Applications of β-open sets

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Abstract. In this paper, we establish the validity of the β-open sets. We introduce and study topological properties of β-limit point, β-derived set, β-interior points, β-border, β-frontier and β-exterior. The existence of their relation is also investigated with examples and counter examples.

Keywords: β-open sets, β-interior points, β-derived set, β-boundary, β-frontier and β-exterior.

1. Introduction

Generalized open sets play a vital role in General Topology and are now the research topics of many topologists worldwide. N. Levine [6] in 1863, introduced the notion of semi-open sets and T.M. Nour [10] in 1998 presented the concept of semi-closure, semi-interior, semi-frontier and semi-exterior. Njastad [9] presented the notion of α-open sets and Caldas [4] further developed the topological properties of α-open sets [11]. One of the generalized forms of open sets is the pre-open set which is given by Mashhour et. al. [8] in 1983. It gave an inspiration to Youngbae Jun et. al. [5] to further generalized the properties of pre-open set. Abd El-Monsef et. al. [1] gave the concept of β-open sets and β-continuity in topological spaces. The concept of nearly open set played a

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significant role in expansions of some advance theories of topological structures such as fuzzy set theory, soft rough set theory, probability theory and are widely research these days due to its wide application.

In this paper, we investigate the fundamental properties of β-limit points, β-derived sets, β-closure of a set, β-interior points, β-border, β-frontier and β-exterior with numerous examples. Moreover, the relation between the properties and existing properties are studied.

2. Preliminaries

Throughout this paper, (X, τ) (or simply X) means topological space. For $A \subseteq X$, closure of A is denoted by Cl(A) and interior of A is denoted Int(A).

Definition 2.1. *Let X be a topological space, then $A \subseteq X$ is called:*
(a) *semi-open [6] if $A \subseteq Cl(Int(A))$;*
(b) *α-open [9] if $A \subseteq Int(Cl(Int(A)))$;*
(c) *pre-open [8] if $A \subseteq Int(Cl(A))$;*
(d) *β-open [1] if $A \subseteq Cl(Int(Cl(A)))$.*

The complement of β-open(resp.α-open, semi-open, pre-open) set is called β-closed set(resp.α-closed set, semi-closed set, pre-closed set). The intersection of all β-closed sets(resp.α-closed sets, semi-closed sets, pre-closed sets) in X containing a subset A in X is called β-closure(resp. α-closure, semi-closure, pre-closure) and is denoted by $Cl_\beta(A)$(resp.$Cl_\alpha(A)$, sCl(A), $Cl_p(A)$). It is well known fact that the set $B \subseteq X$ is β-closed iff $B = Cl_\beta(A)$.

We denote the family of β-open(resp. α-open, pre-open) sets by τ^β(resp. τ^α, τ^p). But τ^β need not be a topology which is explained in Example 3.3.

Example 2.1. (a) Consider a topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ on set $X = \{a, b, c\}$. Then the family of β-open sets, α-open sets and pre-open sets are equal with topology τ on X i.e. $\tau^\beta = \tau^\alpha = \tau = \tau^p$.

(b) Consider a topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ on a set $X = \{a, b, c\}$. Then, $\tau^\beta = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and $\tau^\alpha = \tau = \tau^p$.

3. Applications of β-open sets

Definition 3.1. *Let B be a subset of a topological space (X, τ). A point $b \in B$ is said to be β-limit point of B if $\forall A \in \tau^\beta$ containing b, $A \cap B \setminus \{b\} \neq \emptyset$.*

The set of β-limit points of B is called β-derived set of B and is denoted by $D_\beta(B)$. Note that $D_p(B)$ [5], $D_\alpha(B)$ [4] and $D(B)$ denotes derived set of pre-open set, α-open set and derived set of B respectively.

Example 3.1. (a) Let (X, τ) be the topological space which is described in Example 2.1[a]. Let $A = \{a, b\}$. Then, $D_\beta(A) = \{c\} = D_p(A) = D_\alpha(A) = D(A)$.

(b) Let (X, τ) be the topological space which is described in Example 2.1[b]. Let $A = \{a, b\}$. Then, $D_p(A) = D_\alpha(A) = D(A) = \{c\} = D_\beta(A)$.

Theorem 3.1. *Let B be a subset of X and $b \in X$. Then the following are equivalent:*

(i) *For $b \in A$ and $\forall A \in \tau^\beta$, $B \cap A \neq \emptyset$.*

(ii) *$b \in Cl_\beta(B)$.*

Proof. If $b \notin Cl_\beta(B)$, then there exist β-closed set C such that $B \subseteq C$ and $b \notin C$. Hence, $X \setminus C$ is β-open set containing b and $B \cap X \setminus C \subseteq B \cap X \setminus B = \emptyset$, which is a contradiction to (i). Hence, $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$ is straightforward. □

Corollary 3.1. *For any subset B of X, we have $D_\beta(B) \subseteq Cl_\beta(B)$.*

Proof. Suppose $b \in D_\beta(B)$, then there exists a β-open set A such that $A \cap B \setminus \{b\} \neq \emptyset$ which implies $A \cap B \neq \emptyset$. Hence, $b \in Cl_\beta(B)$. □

Theorem 3.2. *For any subset B of X, $Cl_\beta(B) = B \cup D_\beta(B)$.*

Proof. Let $b \in Cl_\beta(B)$. Assume that $b \notin B$ and let $G \in \tau^\beta$ with $b \in G$. Then $G \cap B \setminus \{b\} \neq \emptyset$ and so $b \in D_\beta(B)$. Hence, $Cl_\beta(B) \subseteq B \cup D_\beta(B)$. For the reverse inclusion, $B \subseteq Cl_\beta(B)$ and by Corollary 3.1, $B \cup D_\beta(B) \subseteq Cl_\beta(B)$. Hence, the proof. □

Corollary 3.2. *A subset B is β-closed set iff it contains the set of β-limit points.*

Lemma 3.1. *If $\{A_i : i \in \Delta\}$ is a family of β-open sets in X, then $\bigcup_{i \in \Delta} A_i$ is a β-open set in X, where Δ is any index set.*

Proof. Straightforward □

Example 3.2. Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \emptyset, \{a\}, \{b\} \{a, b\}\}$. Then, $\tau^\beta = \tau \cup \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}\}$.

So, $\{a, d\} \cap \{b, d\} = \{d\} \notin \tau^\beta$ which means that the intersection of two β-open set is not β-open in general.

Remark 3.1. For any topology τ on a set X, τ^β may not be topology on X.

Example 3.3. Let $X = \{a, b, c, d\}$ be a set with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}, \{a, b, c\}\}$. Then, $\tau^\beta = \tau \cup \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Clearly τ^β is not a topology as $\{b, c\}, \{a, c\} \in \tau^\beta$ but $\{b, c\} \cap \{a, c\} = \{c\} \notin \tau^\beta$. Another reason for τ^β not being topology is explained in Example 3.5.

Theorem 3.3. *Let B_1 and B_2 be subsets of X. If $B_1 \in \tau^\beta$ and τ^β is a topology on X, then $B_1 \cap Cl_\beta(B_2) \subseteq Cl_\beta(B_1 \cap B_2)$.*

Proof. Let $b \in B_1 \cap Cl_\beta(B_2)$. Then, $b \in B_1$ and $b \in Cl_\beta(B_2) = B_2 \cup D_\beta(B_2)$. If $b \in B_2$, then $b \in B_1 \cap B_2 \subseteq Cl_\beta(B_1 \cap B_2)$. If $b \notin B_2$, then $b \in D_\beta(B_2)$ and for all β-open set G containing b, $G \cap B_2 \neq \emptyset$. Since $B_1 \in \tau^\beta$, so $G \cap B_1$ is also a β-open set containing b.

Hence, $G \cap (B_1 \cap B_2) = (G \cap B_1) \cap B_2 \neq \emptyset$ and consequently $b \in D_\beta(B_1 \cap B_2) \subseteq Cl_\beta(B_1 \cap B_2)$. Therefore, $B_1 \cap Cl_\beta(B_2) \subseteq Cl_\beta(B_1 \cap B_2)$. □

The converse of the above theorem is not true in general as seen in the following example.

Example 3.4. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{c\}, \{c, d\}, \{a, b, c\}\}$ be a topology on X and $\tau^\beta = \tau \cup \{\{a, c\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}\}$ is a topology on X. Let $B_1 = \{c, d\}$, $B_2 = \{b, c\} \in \tau^\beta$ and $B_1 \cap B_2 = \{c\}$. Then, $B_1 \cap Cl_\beta(B_2) = \{c, d\} \cap X = \{c, d\}$ and $Cl_\beta(B_1 \cap B_2) = X$. Therefore, converse is not true in general.

Example 3.5. Let (X, τ) be the topological space and τ^β be same as described in Example 3.3. Let $B_1 = \{b, c, d\}$, $B_2 = \{a, b, c\}$ and $B_1 \cap B_2 = \{b, c\}$. Then, $B_1 \cap Cl_\beta(B_2) = \{b, c, d\}$ and $Cl_\beta(B_1 \cap B_2) = \{b, c\}$. Therefore, $B_1 \cap Cl_\beta(B_2) = \{b, c, d\} \nsubseteq \{c, d\} = Cl_\beta(B_1 \cap B_2)$, which implies τ^β is not a topology.

Corollary 3.3. *If B_1 is β-closed in Theorem 3.3, then equality holds i.e. $B_1 \cap Cl_\beta(B_2) = Cl_\beta(B_1 \cap B_2)$.*

Proof. The first implication $B_1 \cap Cl_\beta(B_2) \subseteq Cl_\beta(B_1 \cap B_2)$ is same as in Theorem 3.3. For the other way, $Cl_\beta(B_1) = B_1$ since B_1 is β-closed so, $Cl_\beta(B_1 \cap B_2) \subseteq Cl_\beta(B_1) \cap Cl_\beta(B_2) = B_1 \cap Cl_\beta(B_2)$, which is the desired result. □

Theorem 3.4 (Properties of β-Derived set)**.** *For any subset B_1 and B_2 of topological space (X, τ), the following assertions hold:*

1. *If $B_1 \subseteq B_2$, then $D_\beta(B_1) \subseteq D_\beta(B_2)$.*
2. *$D_\beta(B_1) \cup D_\beta(B_2) \subseteq D_\beta(B_1 \cup B_2)$ and $D_\beta(B_1 \cap B_2) \subseteq D_\beta(B_1) \cap D_\beta(B_2)$.*
3. *$D_\beta(D_\beta(B)) \setminus B \subseteq D_\beta(B)$.*
4. *$D_\beta(B \cup D_\beta(B)) \subseteq B \cup D_\beta(B)$.*

Proof. 1. Let $b \in D_\beta(B_1)$. Then $U \cap B_1 \setminus \{b\} \neq \emptyset$, for any β-open set U containing b. Since $B_1 \subseteq B_2$, $U \cap B_2 \setminus \{b\} \neq \emptyset$, which implies $b \in D_\beta(B_2)$.

2. Follows directly from (1).

3. Let $b \in D_\beta(D_\beta(B)) \setminus B$, then $U \cap D_\beta(B) \setminus \{b\} \neq \emptyset$, for any β-open set U containing b. Let $c \in U \cap D_\beta(B) \setminus \{b\}$. Then, $c \in U$ and $c \in D_\beta(B)$ which implies $U \cap B \setminus \{c\} \neq \emptyset$. Let $d \in U \cap B \setminus \{c\}$. Thus, $d \neq b$, for $d \in B$ and $b \notin B$. Hence, $U \cap B \setminus \{b\} \neq \emptyset$. Hence, $b \in D_\beta(B)$.

4. Let $b \in D_\beta(B \cup D_\beta(B))$. If $b \in B$, the result is obvious. Suppose $b \notin B$, then $G \cap (B \cup D_\beta(B)) \setminus \{b\} \neq \emptyset$, for all $G \in \tau^\beta$ with $b \in G$. Hence, $G \cap B \setminus \{b\} \neq \emptyset$ or $G \cap D_\beta(B) \setminus \{b\} \neq \emptyset$. This implies $b \in D_\beta(B)$ for the first case.

If $G \cap D_\beta(B) \setminus \{b\} \neq \emptyset$, then $b \in D_\beta(D_\beta(B))$. Since, $b \notin B$, it follows from (3) that $b \in D_\beta(D_\beta(B)) \setminus B \subseteq D_\beta(B)$. Hence, the proof. □

Example 3.6. Let $X = \{a, b, c, d, e\}$ with

$$\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}.$$

Then, $\tau^\beta = \{X, \emptyset, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, b, c\}, \{a, c, d\}, \{a, d, e\}, \{a, b, d\}, \{a, c, e\}, \{b, c, d\}, \{b, c, e\}, \{b, d, e\}, \{c, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, c, d, e\}, \{a, b, d, e\}, \{b, c, d, e\}\}$. Consider $B_1 = \{a, c\}$ and $B_2 = \{d, e\}$. Then, $D_\beta(B_1) = \emptyset = D_\beta(B_2)$ and so $D_\beta(B_1) \cup D_\beta(B_2) = \emptyset \subset D_\beta(B_1 \cup B_2) = \{b, e\}$. Hence, converse is not true in the case of Theorem 3.4(2).

Example 3.7. Let $X = \{a, b, c, d\}$ be a set with topology $\tau = \{X, \emptyset, \{c\}, \{c, d\}, \{a, b, c\}\}$. Then, $\tau^\beta = \{X, \emptyset, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}\}$. Let $B = \{a, b, c\}$ be a subset of X. Then, $D_\beta(B) = \{a, b, d\}$ and so $D_\beta(D_\beta(B)) = \emptyset$, which implies converse of part (3) of the Theorem 3.4 need not be true in general. Similarly, $B \cup D_\beta(B) = \{a, b, c, d\}$ and so $D_\beta(B \cup D_\beta(B)) = \{a, b, d\}$. Hence, $B \cup D_\beta(B) \nsubseteq D_\beta(B \cup D_\beta(B))$ which implies the converse of part (4) of the above theorem is not true in general.

Definition 3.2. *Let A be a subset of a topological space X. A point $p \in A$ is called pre-interior point [5] of A if there exists a pre-open set P containing p such that $P \subseteq A$. The set of all pre-interior points of A is known as pre-interior points of A and it is denoted by $Int_p(B)$*

Definition 3.3. *Let B be a subset of a topological space X. A point $b \in B$ is called β-interior point of B if there exists a β-open set G containing b such that $G \subseteq B$. The set of all β-interior points of B is called β-interior points of B and is denoted by $Int_\beta(B)$.*

Theorem 3.5. *Let B be a subset of X. Then, every pre-interior point of B is β-interior point of B, i.e. $Int_p(B) \subseteq Int_\beta(B)$.*

Proof. Let $b \in Int_p(B)$. Then, there exist pre-open set P containing b such that $P \subseteq B$. Every pre-open set is β-open, thus we get a β-open set P containing b such that $P \subseteq B$. It follows that $b \in Int_\beta(B)$.

The converse of this theorem is not true in general given by following example. □

Example 3.8. Let $X=\{a, b, c, d, e\}$ with topology $\tau=\{X, \emptyset, \{b\}, \{d, e\}, \{b, d, e\}\}$. Then, $\tau^p=\tau\cup\{\{d\}, \{e\}, \{b, d\}, \{b, e\}, \{a, b, d\}, \{a, b, e\}, \{b, c, d\}, \{b, c, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{b, c, d, e\}\}$ and $\tau^\beta = \tau^p\cup\{\{a, b\}, \{a, d\}, \{a, e\}, \{b, c\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, b, c\}, \{a, c, d\}, \{a, d, e\}, \{a, c, e\}, \{c, d, e\}, \{a, c, d, e\}\}$.

(i) Consider a subset $B = \{a, c, d\}$. Then, we have $Int_p(B) = \{d\}$ and $Int_\beta(B) = \{a, c, d\}$.

(ii) Consider a subset $B = \{a, c, d, e\}$. Then, we have $Int_p(B) = \{d, e\}$ and $Int_\beta(B) = \{a, c, d, e\}$.

(iii) Consider a subset $B = \{a, b\}$. Then, we have $Int_p(B) = \{b\}$ and $Int_\beta(B) = \{a, b\}$.

Theorem 3.6 (Properties of β-interior)**.** *For subsets B, B_1, B_2 of a topological space X, the following hold:*

(1) *$Int_\beta(B)$ is the largest β-open set contained in B.*

(2) *B is β-open iff $B = Int_\beta(B)$.*

(3) *$Int_\beta(Int_\beta(B)) = Int_\beta(B)$.*

(4) *$Int_\beta(B) = B \setminus D_\beta(X \setminus B)$.*

(5) *$X \setminus Int_\beta(B) = Cl_\beta(X \setminus B)$.*

(6) *$Int_\beta(X \setminus B) = X \setminus Cl_\beta(B)$.*

(7) *If $B_1 \subseteq B_2$, then $Int_\beta(B_1) \subseteq Int_\beta(B_2)$.*

(8) *$Int_\beta(B_1) \cup Int_\beta(B_2) \subseteq Int_\beta(B_1 \cup B_2)$.*

(9) *$Int_\beta(B_1 \cap B_2) \subseteq Int_\beta(B_1) \cap Int_\beta(B_2)$.*

Proof. (1), (2) are straightforward.

(3) Trivially by (1) and (2).

(4) If $b \in B \setminus D_\beta(X \setminus B)$, then $b \notin D_\beta(X \setminus B)$ which implies there exists β-open set U containing b such that $U \cap (X \setminus B) = \emptyset$. Hence, $b \in U \subseteq B$ and $b \in Int_\beta(B)$. On the other hand, if $b \in Int_\beta(B) \subseteq B$ and $Int_\beta(B)$ is β-open set and $Int_\beta(B) \cap (X \setminus B) = \emptyset$. Hence, $b \notin D_\beta(X \setminus B)$. Therefore, $Int_\beta(B) = B \setminus D_\beta(X \setminus B)$.

(5) Using Theorem 3.2 and above part,

$$\begin{aligned} X \setminus Int_\beta(B) &= X \setminus (B \setminus D_\beta(X \setminus B)) \\ &= (X \setminus B) \cup D_\beta(X \setminus B) \\ &= Cl_\beta(X \setminus B). \end{aligned}$$

Hence, the proof.

(6) We have,

$$\begin{aligned} Int_\beta(X \setminus B) &= (X \setminus B) \setminus D_\beta(B) \\ &= (X \setminus B) \cap (D_\beta(B))^c \\ &= (X \setminus B) \cap (X \setminus D_\beta(B)) \\ &= X \setminus (B \cup D_\beta(B)) \\ &= X \setminus Cl_\beta(B). \end{aligned}$$

Hence, the proof.

(7) Let $b \in Int_\beta(B_1)$. Then, by definition, there exists β-open set U such that $b \in U \subseteq B_1$. Since $B_1 \subseteq B_2$ implies $b \in U \subseteq B_2$. Hence, $b \in Int_\beta(B_2)$. Hence, the proof.

(8) Since $B_1 \subseteq B_1 \cup B_2$ therefore, $Int_\beta(B_1) \subseteq B_1 \subseteq B_1 \cup B_2$. Similarly, $Int_\beta(B_2) \subseteq B_2 \subseteq B_1 \cup B_2$. We have, $Int_\beta(B_1) \cup Int_\beta(B_2) \subseteq B_1 \cup B_2$. Now,

$Int_\beta(B_1) \cup Int_\beta(B_2)$ is β-open subset of $B_1 \cup B_2$. As $Int_\beta(B_1 \cup B_2)$ is largest β-open subset of $B_1 \cup B_2$, we have $Int_\beta(B_1) \cup Int_\beta(B_2) \subseteq Int_\beta(B_1 \cup B_2)$. Hence, the proof.

(9) is same as in (8).

Converse of (7), (8) and (9) is not true in general as seen in the following example. □

Example 3.9. 1. Consider a set $X = \{a, b, c, d, e\}$ with same topology $\tau = \{\emptyset, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ and τ^β as in Example 3.6. Let $B_1 = \{a, b, e\}$ and $B_2 = \{a, c, e\}$ be a subset of X. Then $Int_\beta(B_1) = \{a\}$ and $Int_\beta(B_2) = \{a, c, e\}$ which implies $Int_\beta(B_1) \subseteq Int_\beta(B_2)$ while $B_1 \nsubseteq B_2$. Again, let $B_1 = \{b, e\}$ and $B_2 = \{c, d\}$ be a subset of X, then $Int_\beta(B_1) = \emptyset$ and $Int_\beta(B_2) = \{c, d\}$. Hence $Int_\beta(B_1 \cup B_2) = \{b, c, d, e\} \nsubseteq \{c, d\} = Int_\beta(B_1) \cup Int_\beta(B_2)$.

2. Let $X = \{a, b, c, d\}$ be a set with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}, \{a, b, c\}\}$. Then $\tau^\beta = \tau \cup \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, c, d\}, \{b, c, d\}\}$ as in Example 3.3. Consider a subset $B_1 = \{b, c\}$ and $B_2 = \{a, c, d\}$ of X. Then $Int_\beta(B_1) \cap Int_\beta(B_2) = \{c\}$ while $Int_\beta(B_1 \cap B_2) = \emptyset$ which proves that $Int_\beta(B_1) \cap Int_\beta(B_2) \nsubseteq Int_\beta(B_1 \cap B_2)$.

Definition 3.4 ([5])**.** *For any subset A of X, the set*

$$b_p(A) = A \setminus Int_p(A)$$

is called the pre-border of A, and the set

$$Fr_p(A) = Cl_p(A) \setminus Int_p(A)$$

is called the pre-frontier of A.

Definition 3.5. *For any subset B of X, the set,*

$$b_\beta(B) = B \setminus Int_\beta(B)$$

is called the β-border of B, and the set

$$Fr_\beta(B) = Cl_\beta(B) \setminus Int_\beta(B)$$

is called the β-frontier of B.

Theorem 3.7 (Properties of β-Boundary)**.** *For any subset B of X, the following statements hold:*

(1) $b_\beta(B) \subseteq b_p(B)$.
(2) $B = Int_\beta(B) \cup b_\beta(B)$ *and* $Int_\beta(B) \cap b_\beta(B) \neq \emptyset$.
(3) B *is β-open set* $\Leftrightarrow b_\beta(B) = \emptyset$.
(4) $b_\beta(Int_\beta(B)) = \emptyset$.
(5) $Int_\beta(b_\beta(B)) = \emptyset$.
(6) $b_\beta(b_\beta(B)) = b_\beta(B)$.
(7) $b_\beta(B) = B \cap Cl_\beta(X \setminus B)$.
(8) $b_\beta(B) = B \cap D_\beta(X \setminus B)$.

Proof. (1) Since $Int_p(B) \subseteq Int_\beta(B)$, we have $b_\beta(B) = B \setminus Int_\beta(B) \subseteq B \setminus Int_p(B)$, which implies $b_\beta(B) \subseteq b_p(B)$.

Converse of above is not true which is explained in Example 3.10.

(2) Straightforward.

(3) Since $Int_\beta(B) \subseteq B$ and B is β-open $\Leftrightarrow B = Int_\beta(B) \Leftrightarrow b_\beta(B) = B \setminus Int_\beta(B) \Leftrightarrow b_\beta(B) = \emptyset$.

(4) Since $Int_\beta(B)$ is β-open implies directly from (3) that $b_\beta(Int_\beta(B)) = \emptyset$.

(5) Let $b \in Int_\beta(b_\beta(B))$, then $b \in b_\beta(B) \subseteq B$ and so $b \in Int_\beta(B)$ since $Int_\beta(b_\beta(B)) \subseteq Int_\beta(B)$. Thus, $b \in Int_\beta(B) \cap b_\beta(B)$, which is a contradiction as per (2) of Theorem 3.7. Hence, $Int_\beta(b_\beta(B)) = \emptyset$.

(6) Since $b_\beta(b_\beta(B)) = b_\beta(B) \setminus Int_\beta(b_\beta(B)) = b_\beta(B)$, using part (5) Theorem 3.7. Hence, the proof.

(7) Since $b_\beta(B) = B \setminus Int_\beta(B) = B \setminus (X \setminus Cl_\beta(X \setminus B)) = B \cap (X \setminus Cl_\beta(X \setminus B))^c = B \cap Cl_\beta(X \setminus B)$, using part(6) of Theorem 3.6.

(8) By using Theorem 3.2 and above part,

$$\begin{aligned} b_\beta(B) &= B \cap Cl_\beta(X \setminus B) \\ &= B \cap ((X \setminus B) \cup D_\beta(X \setminus B)) \\ &= (B \cap X \setminus B) \cup (B \cap D_\beta(X \setminus B)) \\ &= \emptyset \cup (B \cap D_\beta(X \setminus B)) \\ &= B \cap D_\beta(X \setminus B). \end{aligned}$$

Hence, the proof. □

Example 3.10. Let $X = \{a, b, c, d, e\}$ be a set with topology $\tau = \{X, \emptyset, \{b\}, \{d, e\}, \{b, d, e\}\}$. Then $\tau^p = \tau \cup \{\{d\}, \{e\}, \{b, d\}, \{b, e\}, \{a, b, d\}, \{a, b, e\}, \{b, c, d\}, \{b, c, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{b, c, d, e\}\}$ and $\tau^\beta = \tau^p \cup \{\{a, b\}, \{a, d\}, \{a, e\}, \{b, c\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, b, c\}, \{a, c, d\}, \{a, d, e\}, \{a, c, e\}, \{c, d, e\}, \{a, c, d, e\}$. Consider a subset $B = \{a, c, d\}$. Then $b_p(B) = \{a, c\}$ and $b_\beta(B) = \emptyset$ which implies that the converse of Theorem 3.7(1) is not true in general.

Lemma 3.2. *Let B be a subset of topological space X, then B is β-closed if and only if $Fr_\beta(B) \subseteq B$.*

Proof. Let B be β-closed. Then, $Fr_\beta(B) = Cl_\beta(B) \setminus Int_\beta(B) = B \setminus Int_\beta(B) \subseteq B$. Conversely, suppose $Fr_\beta(B) \subseteq B$. Then, $Cl_\beta(B) \setminus Int_\beta(B) \subseteq B$ and so $Cl_\beta(B) \subseteq B$. Hence, $B = Cl_\beta(B)$ and so B is β-closed, which completes the proof. □

Theorem 3.8 (Properties of β-Frontier)**.** *Let B be a subset of X, then the following assertions hold:*

(1) $Fr_\beta(B) \subseteq Fr_p(B)$.

(2) $Cl_\beta(B) = Int_\beta(B) \cup Fr_\beta(B)$ *and* $Int_\beta(B) \cup Fr_\beta(B) = \emptyset$.

(3) $b_\beta(B) \subseteq Fr_\beta(B)$.

(4) $Fr_\beta(B) = b_\beta(B) \cup (D_\beta(B) \setminus Int_\beta(B))$.

(5) B *is* β*-open* $\Leftrightarrow Fr_\beta(B) = b_\beta(X \setminus B)$.
(6) $Fr_\beta(B) = Cl_\beta(B) \cap Cl_\beta(X \setminus B)$.
(7) $Fr_\beta(B) = Fr_\beta(X \setminus B)$.
(8) $Fr_\beta(B)$ *is* β*-closed.*
(9) $Int_\beta(B) = B \setminus Fr_\beta(B)$.
(10) $Fr_\beta(Fr_\beta(B)) \subseteq Fr_\beta(B)$.
(11) $Fr_\beta(Int_\beta(B)) \subseteq Fr_\beta(B)$.
(12) $Fr_\beta(Cl_\beta(B)) \subseteq Fr_\beta(B)$.

Proof. (1) Since $Fr_\beta(B) = Cl_\beta(B) \setminus Int_\beta(B) \subseteq Cl_p(B) \setminus Int_\beta(B) \subseteq Cl_p(B) \setminus Int_p(B) = Fr_p(B)$.

(2) The first part is direct. For the next, we have $Int_\beta(B) \cup F_\beta(B) = Int_\beta(B) \cup (Cl_\beta(B) \setminus Int_\beta(B)) = \emptyset$ (Obviously).

(3) Since $B \subseteq Cl_\beta(B)$ and $b_\beta(B) = B \setminus Int_\beta(B) \subseteq Cl_\beta(B) \setminus Int_\beta(B) = Fr_\beta(B)$.

(4) By using the definition of β-boundary of B and Theorem 3.2, we have

$$\begin{aligned}
Fr_\beta(B) &= Cl_\beta(B) \setminus Int_\beta(B)\\
&= (B \cup D_\beta(B)) \setminus Int_\beta(B)\\
&= (B \cup D_\beta(B)) \cap (X \setminus Int_\beta(B))\\
&= (B \cap (X \setminus Int_\beta(B)) \cup (D_\beta(B) \cap (X \setminus Int_\beta(B))\\
&= (B \setminus Int_\beta(B)) \cup (D_\beta(B) \setminus Int_\beta(B))\\
&= b_\beta(B) \cup (D_\beta(B) \setminus Int_\beta(B)),
\end{aligned}$$

which completes the proof.

(5) Suppose B is β-open. Then,

$$\begin{aligned}
Fr_\beta(B) &= b_\beta(B) \cup (D_\beta(B) \setminus Int_\beta(B))\\
&= \emptyset \cup (D_\beta(B) \setminus B)\\
&= D_\beta(B) \setminus B\\
&= D_\beta(B) \cap (X \setminus B)\\
&= b_\beta(X \setminus B),
\end{aligned}$$

using part (3) and (8) of Theorem 3.7.

Conversely, suppose $Fr_\beta(B) = b_\beta(X \setminus B)$. Then

$$\begin{aligned}
\emptyset &= Fr_\beta(B) \setminus b_\beta(X \setminus B)\\
&= (Cl_\beta(B) \setminus Int_\beta(B)) \setminus (X \setminus B \setminus Int_\beta(B))\\
&= B \setminus Int_\beta(B),
\end{aligned}$$

which implies $B \subseteq Int_\beta(B)$. In general, $Int_\beta(B) \subseteq B$. Hence, $Int_\beta(B) = B$.

(6) Using the part (5) of Theorem 3.6, we have

$$\begin{aligned} Cl_\beta(B) \cap Cl_\beta(X \setminus B) &= Cl_\beta(B) \cap (X \setminus Int_\beta(B)) \\ &= Cl_\beta(B) \cap (Int_\beta(B))^c \\ &= Cl_\beta(B) \setminus Int_\beta(B) \\ &= Fr_\beta(B), \end{aligned}$$

which complete the proof.

(7)Same as (6).

(8) We need to show that $Cl_\beta(Fr_\beta(B)) = Fr_\beta(B)$. Clearly, $Fr_\beta(B) \subseteq Cl_\beta(Fr_\beta(B))$. Next, we shall show that $Cl_\beta(Fr_\beta(B) \subseteq Fr_\beta(B)$. We have,

$$\begin{aligned} Cl_\beta(Fr_\beta(B)) &= Cl_\beta(Cl_\beta(B) \cap Cl_\beta(X \setminus B)) \\ &\subseteq Cl_\beta(Cl_\beta(B)) \cap Cl_\beta(Cl_\beta(X \setminus B)) \\ &= Cl_\beta(B) \cap Cl_\beta(X \setminus B) \\ &= Fr_\beta(B), \end{aligned}$$

which implies $Fr_\beta(B)$ is closed set.

(9) Using the definition of β-frontier of B and basic property of set theory, we have

$$\begin{aligned} B \setminus Fr_\beta(B) &= B \setminus (Cl_\beta(B) \setminus Int_\beta(B)) \\ &= (B \setminus Cl_\beta(B)) \cup (B \cap Cl_\beta(B) \cap Int_\beta(B)) \\ &= (B \setminus Cl_\beta(B)) \cup Int_\beta(B) \\ &= \emptyset \cup Int_\beta(B) \\ &= Int_\beta(B). \end{aligned}$$

This completes the proof.

(10) Since $Fr_\beta(B)$ is β-closed and so by Lemma 3.2, $Fr_\beta(Fr_\beta(B)) \subseteq Fr_\beta(B)$.

(11) We have,

$$\begin{aligned} Fr_\beta(Int_\beta(B)) &= Cl_\beta(Int_\beta(B)) \setminus Int_\beta(Int_\beta(B)) \\ &\subseteq Cl_\beta(B) \setminus Int_\beta(B) \\ &= Fr_\beta(B). \end{aligned}$$

(12)We have,

$$\begin{aligned} Fr_\beta(Cl_\beta(B)) &= Cl_\beta(Cl_\beta(B)) \setminus Int_\beta(Cl_\beta(B)) \\ &\subseteq Cl_\beta(B) \setminus Int_\beta(B) \\ &= Fr_\beta(B). \end{aligned}$$

Hence, the proof. $\square$

Example 3.11. Let $X = \{a, b, c, d\}$ be a set with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}, \{a, b, c\}\}$. Then $\tau^p = \tau$ and $\tau^\beta = \tau^p \cup \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{b, c, d\}\ \{a, c, d\}\}$.

Consider a subset $A = \{c, d\}$ and $B = \{a, c\}$ of X, then $Fr_\beta(A) = \{c, d\} = Fr_p(A)$. Also, $Fr_\beta(B) = \emptyset$ while $Fr_p(B) = \{c, d\}$ which implies equality in Theorem 3.8(1) may not hold.

Example 3.12. Consider $X = \{a, b, c, d\}$ with same topology τ and τ^β as in Example 3.2. Let $B = \{a, b, c\}$, then $b_\beta(B) = \emptyset$ while $Fr_\beta(B) = \{d\}$, which shows that the converse of Theorem 3.8(3) is not true in general.

Definition 3.6. *Let B be a subset of X, $Ext_\beta(B) = Int_\beta(X \setminus B)$ is said to be β-exterior of B.*

We denote $Ext_p(B)$ to be pre-exterior [5] of B.

Example 3.13. Let $X = \{a, b, c, d, e\}$ with

$$\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}.$$

Then, $\tau^\beta = \{X, \emptyset, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, b, c\}, \{a, c, d\}, \{a, d, e\}, \{a, b, d\}, \{a, c, e\}, \{b, c, d\}, \{b, c, e\}, \{b, d, e\}, \{c, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, c, d, e\}, \{a, b, d, e\}, \{b, c, d, e\}\}$. Consider a subset $A = \{b, c, d\}$ and $B = \{a, c, d, e\}$ of set X , then $Ext_\beta(A) = Int_\beta(X \setminus A) = \{a\}$and $Ext_\beta(B) = Int_\beta(X \setminus B) = \emptyset$.

Theorem 3.9. *For a subset B, B_1, B_2 of X, the following assertion are valid.*

(1) $Ext_p(B) \subseteq Ext_\beta(B)$.
(2) $Ext_\beta(B)$ *is a* β*-open.*
(3) $Ext_\beta(B) = X \setminus Cl_\beta(B)$.
(4) $Ext_\beta(Ext_\beta(B)) = Int_\beta(Cl_\beta(B)) \supseteq Int_\beta(B)$.
(5) *If* $B_1 \subseteq B_2$, *then* $Ext_\beta(B_1) \subseteq Ext_\beta(B_2)$.
(6) $Ext_\beta(B_1 \cup B_2) \subseteq Ext_\beta(B_1) \cap Ext_\beta(B_2)$.
(7) $Ext_\beta(B_1) \cup Ext_\beta(B_2) \subseteq Ext_\beta(B_1 \cap B_2)$.
(8) $Ext_\beta(X) = \emptyset, Ext_\beta(\emptyset) = X$.
(9) $Ext_\beta(B) = Ext_\beta(X \setminus Ext_\beta(B))$.
(10) $B = Int_\beta(B) \cup Ext_\beta(B) \cup Fr_\beta(B)$.

Proof. (1) Clearly by Theorem 3.5, $Int_p(B) \subseteq Int_\beta(B)$, we have $Ext_p(B) = Int_\beta(X \setminus B) \subseteq Int_\beta(X \setminus B) = Ext_\beta(B)$.

(2) Straightforward.

(3) By part(6) of Theorem 3.6, $X \setminus Cl_\beta(B) = Int_\beta(X \setminus B) = Ext_\beta(X \setminus B)$.

(4) By Theorem 3.5 and part (5) of Theorem 3.6,

$$\begin{aligned} Ext_\beta(Ext_\beta(B)) &= Ext_\beta(Int_\beta(X \setminus B)) \\ &= Int_\beta(X \setminus Int_\beta(X \setminus B)) \\ &= Int_\beta(Cl_\beta(X \setminus (X \setminus B))) \\ &= Int_\beta(Cl_\beta(B)) \supseteq Int_\beta(B). \end{aligned}$$

(5) Let $B_1 \subseteq B_2$. Then, $Ext_\beta(B_2) = Int_\beta(X \setminus B_2) \subseteq Int_\beta(X \setminus B_1) = Ext_\beta(B_1)$.

(6) By using part (9) of Theorem 3.6, we have

$$\begin{aligned} Ext_\beta(B_1 \cup B_2) &= Int_\beta(X \setminus (B_1 \cup B_2)) \\ &= Int_\beta((X \setminus B_1) \cap (X \setminus B_2)) \\ &\subseteq Int_\beta(X \setminus B_1) \cap Int_\beta(X \setminus B_2) \\ &= Ext_\beta(B_1) \cap Ext_\beta(B_2), \end{aligned}$$

which completes the proof.

(7) By using part (8) of Theorem 3.6, we have

$$\begin{aligned} Ext_\beta(B_1) \cup Ext_\beta(B_2) &= Int_\beta(X \setminus B_1) \cup Int_\beta(X \setminus B_2) \\ &\subseteq Int_\beta((X \setminus B_1) \cup (X \setminus B_2)) \\ &= Int_\beta(X \setminus (B_1 \cap B_2)) \\ &= Ext_\beta(B_1 \cap B_2), \end{aligned}$$

hence the proof.

(8) Straightforward.

(9) By using the definition of β-exterior of B, we have

$$\begin{aligned} Ext_\beta(X \setminus Ext_\beta(B)) &= Ext_\beta(X \setminus Int_\beta(X \setminus B)) \\ &= Int_\beta(Int_\beta(X \setminus B)) \\ &= Int_\beta(X \setminus B) \\ &= Ext_\beta(B). \end{aligned}$$

Hence, the proof.

(10) Trivial. □

Example 3.14. Let (X, τ) be a topological space same as given in Example 3.13. Consider $B_1 = \{b, c, d\}$ and $B_2 = \{b, c, e\}$, then $Ext_\beta(B_1) = Int_\beta(X \setminus B_1) = \{a\}$ and $Ext_\beta(B_2) = Int_\beta(X \setminus B_2) = \{a, d\}$, which implies $Ext_\beta(B_1) \subseteq Ext_\beta(B_2)$ but $B_1 \nsubseteq B_2$. This shows that the converse of Theorem 3.9(5) is not true.

Example 3.15. Let (X, τ) be a topological space same as given in Example 3.13. Let $B_1 = \{d, e\}$ and $B_2 = \{c\}$. Then, $Ext_\beta(B_1 \cup B_2) = \{a\} \neq \{a, b\} = \{a, b, c\} \cap \{a, b, d, e\} = Ext_\beta(B_1) \cap Ext_\beta(B_2)$, which implies that the equality in the Theorem 3.9(6) is not true.

Example 3.16. Let (X, τ) be a topological space same as given in Example 3.13. Let $B_1 = \{a, c, d\}$ and $B_2 = \{b, e\}$. Then, $Int_\beta(X \setminus B_1) = \emptyset$ and $Int_\beta(X \setminus B_2) = \{a, c, d\}$. Hence, $Ext_\beta(B_1) \cup Ext_\beta(B_2) = \emptyset \cup \{a, c, d\} = \{a, c, d\} \subseteq Ext_\beta(B_1 \cap B_2) = X$ which shows that the equality in Theorem 3.9(7) is not valid.

4. Conclusion

This paper begins with a brief survey of the notion of β-open sets and β-continuity introduced by Abd El-Monsef et al. [1]. We also recall some other generalized open sets in topological spaces, like semi-open sets [6], pre-open sets [8] and α-open sets [9] so as to compare these sets to β-open sets.

The authors studied β-limit points and β-derived sets in topological spaces and proved many results on β-derived sets. Some characteristics of β-interiors and β-closures of sets are also investigated.

Moreover, β-exterior, β-frontier and β-boundary of sets are also studied. Several examples are given to indicate the connections among these concepts. Some properties of these concepts are also discussed which will open the way for more applications of β-open sets in real-life problems. Also, all these properties of β-open sets in topological spaces can be very handy for studying compactness, connectedness, separation axioms via β-open sets.

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