A unified generalization of some refinements of Jensen's inequality

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Abstract. In this paper, we establish a unified generalization of three refinements of Jensen's inequality by introducing several parameters. As applications, we illustrate that the improved Jensen's inequality can generate some new inequalities for special means such as arithmetic mean, geometric mean and logarithmic mean.

Keywords: Jensen's inequality, convex function, generalization, refinement, special means.

1. Introduction and main result

Let f be a convex function on $[a,b] \subset \mathbb{R}$. The classical Jensen's inequality reads as

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2}. \tag{1}$$

The Jensen's inequality, which was first proposed by Jensen in 1905, is one of the most important inequalities in pure and applied mathematics (see [1, 2]). For over 100 years, this celebrated inequality has generated lots of extensions and applications, see [3, 4, 5, 6, 7, 8, 9] and references cited therein. Besides these, there are some papers dealing with refinements of Jensen's inequality, the most famous of which is the Hermite-Hadamard inequality below:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{2}$$

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In [10], Wu provided two refinements of Jensen's inequality, as follows:

$$f\left(\frac{a+b}{2}\right) \le \frac{8}{(b-a)^2}\left[\frac{F(a)+F(b)}{2} - F\left(\frac{a+b}{2}\right)\right] \le \frac{f(a)+f(b)}{2}, \tag{3}$$

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{2}{(b-a)}\int_a^b f(x)dx \\ &- \frac{8}{(b-a)^2}\left[\frac{F(a)+F(b)}{2} - F\left(\frac{a+b}{2}\right)\right] \le \frac{f(a)+f(b)}{2}, \end{aligned} \tag{4}$$

where f is a convex function and F is a differentiable function such that $F''(x) = f(x)$ on $[a,b]$.

Inspired by inequalities (2), (3) and (4) above, a natural and interesting problem is whether we can establish a unified generalization of these refined Jensen's inequalities. In this paper we address this issue. Specifically, we shall construct a new inequality by introducing several parameters. Moreover, we will apply the inequality obtained to establish some inequalities for special means involving arithmetic mean, geometric mean, logarithmic mean and generalized logarithmic mean.

Our main result is stated in the following theorem.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$, and let F be a differentiable function such that $F''(x) = f(x)$ on $[a,b]$. Then, for $\mu \ge \max\{0,\lambda\}$ and $\lambda \in \mathbb{R}$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{2\mu-2\lambda}{(2\mu-\lambda)(b-a)}\int_a^b f(x)dx \\ &+ \frac{8\lambda}{(2\mu-\lambda)(b-a)^2}\left[\frac{F(a)+F(b)}{2} - F\left(\frac{a+b}{2}\right)\right] \le \frac{f(a)+f(b)}{2}. \end{aligned} \tag{5}$$

Remark 1.1. As a direct consequence of Theorem 1.1, if we put $\lambda = 0,\ \mu = 1$ in (5), we acquire the Hermite-Hadamard inequality; if we take $\lambda = 1,\ \mu = 1$ in (5), we obtain inequality (3); if we choose $\lambda = -1,\ \mu = 0$ in (5), we get inequality (4).

2. Proof of Theorem 1.1

Let us first transcribe a lemma that we will need in the proof of Theorem 1.1.

Lemma 2.1 ([10])**.** *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$, let g be a nonnegative, integrable function on $[a,b]$ and let*

$$\eta = \Big(\int_a^b (b-x)g(a+b-x)dx\Big) \Big/ \Big(\int_a^b (b-x)g(x)dx\Big).$$

Then

$$f\left(\frac{a+\eta b}{1+\eta}\right) \le \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \le \frac{f(a)+\eta f(b)}{1+\eta}. \tag{6}$$

Proof of Theorem 1.1. Choosing a function $g:\ [a,b]\to\mathbb{R}$ defined by

$$\begin{aligned} g(x) &:= (\mu-\lambda)\left|2x-a-b\right| + \mu(b-a-\left|2x-a-b\right|) \\ &= \begin{cases} (\mu-\lambda)(a+b-2x)+2\mu(x-a), & a\le x\le \dfrac{a+b}{2} \\ (\mu-\lambda)(2x-a-b)+2\mu(b-x), & \dfrac{a+b}{2}<x\le b. \end{cases} \end{aligned}$$

In view of the assumption $\mu\ge\max\{0,\lambda\}$, it is easy to verify that $g(x)$ is nonnegative and integrable on $[a,b]$. Then, one has

$$\begin{aligned} \eta &= \frac{\int_a^b (b-x)g(a+b-x)dx}{\int_a^b (b-x)g(x)dx} \\ &= \frac{\int_a^b (b-x)[(\mu-\lambda)\left|a+b-2x\right|+\mu(b-a-\left|a+b-2x\right|)]dx}{\int_a^b (b-x)[(\mu-\lambda)\left|2x-a-b\right|+\mu(b-a-\left|2x-a-b\right|)]dx} \\ &= 1 \end{aligned} \tag{7}$$

and

$$\begin{aligned} \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} &= \frac{\int_a^b f(x)[(\mu-\lambda)\left|2x-a-b\right|+\mu(b-a-\left|2x-a-b\right|)]dx}{\int_a^b [(\mu-\lambda)\left|2x-a-b\right|+\mu(b-a-\left|2x-a-b\right|)]dx} \\ &= \frac{\int_a^b f(x)[\mu(b-a)-\lambda\left|2x-a-b\right|]dx}{\int_a^b [\mu(b-a)-\lambda\left|2x-a-b\right|]dx} \\ &= \frac{\mu(b-a)\int_a^b f(x)dx-\lambda\int_a^b f(x)\left|2x-a-b\right|dx}{\mu(b-a)^2-\lambda\int_a^b \left|2x-a-b\right|dx}. \end{aligned} \tag{8}$$

Note that

$$\begin{aligned} \int_a^b \left|2x-a-b\right|dx &= \int_a^{\frac{a+b}{2}}(a+b-2x)dx+\int_{\frac{a+b}{2}}^b (2x-a-b)dx \\ &= \frac{(b-a)^2}{2} \end{aligned} \tag{9}$$

and

$$\begin{aligned} \int_a^b f(x)\left|2x-a-b\right|dx &= \int_a^{\frac{a+b}{2}} f(x)(a+b-2x)dx+\int_{\frac{a+b}{2}}^b f(x)(2x-a-b)dx \\ &= \int_a^{\frac{a+b}{2}}(a+b-2x)dF'(x)+\int_{\frac{a+b}{2}}^b (2x-a-b)dF'(x) \end{aligned}$$

$$
\begin{aligned}
&= -(b-a)F'(a)+2\int_a^{\frac{a+b}{2}} F'(x)dx+(b-a)F'(b)-2\int_{\frac{a+b}{2}}^b F'(x)dx \\
&= (b-a)[F'(b)-F'(a)]+2\int_a^{\frac{a+b}{2}} dF(x)-2\int_{\frac{a+b}{2}}^b dF(x) \\
&= (b-a)[F'(b)-F'(a)]+2F(\frac{a+b}{2})-2F(a)-2F(b)+2F(\frac{a+b}{2}) \\
&= (b-a)\int_a^b f(x)dx+4F(\frac{a+b}{2})-2[F(a)+F(b)]. \qquad (10)
\end{aligned}
$$

Applying equalities (9) and (10) to (8), we obtain

$$
\begin{aligned}
\frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} &= \frac{2\mu-2\lambda}{(2\mu-\lambda)(b-a)}\int_a^b f(x)dx \\
&+ \frac{8\lambda}{(2\mu-\lambda)(b-a)^2}\left[\frac{F(a)+F(b)}{2}-F\left(\frac{a+b}{2}\right)\right]. \qquad (11)
\end{aligned}
$$

Combining (6), (7) and (11) leads to the desired inequality (5). The proof of Theorem 1.1 is complete. □

3. Some applications

A growing number of inequalities for special means have been found significant applications in theory and practice (see [11, 12, 13, 14, 15, 16, 17, 18, 19]). To demonstrate usefulness of Theorem 1.1, in this section, we derive some inequalities for special means via the inequalities of Theorem 1.1.

Let us recall the arithmetic mean, geometric mean, logarithmic mean and generalized logarithmic mean for positive numbers α and β which are defined as follows:

$$
\begin{aligned}
A(\alpha,\beta) &= \frac{\alpha+\beta}{2}, && \text{arithmetic mean,} \\
G(\alpha,\beta) &= \sqrt{\alpha\beta} && \text{geometric mean,} \\
L(\alpha,\beta) &= \frac{\beta-\alpha}{\ln\beta-\ln\alpha}, && \text{logarithmic mean,} \\
L_p(\alpha,\beta) &= \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}},\ p\neq -1,\ 0, && \text{generalized logarithmic mean.}
\end{aligned}
$$

We have the following results:

Theorem 3.1. *Let a, b be positive real numbers, $\mu \geq \max\{0, \lambda\}$ and $\lambda \in \mathbb{R}$. Then, for $p \geq 1$ or $p < 0$ $(p \neq -1, -2)$, the following inequalities hold*

$$\begin{aligned}(A(a,b))^p &- \Big(\frac{2\mu-2\lambda}{2\mu-\lambda}\Big)(L_p(a,b))^p \\ &\leq \frac{8\lambda}{(2\mu-\lambda)(p+1)(p+2)(b-a)^2}\left[A(a^{p+2},b^{p+2})-(A(a,b))^{p+2}\right] \\ &\leq A(a^p,b^p)-\Big(\frac{2\mu-2\lambda}{2\mu-\lambda}\Big)(L_p(a,b))^p. \qquad (12)\end{aligned}$$

Furthermore, inequality (12) *is reversed for* $0 < p < 1$.

Proof of Theorem 3.1. It is clear that inequality (12) is symmetric with respect to variable a and b. Without loss of generality we assume that $b > a > 0$. Note that the function $f(x) = x^p$ is convex on $(0, +\infty)$ for $p \geq 1$ or $p < 0$, and the function $f(x) = -x^p$ is convex on $(0, +\infty)$ for $0 < p < 1$. We obtain immediately inequality (12) and its reverse version by applying these functions to Theorem 1.1. This completes the proof of Theorem 3.1. □

As a consequence of Theorem 3.1, taking $\lambda = 0,\ \mu = 1$; $\lambda = 1,\ \mu = 1$ and $\lambda = -1,\ \mu = 0$ respectively in (12), we obtain the following inequalities.

Corollary 3.1. *If a, b are positive real numbers, $p \geq 1$ or $p < 0$ $(p \neq -1, -2)$, then we have*

$$(A(a,b))^p \leq (L_p(a,b))^p \leq A(a^p,b^p), \qquad (13)$$

$$\begin{aligned}(A(a,b))^p &\leq \frac{8}{(p+1)(p+2)(b-a)^2}\left[A(a^{p+2},b^{p+2})-(A(a,b))^{p+2}\right] \\ &\leq A(a^p,b^p), \qquad (14)\end{aligned}$$

$$\begin{aligned}(A(a,b))^p - 2(L_p(a,b))^p &\leq \frac{8}{(p+1)(p+2)(b-a)^2}\left[(A(a,b))^{p+2}-A(a^{p+2},b^{p+2})\right] \\ &\leq A(a^p,b^p)-2(L_p(a,b))^p. \qquad (15)\end{aligned}$$

All of the above inequalities are reversed for $0 < p < 1$.

Theorem 3.2. *Let a, b be positive real numbers, $\mu \geq \max\{0, \lambda\}$ and $\lambda \in \mathbb{R}$. Then the following inequalities hold*

$$\begin{aligned}G(a,b)-\Big(\frac{2\mu-2\lambda}{2\mu-\lambda}\Big)L(a,b) &\leq \Big(\frac{8\lambda}{2\mu-\lambda}\Big)\Big(\frac{L(a,b)}{b-a}\Big)^2[A(a,b)-G(a,b)] \\ &\leq A(a,b)-\Big(\frac{2\mu-2\lambda}{2\mu-\lambda}\Big)L(a,b). \qquad (16)\end{aligned}$$

Proof of Theorem 3.2. Without loss of generality we assume that $b > a > 0$. Note that $f(x) = e^x$ is convex on $[\ln a, \ln b]$. Using Theorem 1.1 with $f(x) = e^x, x \in [\ln a, \ln b]$, we acquire inequality (16) described in Theorem 3.2. $\square$

As a consequence of Theorem 3.2, putting $\lambda = 0,\ \mu = 1$; $\lambda = 1,\ \mu = 1$ and $\lambda = -1,\ \mu = 0$ respectively in (16), we get the following inequalities.

Corollary 3.2. *If a, b are positive real numbers, then we have*

$$G(a,b) \leq L(a,b) \leq A(a,b), \tag{17}$$

$$G(a,b) \leq 8\Big(\frac{L(a,b)}{b-a}\Big)^2[A(a,b) - G(a,b)] \leq A(a,b), \tag{18}$$

$$G(a,b) - 2L(a,b) \leq 8\Big(\frac{L(a,b)}{b-a}\Big)^2[G(a,b) - A(a,b)] \leq A(a,b) - 2L(a,b). \tag{19}$$

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