_a^b f^p(x)dx\big)^{q-1} f^p(a)\big(\int_a^b g^q(x)dx\big)^p\big(\int_a^b f(x)g(x)dx\big)^{p+q}}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}}\\
&- \frac{p\big(\int_a^b g^q(x)dx\big)^{p-1} g^q(a)\big(\int_a^b f^p(x)dx\big)^q\big(\int_a^b f(x)g(x)dx\big)^{p+q}}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}}\\
&+ \frac{(p+q)\big(\int_a^b f(x)g(x)dx\big)^{p+q-1} f(a)g(a)\big(\int_a^b f^p(x)dx\big)^q\big(\int_a^b g^q(x)dx\big)^p}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\Delta_2 :=& (b-a)\Big(\frac{\partial H_3}{\partial b} - \frac{\partial H_3}{\partial a}\Big)\\
=& \frac{b-a}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}}\Big[q\Big(\int_a^b f^p(x)dx\Big)^{q-1}\Big(\int_a^b g^q(x)dx\Big)^p
\end{aligned}$$

$$\times \Big(\int_a^b f(x)g(x)dx \Big)^{p+q} \big(f^p(b) + f^p(a) \big) + p \Big(\int_a^b g^q(x)dx \Big)^{p-1}$$
$$\times \Big(\int_a^b f^p(x)dx \Big)^q \Big(\int_a^b f(x)g(x)dx \Big)^{p+q} \big(g^q(b) + g^q(a) \big) - (p+q)$$
$$\times \Big(\int_a^b f(x)g(x)dx \Big)^{p+q-1} \Big(\int_a^b f^p(x)dx \Big)^q \Big(\int_a^b g^q(x)dx \Big)^p \big(f(b)g(b) + f(a)g(a) \big) \Big]$$
$$= \frac{b-a}{\big(\int_a^b f(x)g(x)dx \big)^{2(p+q)}} \Big(\int_a^b f(x)g(x)dx \Big)^{p+q-1} \Big(\int_a^b f^p(x)dx \Big)^{q-1}$$
$$\times \Big(\int_a^b g^q(x)dx \Big)^{p-1} \Big[\Big(\int_a^b f(x)g(x)dx \Big) \Big(q \int_a^b g^q(x)dx (f^p(b) + f^p(a))$$
$$+ p \int_a^b f^p(x)dx (g^q(b) + g^q(a)) \Big)$$
$$- (p+q) \Big(\int_a^b f^p(x)dx \int_a^b g^q(x)dx \Big) (f(b)g(b) + f(a)g(a)) \Big].$$

Using the assumption condition of Theorem 1.3 and the non-negativity of

$$\frac{b-a}{\big(\int_a^b f(x)g(x)dx \big)^{2(p+q)}} \Big(\int_a^b f(x)g(x)dx \Big)^{p+q-1} \Big(\int_a^b f^p(x)dx \Big)^{q-1} \Big(\int_a^b g^q(x)dx \Big)^{p-1},$$

we deduce that $\Delta_2 \geq (\leq)\ 0$ if and only if

$$\Big(\int_a^b f(x)g(x)dx \Big) \Big[q \int_a^b g^q(x)dx (f^p(b) + f^p(a)) + p \int_a^b f^p(x)dx (g^q(b) + g^q(a)) \Big]$$
$$\geq (\leq) \Big(\int_a^b f^p(x)dx \int_a^b g^q(x)dx \Big) (f(b)g(b) + f(a)g(a))(p+q)$$
$$\iff \frac{q(f^p(b) + f^p(a))}{\int_a^b f^p(x)dx} + \frac{p(g^q(b) + g^q(a))}{\int_a^b g^q(x)dx} \geq (\leq) \frac{(f(b)g(b) + f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx}.$$

Hence, $H_3(a,b)$ is Schur-convex (Schur-concave) on I^2 if and only if

$$\frac{q(f^p(b) + f^p(a))}{\int_a^b f^p(x)dx} + \frac{p(g^q(b) + g^q(a))}{\int_a^b g^q(x)dx} \geq (\leq) \frac{(f(b)g(b) + f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx}.$$

This completes the proof of Theorem 1.3. □

4. Some corollaries

In this section, we give some consequences of Theorem 1.3.

Corollary 4.1. *Let $f(x), g(x)$ be two nonnegative convex functions on I, $f''g + g''f + 2f'g' \leq 0$, and let $\int_a^b f(x)g(x)dx \neq 0$, $\int_a^b (f(x))^p dx \neq 0$, $\int_a^b (g(x))^q dx \neq 0$, for any $a, b \in I$ $(a \neq b)$. If $p \geq 1, q \geq 1$, then $H_3(a,b)$ is Schur-convex on I^2.*

Proof. Direct computation gives
$(f^p)'' = pf^{p-2}[(p-1)(f')^2 + ff'']$, $(g^q)'' = qg^{q-2}[(q-1)(g')^2 + gg'']$,
$(fg)'' = f''g + g''f + 2f'g'$.

Since $f(x), g(x)$ are convex function on I, $p \geq 1, q \geq 1$, we have $(f^p(x))'' \geq 0$, $(g^q(x))'' \geq 0$ for $x \in I$, so $f^p(x), g^q(x)$ are convex functions on I. In addition, form the assumption $f''g+g''f+2f'g' \leq 0$, we conclude that $f(x)g(x)$ is concave function on I.

By using Lemma 2.3 (Hermite-Hadamard inequality), we obtain

$$\frac{q(f^p(b)+f^p(a))}{\int_a^b f^p(x)dx} + \frac{p(g^q(b)+g^q(a))}{\int_a^b g^q(x)dx} - \frac{(f(b)g(b)+f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx}$$
$$\geq \frac{2q}{b-a} + \frac{2p}{b-a} - \frac{(f(b)g(b)+f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx}$$
$$= (p+q)\Big[\frac{2}{b-a} - \frac{(f(b)g(b)+f(a)g(a))}{\int_a^b f(x)g(x)dx}\Big] \geq 0.$$

We deduce from Theorem 1.3 that $H_3(a,b)$ is Schur-convex on I^2. The proof of Corollary 4.1 is complete. □

Corollary 4.2. *Let $f(x), g(x)$ be two nonnegative and opposite monotonicity concave functions, and let $\int_a^b f(x)g(x)dx \neq 0$, $\int_a^b (f(x))^p dx \neq 0$, $\int_a^b (g(x))^q dx \neq 0$, for any $a, b \in I$ $(a \neq b)$. If $p < 0, q < 0$, then $H_3(a,b)$ is Schur-concave on I^2.*

Proof. In light of

$$(f^p)'' = p[(p-1)(f')^2 + ff'']f^{p-2}, (g^q)'' = q[(q-1)(g')^2 + gg'']g^{q-2},$$
$$(fg)'' = f''g + g''f + 2f'g',$$

we conclude that $(f^p(x))'' \geq 0$, $(g^q(x))'' \geq 0$, so $f^p(x)$ and $g^q(x)$ are convex functions on I. Since $f(x), g(x)$ are opposite monotonicity concave functions, which implies that $f(x)g(x)$ is concave function on I. By Hermite-Hadamard inequality, we have

$$\frac{q(f^p(b)+f^p(a))}{\int_a^b f^p(x)dx} + \frac{p(g^q(b)+g^q(a))}{\int_a^b g^q(x)dx} - \frac{(f(b)g(b)+f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx}$$
$$\leq \frac{2q}{b-a} + \frac{2p}{b-a} - \frac{(f(b)g(b)+f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx}$$
$$= (p+q)\Big[\frac{2}{b-a} - \frac{(f(b)g(b)+f(a)g(a))}{\int_a^b f(x)g(x)dx}\Big] \leq 0.$$

It follows from Theorem 1.3 that $H_3(a,b)$ is Schur-concave on I^2. Corollary 4.2 is proved. □

Corollary 4.3. *Let $f(x), g(x)$ be two nonnegative and opposite monotonicity concave functions, and let $\int_a^b f(x)g(x)dx \neq 0$, $\int_a^b (f(x))^p dx \neq 0$, $\int_a^b (g(x))^q dx \neq 0$, for any $a, b \in I$ $(a \neq b)$. If $p < 0$, $0 < q \leq 1$ and $p + q \geq 0$, then $H_3(a, b)$ is Schur-convex on I^2.*

Proof. In view of

$$(f^p)'' = p[(p-1)(f')^2 + ff'']f^{p-2}, (g^q)'' = q[(q-1)(g')^2 + gg'']g^{q-2},$$
$$(fg)'' = f''g + g''f + 2f'g',$$

we deduce that $f^p(x)$ is convex function for $p < 0$, $g^q(x)$ is concave function for $0 < q \leq 1$, $f(x)g(x)$ is concave function on I. By using Hermite-Hadamard inequality, we obtain

$$\begin{aligned}
&\frac{q(f^p(b) + f^p(a))}{\int_a^b f^p(x)dx} + \frac{p(g^q(b) + g^q(a))}{\int_a^b g^q(x)dx} - \frac{(f(b)g(b) + f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx} \\
&\geq \frac{2q}{b-a} + \frac{2p}{b-a} - \frac{(f(b)g(b) + f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx} \\
&= (p+q)\Big[\frac{2}{b-a} - \frac{(f(b)g(b) + f(a)g(a))}{\int_a^b f(x)g(x)dx}\Big] \geq 0.
\end{aligned}$$

We deduce from Theorem 1.3 that $H_3(a, b)$ is Schur-convex on I^2. Corollary 4.3 is proved. □

5. Applications to inequalities of Hölder type

Firstly, we establish two discrete Hölder-type inequality involving power mean and arithmetic mean.

Theorem 5.1. *Let $a_k \geq 0, b_k \geq 0, k = 1, 2, \ldots, n$, and let p, q be two non-zero real numbers.*

(i) If $p \geq 1, q \geq 1$, then

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \geq n^{\frac{1}{p}+\frac{1}{q}} A_n(\boldsymbol{a}) A_n(\boldsymbol{b}); \tag{10}$$

(ii) If $p \leq 1, q \leq 1$, then

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \leq n^{\frac{1}{p}+\frac{1}{q}} A_n(\boldsymbol{a}) A_n(\boldsymbol{b}), \tag{11}$$

where $A_n(\boldsymbol{a}) = \frac{1}{n}\sum_{k=1}^n a_k$, $A_n(\boldsymbol{b}) = \frac{1}{n}\sum_{k=1}^n b_k$.

Proof. (i) By Lemma 2.5, one has the majorization relationship

$$(a_1, a_2, \cdots, a_n) \succ (A_n(\boldsymbol{a}), A_n(\boldsymbol{a}), \cdots, (A_n(\boldsymbol{a}))\,.$$

From Theorem 1.1, we know that, for $p \geq 1$, $H_1(\boldsymbol{a})$ is Schur-convex on $\mathbb{R}^n_+$. It follows from Definition 2.1 that $H_1(\boldsymbol{a}) \geq H_1(A_n(\boldsymbol{a}))$ for $p \geq 1$. Hence

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \geq (n(A_n(\boldsymbol{a}))^p)^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} = n^{\frac{1}{p}}A_n(\boldsymbol{a})\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}}.$$

On the other hand, by Theorem 1.2, we obtain that, for $q \geq 1$, $H_2(\boldsymbol{b})$ is Schur-convex on $\mathbb{R}^n_+$. Now, from the majorization relation

$$(b_1, b_2, \cdots, b_n) \succ \Big(A_n(\boldsymbol{b}), A_n(\boldsymbol{b}), \cdots, A_n(\boldsymbol{b})\Big),$$

we have $H_2(\boldsymbol{b}) \geq H_2(A_n(\boldsymbol{b}))$ for $q \geq 1$, that is

$$n^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}}A_n(\boldsymbol{a}) \geq n^{\frac{1}{p}}A_n(\boldsymbol{a})(n(A_n(\boldsymbol{b}))^q)^{\frac{1}{q}} = n^{\frac{1}{p}+\frac{1}{q}}A_n(\boldsymbol{a})A_n(\boldsymbol{b}).$$

Hence, we get

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \geq n^{\frac{1}{p}}A_n(\boldsymbol{a})\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \geq n^{\frac{1}{p}+\frac{1}{q}}A_n(\boldsymbol{a})A_n(\boldsymbol{b}),$$

which implies the required inequality (10).

(ii) By the same way as the proof of inequality (10), we can prove the inequality (11). This completes the proof of Theorem 5.1. □

Nextly, we provide two refined versions of discrete Hölder-type inequality under certain specified conditions.

Theorem 5.2. *Let $a_k \geq 0$, $b_k \geq 0, k = 1, 2, \ldots, n$, p, q be two non-zero real numbers.*

(i) If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\{a_k\}, \{b_k\}$ $(k = 1, 2, \ldots, n)$ have opposite monotonicity, then

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \geq nA_n(\boldsymbol{a})A_n(\boldsymbol{b}) \geq \sum_{k=1}^n a_k b_k; \tag{12}$$

(ii) If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\{a_k\}, \{b_k\}$ $(k = 1, 2, \ldots, n)$ have same monotonicity, then

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \leq nA_n(\boldsymbol{a})A_n(\boldsymbol{b}) \leq \sum_{k=1}^n a_k b_k. \tag{13}$$

Proof. (i) For $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, by utilizing Theorem 1.1, we have

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \geq n^{\frac{1}{p}+\frac{1}{q}}A_n(\boldsymbol{a})A_n(\boldsymbol{b}) = nA_n(\boldsymbol{a})A_n(\boldsymbol{b}).$$

Moreover, using Lemma 2.2 (Chebyshev inequality) gives

$$nA_n(\boldsymbol{a})A_n(\boldsymbol{b}) = \frac{\sum_{k=1}^n a_k \sum_{k=1}^n b_k}{n} \geq \sum_{k=1}^n a_k b_k.$$

Hence, we have

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \geq nA_n(\boldsymbol{a})A_n(\boldsymbol{b}) \geq \sum_{k=1}^n a_k b_k,$$

which implies the required inequality (12).

(ii) In the same way as the proof of inequality (12), we can verify the validity of inequality (13). The proof of Theorem 5.2 is complete. □

In Theorems 5.3, 5.4 and 5.5 below, we will give some refined versions of integral Hölder-type inequality under certain specified conditions.

Theorem 5.3. *Let $f(x), g(x)$ be two integrable and nonnegative functions on $[a, b]$, and let p, q be two non-zero real numbers.*

(i) If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f(x), g(x)$ have opposite monotonicity, then

$$\begin{aligned}&\Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}}\\ &\geq \frac{1}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx \geq \int_a^b f(x)g(x)dx.\end{aligned} \tag{14}$$

(ii) If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f(x), g(x)$ have same monotonicity, then

$$\begin{aligned}&\Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}}\\ &\leq \frac{1}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx.\end{aligned} \tag{15}$$

Proof. (i) If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_k \geq 0, b_k \geq 0$ and $\{a_k\}, \{b_k\}$ $(k = 1, 2, \ldots, n)$ have opposite monotonicity, then by Theorem 5.2, we obtain

$$\begin{aligned}&\Big(\frac{b-a}{n}\sum_{k=1}^n f^p\Big(a+\frac{k(b-a)}{n}\Big)\Big)^{\frac{1}{p}}\Big(\frac{b-a}{n}\sum_{k=1}^n g^q\Big(a+\frac{k(b-a)}{n}\Big)\Big)^{\frac{1}{q}}\\ &\geq \frac{1}{b-a}\Bigg(\frac{b-a}{n}\sum_{k=1}^n f\Big(a+\frac{k(b-a)}{n}\Big)\Bigg)\Bigg(\frac{b-a}{n}\sum_{k=1}^n g\Big(a+\frac{k(b-a)}{n}\Big)\Bigg)\\ &\geq \frac{b-a}{n}\sum_{k=1}^n f\Big(a+\frac{k(b-a)}{n}\Big)g\Big(a+\frac{k(b-a)}{n}\Big).\end{aligned}$$

Letting $n \to \infty$ in both sides of the above inequalities, we obtain

$$\Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}} \geq \frac{1}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx \geq \int_a^b f(x)g(x)dx,$$

which is the desired inequality (14).

(ii) By the same way as the proof of inequality (14), one can prove the inequality (15). This completes the proof of Theorem 5.3. □

Obviously, inequalities (12), (13), (14), (15) are the sharpened versions of Hölder's inequality under some specified conditions.

Theorem 5.4. *Let $f(x), g(x)$ be two nonnegative convex functions on I, $f''g + g''f + 2f'g' \le 0$, and let $\int_a^b f(x)g(x)dx \neq 0$, $\int_a^b (f(x))^p dx \neq 0$, $\int_a^b (g(x))^q dx \neq 0$, for any $a, b \in I$ $(a \neq b)$. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$(i) \quad \int_a^b f(x)g(x)dx \le \frac{\int_{u(t)}^{v(t)} f(x)g(x)dx}{\big(\int_{u(t)}^{v(t)} f^p(x)dx\big)^{\frac{1}{p}} \big(\int_{u(t)}^{v(t)} g^q(x)dx\big)^{\frac{1}{q}}}$$

$$\times \Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}} \Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}} \le \Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}} \Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}}, \tag{16}$$

where $u(t) = tb + (1-t)a, v(t) = ta + (1-t)b$, $0 \le t \le 1, t \neq \frac{1}{2}$.

$$(ii) \quad \Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}} \Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}} \ge \frac{1}{b-a} \int_a^b f(x)dx \int_a^b g(x)dx$$

$$\ge f\big(\frac{a+b}{2}\big) g\big(\frac{a+b}{2}\big)(b-a) \ge \int_a^b f(x)g(x)dx. \tag{17}$$

Proof. (i) Since $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(x)$ are nonnegative convex functions with $f''g + g''f + 2f'g' \le 0$ on I, it follows from Corollary 4.1 that $H_3(a,b)$ is Schur-convex on I^2. Additionally, from Lemma 2.4, one has, for $0 \le t \le 1, t \neq \frac{1}{2}$, the relation $\big(\frac{a+b}{2}, \frac{a+b}{2}\big) \prec (u(t), v(t)) \prec (a,b)$. Hence, we obtain

$$H_3(a,b) \ge H_3(u(t), v(t)) \ge H_3\Big(\frac{a+b}{2}, \frac{a+b}{2}\Big) = \Big(f\big(\frac{a+b}{2}\big) g\big(\frac{a+b}{2}\big)\Big)^{pq-p-q} = 1,$$

which implies that

$$\frac{\big(\int_a^b f^p(x)dx\big)^q \big(\int_a^b g^q(x)dx\big)^p}{\big(\int_a^b f(x)g(x)dx\big)^{p+q}} \ge \frac{\big(\int_{u(t)}^{v(t)} f^p(x)dx\big)^q \big(\int_{u(t)}^{v(t)} g^q(x)dx\big)^p}{\big(\int_{u(t)}^{v(t)} f(x)g(x)dx\big)^{p+q}} \ge 1$$

$$\Longleftrightarrow$$

$$\begin{aligned} &\big(\int_a^b f(x)g(x)dx\big)^{p+q} \\ &\le \frac{\big(\int_{u(t)}^{v(t)} f(x)g(x)dx\big)^{p+q}}{\big(\int_{u(t)}^{v(t)} f^p(x)dx\big)^q \big(\int_{u(t)}^{v(t)} g^q(x)dx\big)^p} \big(\int_a^b f^p(x)dx\big)^q \big(\int_a^b g^q(x)dx\big)^p \\ &\le \big(\int_a^b f^p(x)dx\big)^q \big(\int_a^b g^q(x)dx\big)^p. \end{aligned}$$

It follows from $\frac{1}{p}+\frac{1}{q}=1$ that $p+q=pq$, taking the $\frac{1}{pq}$ power of two sides in the above inequalities, we derive the desired inequality (16).

(ii) Using Hölder's inequality (2) gives

$\big(b-a\big)^{\frac{1}{q}}\Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}} \ge \int_a^b f(x)dx,\ \big(b-a\big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}} \ge \int_a^b g(x)dx.$

Hence, we have

$$\Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}} \ge \frac{1}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx.$$

In addition, from the assumption conditions, we find that $f(x), g(x)$ are convex on I, $f(x)g(x)$ is concave on I, thus we deduce from the Hermite-Hadamard inequality that

$$\frac{1}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx \ge f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big)(b-a) \ge \int_a^b f(x)g(x)dx.$$

The proof of Theorem 5.4 is complete. □

It is worth noting that inequalities (16) and (17) are the refined versions of Hölder's inequality under a specified condition.

Theorem 5.5. *Let $f(x), g(x)$ be two nonnegative and opposite monotonicity concave functions, and let $\int_a^b f(x)g(x)dx \neq 0$, $\int_a^b (f(x))^p dx \neq 0$, $\int_a^b (g(x))^q dx \neq 0$, for any $a, b \in I$ $(a \neq b)$. If $p<0$, $q<0$, then*

$$\begin{aligned}&\Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}}\\ &\le \Big(f\big(\frac{a+b}{2}\big)g\big(\frac{a+b}{2}\big)\Big)^{1-\frac{1}{p}-\frac{1}{q}}\Big(\int_a^b f(x)g(x)dx\Big)^{\frac{1}{p}+\frac{1}{q}}.\end{aligned}\tag{18}$$

Proof. By the aid of Corollary 4.2, we observe that $H_3(a,b)$ is Schur-concave on I^2, in addition, from Lemma 2.5, one has $\big(\frac{a+b}{2},\frac{a+b}{2}\big) \prec (a,b)$. We thus have

$$\begin{aligned}&H_3(a,b) \le H_3\Big(\frac{a+b}{2},\frac{a+b}{2}\Big) = \Big(f\big(\frac{a+b}{2}\big)g\big(\frac{a+b}{2}\big)\Big)^{pq-p-q}\\ &\Longleftrightarrow\\ &\frac{\big(\int_a^b f^p(x)dx\big)^q\big(\int_a^b g^q(x)dx\big)^p}{\big(\int_a^b f(x)g(x)dx\big)^{p+q}} \le \Big(f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big)\Big)^{pq-p-q},\end{aligned}$$

taking the $\frac{1}{pq}$ power of the two-sides inequality above, we obtain the required inequality (18). Theorem 5.5 □

6. Applications to inequalities for special means

Let $b > a > 0$, the Stolarsky mean is defined as follows (see [12])

$$L_p(a,b) = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, \quad p \neq -1, 0.$$

The arithmetic mean, geometric mean and logarithmic mean are respectively defined by

$$A(a,b) = \frac{a+b}{2}, \;\; G(a,b) = \sqrt{ab}, \;\; L(a,b) = \frac{b-a}{\log b - \log a}.$$

Theorem 6.1. *Let $b > a > 0$, $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) If $p > 1$, then

$$L_p(a,b) \geq A(a,b)L(a,b)L_{-q}(a,b) \geq L_{-q}(a,b). \tag{19}$$

(ii) If $0 < p < 1$, then

$$L_p(a,b) \leq (A(a,b))^2 (L_q(a,b))^{-1} \leq (L_2(a,b))^2 (L_q(a,b))^{-1}. \tag{20}$$

Proof. Note that

$$\left(\frac{1}{b-a}\int_a^b x^p dx\right)^{\frac{1}{p}} = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} = L_p(a,b),$$

$$\left(\frac{1}{b-a}\int_a^b x^{-q} dx\right)^{\frac{1}{q}} = \left(\frac{b^{-q+1} - a^{-q+1}}{(-q+1)(b-a)}\right)^{\frac{1}{q}} = (L_{-q}(a,b))^{-1}.$$

(i) For $p > 1$, by Theorem 5.3, we have

$$\begin{aligned}&\left(\frac{1}{b-a}\int_a^b f^p(x)\right)^{\frac{1}{p}}\left(\frac{1}{b-a}\int_a^b g^q(x)dx\right)^{\frac{1}{q}}\\ \geq&\left(\frac{1}{b-a}\right)^2\int_a^b f(x)dx\int_a^b g(x)dx \geq \frac{1}{b-a}\int_a^b f(x)g(x)dx.\end{aligned}$$

Taking $f(x) = x, g(x) = x^{-1}$ in the above inequality, it follows that

$$L_p(a,b)(L_{-q}(a,b))^{-1} \geq \frac{1}{(b-a)^2}\int_a^b xdx\int_a^b x^{-1}dx \geq \frac{1}{b-a}\int_a^b dx,$$

that is

$$L_p(a,b) \geq A(a,b)L(a,b)L_{-q}(a,b) \geq L_{-q}(a,b).$$

(ii) For $0 < p < 1$, by Theorem 5.3, we have

$$\begin{aligned}&\left(\frac{1}{b-a}\int_a^b f^p(x)\right)^{\frac{1}{p}}\left(\frac{1}{b-a}\int_a^b g^q(x)dx\right)^{\frac{1}{q}}\\ \leq&\left(\frac{1}{b-a}\right)^2\int_a^b f(x)dx\int_a^b g(x)dx \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx.\end{aligned}$$

Taking $f(x) = x, g(x) = x$, we obtain

$$L_p(a,b) \le (A(a,b))^2 (L_q(a,b))^{-1} \le (L_2(a,b))^2 (L_q(a,b))^{-1}.$$

The proof of Theorem 6.1 is complete. □

Theorem 6.2. *Let $b > a > 0$, $u(t) = tb + (1-t)a, v(t) = ta + (1-t)b$, $0 \le t \le 1$, $t \ne \frac{1}{2}$. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$L_{-q}(a,b) \le \frac{L_{-q}(u(t),v(t))}{L_p(u(t),v(t))} L_p(a,b) \le L_p(a,b). \tag{21}$$

Proof. Using Theorem 5.4 with a substitution of $f(x) = x, \ g(x) = x^{-1}$ in inequality (16), we obtain

$$\begin{aligned}\int_a^b dx \le & \frac{\int_{u(t)}^{v(t)} dx}{\left(\int_{u(t)}^{v(t)} x^p dx\right)^{\frac{1}{p}} \left(\int_{u(t)}^{v(t)} (x^{-1})^q dx\right)^{\frac{1}{q}}} \left(\int_a^b x^p dx\right)^{\frac{1}{p}} \left(\int_a^b (x^{-1})^q dx\right)^{\frac{1}{q}} \\ \le & \left(\int_a^b x^p dx\right)^{\frac{1}{p}} \left(\int_a^b (x^{-1})^q dx\right)^{\frac{1}{q}},\end{aligned}$$

that is

$$\begin{aligned}(b-a) \le & \frac{(v(t)-u(t))(b-a)^{\frac{1}{p}+\frac{1}{q}} L_p(a,b)(L_{-q}(a,b))^{-1}}{(v(t)-u(t))^{\frac{1}{p}+\frac{1}{q}} L_p(u(t),v(t))(L_{-q}(u(t),v(t)))^{-1}} \\ \le & (b-a)^{\frac{1}{p}+\frac{1}{q}} L_p(a,b)(L_{-q}(a,b))^{-1},\end{aligned}$$

which leads to the desired inequality

$$L_{-q}(a,b) \le \frac{L_{-q}(u(t),v(t))}{L_p(u(t),v(t))} L_p(a,b) \le L_p(a,b).$$

This completes the proof of Theorem 6.2. □

7. Conclusion

In this work, we provided a new approach to refine Hölder's inequality. Firstly, we constructed some functions associated with Hölder's inequality and verified their Schur convexities, meanwhile, in Theorems 1.1 and 1.2, we proved the Schur convexity of functions associated with discrete Hölder's inequality, we derived the Schur convexity of function connected to integral Hölder's inequality in Theorem 1.3. Nextly, with the help of the Schur convexity of functions, in Theorem 5.1 we acquired two discrete Hölder-type inequality involving power mean and arithmetic mean; in Theorem 5.2 we provided two refined versions of discrete Hölder-type inequality; in Theorems 5.3, 5.4 and 5.5, we offered some refined versions of integral Hölder-type inequality. Finally, we illustrated the applications of the obtained Hölder-type inequalities, some novel comparison inequalities for Stolarsky mean, arithmetic mean, geometric mean and logarithmic mean are derived respectively in Theorems 6.1 and 6.2.

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