# Approximate solution of Fredholm type fractional integro-differential equations using Bernstein polynomials

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**Abstract.** The main goal of this paper is to find an approximate solution for a certain type of Fredholm fractional integro-differential equation by using Bernstein polynomials. In the last section, some examples have been presented to compare their approximate and exact solutions.

**Keywords:** Caputo derivative, fractional integro-differential equations, Bernstein polynomials.

## 1. Introduction

Fractional differential equations have been implemented to model various problems in several fields, [2], [3], [4], [6] and [10]. Any system containing fractional derivatives is more practical than the regular system because of the non-locality of the fractional derivative. Recently, mathematicians have shown a lot of interest in studying new types of equations having non-local fractional derivatives. The study of any type of fractional integro-differential equation depends on the

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type of the fractional derivative. Therefore, many researchers have shown great interest in studying new types of the Caputo fractional differential equations and their applications, see [12] and [13]. Fractional integro-differential equations of Fredholm type have been studied by many researchers to find their approximate solutions using many types of methods and polynomials, see [1], [5], [12], [14], [18] and [19]. The Bernstein polynomials [7] is one of the methods for computing the approximate solution of fractional equation, see [13], [14], [17]. In [8], a solution of a special type of fractional integro-differential equations using Jacobi wavelet operational matrix of fractional integration presented and the same authors in [9], discussed numerical Solution of a Fredholm Fractional Integro-differential equation. Recently, Mansouri and Azimzadeh in [11], introduced an approximate solution of fractional delay Volterra integro-differential equations by Bernstein polynomials . Also, in [16], numerical solution method for multi-term variable order fractional differential equations by shifted Chebyshev polynomials of the third kind is given.

In this article, we study how to find approximate solutions to a class of Fredholm fractional integro-differential equations that contains the Caputo fractional derivative of order $n-1 < \alpha \leq n$. Finally, some examples are given to find their approximate solutions.

## 2. Preliminaries

In this section, we present some necessary definitions and results which will be used in other sections. We start with the definition and main properties of the fractional derivative. For more details on the subject see [15] and [4].

**Definition 2.1** ([15])**.** Let $y = f(x)$ be a function, then the fractional derivative of $y$ in Caputo sense of order $\alpha > 0$ is defined as:

$$
{}_a^cD_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1+n}} dt, & n-1 < \alpha < n, \quad n \in N, \\ \dfrac{d^n}{dx^n} f(x), & \alpha = n \in N. \end{cases}
$$

If $f(x)$ is a constant function, then ${}_a^cD_x^\alpha f(x) = 0$.

The Caputo derivative of $f(x) = (x-a)^j$ is defined as: (see [15])

$$
{}_a^cD_x^\alpha (x-a)^j = \begin{cases} 0, & \text{for} \quad j \in N \cup \{0\} \quad \text{and} \quad j < \lceil \alpha \rceil, \\ \dfrac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}(x-a)^{j-\alpha}, & \text{for} \quad j \in N \quad \text{and} \quad j \geq \lceil \alpha \rceil \\ & \text{or} \quad j \notin N \quad \text{and} \quad j > \lfloor \alpha \rfloor. \end{cases}
$$

Here, $\lceil \alpha \rceil$ is denoted to be the smallest integer greater than or equal to $\alpha$ and $\lfloor \alpha \rfloor$ is the largest integer less than or equal to $\alpha$.

**Lemma 2.1** ([15]). *The Caputo fractional differentiation is a linear operation, that is for any two constants $a_1, a_2$ and any two functions $y_1, y_2$, we have*

$$ {}_a^cD_x^\alpha(a_1y_1 + a_2y_2) = a_1({}_a^cD_x^\alpha(y_1)) + a_2({}_a^cD_x^\alpha(y_2)). $$

**Definition 2.2** ([7]). The Bernstein polynomials of degree $n$ are denoted by $B_{i,n}(x)$ and defined as:

$$ B_{i,n}(x) = \frac{\binom{n}{i}(x-a)^i(b-x)^{n-i}}{(b-a)^n}, \quad x \in [a,b] \subseteq \mathbb{R}, \quad i = 0,1,2,...,n. \tag{1} $$

Particularly, if $x \in [0,1]$ then $B_{i,n}(x)$ are defined as:

$$ B_{i,n}(x) = \binom{n}{i} x^i(1-x)^{n-i}, \quad i = 0,1,2,...,n. $$

Since $(b-x)^{n-i} = [(b-a)-(x-a)]^{(n-i)}$, equation (1) can be written as:

$$ B_{i,n}(x) = \sum_{j=i}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{i} \binom{n-i}{j-i} (x-a)^j. \tag{2} $$

Hence,

$$ B_{i,n}(x) = \sum_{j=i}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} (x-a)^j. \tag{3} $$

**Lemma 2.2** ([7]). *The derivatives of Bernstein polynomials of degree $n$ can be written as a linear combination of Bernstein polynomials of degree $n-1$ which is given by:*

$$ \frac{d}{dx} B_{i,n}(x) = n(B_{i-1,n-1}(x) - B_{i,n-1}(x)). \tag{4} $$

**Lemma 2.3.** The fractional derivative of order $0 < \alpha \in \mathbb{R} \setminus \mathbb{N}$ of the Bernstein polynomials of degree $n$ in the Caputo sense is given by:

$$ {}_a^cD_x^\alpha B_{i,n}(x) = \sum_{j=i}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} {}_a^cD_x^\alpha (x-a)^j. \tag{5} $$

Since ${}_a^cD_x^\alpha (x-a)^j = 0$ for each $j < \alpha$ , we have

$$ {}_a^cD_x^\alpha B_{i,n}(x) = \sum_{j=\lceil \alpha \rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} (x-a)^{j-\alpha}. \tag{6} $$

**Proof.** Follows from applying Definition 2.1 to equation (3). □

## 3. Approximation method

In this section, we propose the following fractional integro-differential equation and provide approximate solutions to this equation:

$$\begin{aligned}
&{}_a^cD_x^{\alpha} y(x) + \sum_{k=2}^{n} g_k(x)\ {}_a^cD_x^{(\frac{\alpha}{k})} y(x) + g_0(x)y(x) \\
&= f(x) + \sum_{m=1}^{n} \int_a^b K_m(x,t)\ {}_a^cD_t^{\frac{\beta}{m}} y(t)dt,
\end{aligned} \tag{7}$$

where $n-1 < \alpha \le n$, $\beta \le \alpha$ and $a \le t, x \le b$. Subject to the conditions $y^{(i)}(a) = \lambda_i$, $i = 0, 1, 2, ..., n-1$.

The solution of equation (7) is the function $y(x)$ which is a continuous function and its approximate solution can be expressed in terms of $n^{th}$-degree of Bernstein polynomial

$$y_n(x) = \sum_{i=0}^{n} c_i B_{i,n}(x). \tag{8}$$

From the initial condition, we have $\lambda_0 = y_n(a) = \sum_{i=0}^{n} c_i B_{i,n}(a)$, which implies that

$$c_0 = \lambda_0. \tag{9}$$

Again, from equation (3), we have

$$y_n'(x) = \sum_{i=0}^{n} c_i B_{i,n}'(a) = \sum_{i=0}^{n} c_i \sum_{j=i}^{n-i} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} j(x-a)^{j-1}.$$

This implies that all the terms are zero at $x = a$ except when $j = 1$. Hence, we obtain that

$$\lambda_1 = y_n'(a) = \sum_{i=0}^{n} c_i \frac{(-1)^{1-i}}{(b-a)} \binom{n}{1} \binom{1}{i}.$$

Therefore,

$$\lambda_1 = \frac{-n}{b-a} c_0 + \frac{n}{b-a} c_1.$$

Hence,

$$c_1 = \lambda_0 + \frac{(b-a)\lambda_1}{n}. \tag{10}$$

Thus, in general, if $n \ge m \in N$ we have

$$y_n^{(m)}(x) = \sum_{i=0}^{n} c_i B_{i,n}^{m}(x) = \sum_{i=0}^{n} c_i \sum_{j=i}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} m! \binom{j}{m} (x-a)^{j-m},$$

when $x = a$ all the terms are zero except $j = m$. Hence,

$$\lambda_m = y_n^{(m)}(a) = \sum_{i=0}^{m} c_i \frac{(-1)^{m-i} m!}{(b-a)^m} \binom{n}{m} \binom{m}{i}. \tag{11}$$

From equation (11) and solving for the coefficients $c_i$, $i = 0, 1, ..., m$, we obtain that:

$$c_i = \sum_{k=0}^{i} \frac{\binom{i}{k}}{\binom{n}{k}} \times \frac{(b-a)^i \; y^{(i)}(a)}{k!}. \tag{12}$$

Now, by substituting equations (2), (4), (12) in equation (7), we get an algebraic equation with unknown constants $c_i$, $i = m+1, m+2, ..., n$ and by a suitable way we can find a matrix equation of the form $AC = B$, where $A$ ia an $(n-m) \times (n-m)$ matrix and $C^T = [c_{m+1}, c_{m+2}, ..., c_n]$. Then $C = A^{-1}B$. Substituting the $c_i$'s in equation (8) we get the approximate solution of equation (7).

## 4. Illustrative examples

In this section, we discuss the approximate solution of some examples for distinct fractional derivatives $\alpha$ and $\beta$, where $n-1 < \alpha \leq n$ and $\beta \leq \alpha$ and compare them with their exact solutions. We start with the following example:

**Example 4.1.** Consider the integro-differential equation

$$ {}_1^cD_x^\alpha y(x) = f(x) + 3\int_1^2 (xt) \; {}_1^cD_t^\beta y(t)dt, \tag{13}$$

where $f(x) = \frac{2}{\Gamma(3-\alpha)}(x-1)^{2-\alpha} - \frac{6x(\beta-3)(2\beta-9)}{\Gamma(5-\beta)}$, $1 < \alpha \leq 2$, $\beta \leq \alpha$ and $1 \leq t, x \leq 2$. Subject to the conditions $y(1) = y'(1) = 2$.

Using Bernstein polynomials of degree $n = 3$, we approximate the solution as:

$$y(x) = \sum_{i=0}^{3} c_i B_{i,3}(x). \tag{14}$$

From equations (9), (10), we obtain that $c_0 = 2$ and $c_1 = \frac{8}{3}$.

Applying equation (6) on $y(x)$ and substituting in equation (13), we get

$$ {}_1^cD_x^\alpha \sum_{i=0}^{3} c_i B_{i,3}(x) = f(x) + 3\int_1^2 (xt) \; {}_1^cD_t^\beta \sum_{i=0}^{3} c_i B_{i,3}(t)dt. \tag{15}$$

Hence

$$\sum_{i=0}^{3} c_i \{ {}_1^cD_x^\alpha B_{i,3}(x) - 3\int_1^2 (xt) \; {}_1^cD_t^\beta B_{i,3}(t)dt\} = f(x). \tag{16}$$

Applying equation (6) , we get

$$\sum_{i=0}^{3} c_i \{ \sum_{j=\lceil\alpha\rceil}^{3} \frac{(-1)^{j-i}}{(b-a)^j} \binom{3}{j} \binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} (x-1)^{j-\alpha}$$
$$-3 \int_1^2 (xt) \sum_{j=\lceil\beta\rceil}^{3} \frac{(-1)^{j-i}}{(b-a)^j} \binom{3}{j} \binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+1-\beta)} (x-1)^{j-\beta} dt \} = f(x).$$

As a particular case, if we take $\alpha = 2$ and $\beta = 1$ the exact solution of equation (13) is $y(x) = x^2 + 1$. After integrating and simplifying the above equation, we get the following equation:

$$\begin{aligned} & c_0\Big[12 - 6x\Big] + c_1\Big[-30 + 18x\Big] + c_2\Big[24 - 18x\Big] + c_3\Big[-6 + 6x\Big] \\ & -3x \int_1^2 \{c_0\Big[-12t + 12t^2 - 3t^3\Big] + 3c_1\Big[8t - 10t^2 + 3t^3\Big] \\ & -3c_2\Big[-5t + 8t^2 - 3t^3\Big] + c_3\Big[3t - 6t^2 + 3t^3\Big]\} dt = 2 - 14x. \end{aligned} \tag{17}$$

Integrating the last equation and substituting for $c_0$ and $c_1$ and simplifying, we get

$$c_2[24 - \frac{69}{4}x] + c_3[-6 + \frac{3}{4}x] = 58 - \frac{119}{2}x.$$

Solving for $c_2$ and $c_3$, we obtain that $c_2 = 3.666$ and $c_3 = 4.997$. The approximate solution of equation (13) is

$$y(x) \approx 2(2-x)^3 + 8(x-1)(2-x)^2 + 3 \times (3.66)(x-1)^2(2-x) + 4.997(x-1)^3.$$

The following table describes the relation between the exact and approximate solution of some selected values of $x$, where $n = 3$, $\alpha = 2$ and $\beta = 1$.

Table 1: Exact and approximate solution when $\alpha = 2$ and $\beta = 1$

| $x$ | $y_{Approx}$ | $y_{Exact}$ |
|---|---|---|
| 1.1 | 2.20998 | 2.21 |
| 1.2 | 2.43991 | 2.44 |
| 1.3 | 2.68979 | 2.69 |
| 1,4 | 2.95962 | 2.95999999999999 |
| 1.5 | 3.24938 | 3.25 |
| 1.6 | 3.55906 | 3.55999999999999 |
| 1.7 | 3.88868 | 3.88999999999999 |
| 1.8 | 4.23821 | 4.24 |
| 1.9 | 4.60765 | 4.60999999999999 |
| 2 | 4.9971 | 4.99999999999999 |

Now, if we take $\alpha = \frac{3}{2}$ and $\beta = 0.5$, we have

$$\sum_{i=0}^{3} c_i \{ \sum_{j=\lceil\alpha\rceil}^{3} \frac{(-1)^{j-i}}{(b-a)^j} \binom{3}{j} \binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j-\frac{1}{2})} (x-1)^{j-\frac{3}{2}}$$
$$- 3\int_1^2 (xt) \sum_{j=\lceil\beta\rceil}^{3} \frac{(-1)^{j-i}}{(b-a)^j} \binom{3}{j} \binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+\frac{1}{2})} (x-1)^{j-\frac{1}{2}} dt\}$$
$$= \frac{4}{\sqrt{\pi}} (\sqrt{x-1} - \frac{64x}{7}).$$

Substituting and simplifying, we get

$$c_2 \left[ \frac{6}{\Gamma(\frac{3}{2})} (x-1)^{\frac{1}{2}} - \frac{18}{\Gamma(\frac{5}{2})} (x-1)^{\frac{3}{2}} - \frac{64x}{35\sqrt{\pi}} \right] + c_3 \left[ \frac{6}{\Gamma(\frac{5}{2})} (x-1)^{\frac{3}{2}} - \frac{64x}{7\Gamma(\frac{7}{2})} \right]$$
$$= -\frac{320x}{21\sqrt{\pi}} + \frac{256x}{105\sqrt{\pi}} + \frac{24}{\Gamma(\frac{3}{2})} (x-1)^{\frac{1}{2}} - \frac{36}{\Gamma(\frac{5}{2})} (x-1)^{\frac{3}{2}} - \frac{48}{\Gamma(\frac{3}{2})} (x-1)^{\frac{1}{2}}$$
$$- \frac{48}{\Gamma(\frac{5}{2})} (x-1)^{\frac{3}{2}} + \frac{4}{\sqrt{\pi}} (\sqrt{x-1} - \frac{64x}{7}).$$

We get $1.0316c_2 + 2.7511c_3 = 9.2489$ and $8.8335c_2 - 0.98876c_3 = -2.4978$. Solving for $c_2$ and $c_3$, we get $c_2 = 0.0897$ and $c_3 = 3.3283$.

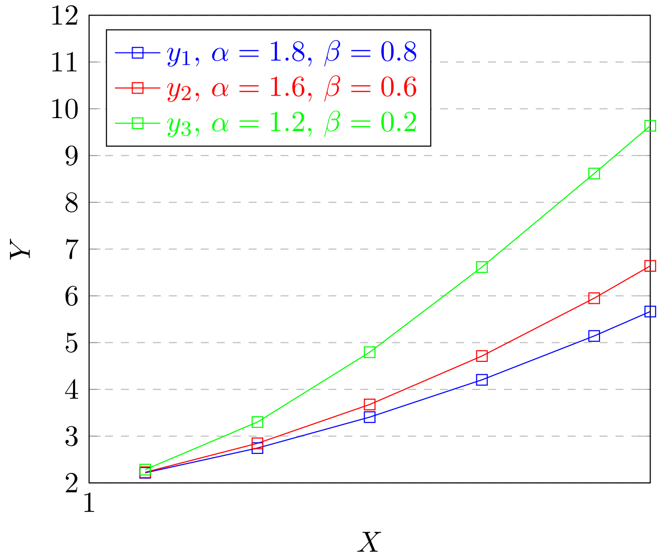
The following table describes the approximate solution of equation (13) for some selected values of $n$, $\alpha$ and $\beta$. Here, $y_1$, $y_2$ and $y_3$ represent the approximate solution when $n = 3$, ($\alpha = 1.8$, $\beta = 0.8$), ($\alpha = 1.6$, $\beta = 0.6$) and ($\alpha = 1.2$, $\beta = 0.2$), respectively. While $y_4$, $y_5$ and $y_6$ represent the approximate solution when $n = 7$, ($\alpha = 1.8$, $\beta = 0.8$), ($\alpha = 1.6$, $\beta = 0.6$) and ($\alpha = 1.2$, $\beta = 0.2$).

Table 2: Approximate solution when ($n = 3$) and($n = 7$)

| $x$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |
|---|---|---|---|---|---|---|
| 1.1 | 2.215946531 | 2.227593549 | 2.284126883 | 2.216914203 | 2.233433219 | 2.337681516 |
| 1.2 | 2.464096576 | 2.509845878 | 2.724168086 | 2.466686882 | 2.524408268 | 2.836704064 |
| 1.3 | 2.744915811 | 2.84596451 | 3.301614439 | 2.748712324 | 2.865521894 | 3.420408092 |
| 1.4 | 3.058869912 | 3.235156969 | 3.997956773 | 3.063062995 | 3.254315643 | 4.07293886 |
| 1.5 | 3.406424555 | 3.676630777 | 4.794685919 | 3.410119352 | 3.690280056 | 4.793196618 |
| 1.6 | 3.788045417 | 4.169593457 | 5.673292706 | 3.790382296 | 4.173485633 | 5.579198956 |
| 1.7 | 4.204198174 | 4.713252534 | 6.615267966 | 4.204405053 | 4.704224102 | 6.429009677 |
| 1.8 | 4.655348502 | 5.306815529 | 7.602102528 | 4.652781212 | 5.283043554 | 7.344388516 |
| 1.9 | 5.141962077 | 5.949489967 | 8.615287224 | 5.136125705 | 5.910560971 | 8.323316056 |
| 2 | 5.664504577 | 6.640483371 | 9.636312884 | 5.654985475 | 6.586435719 | 9.327548172 |

The following graphs represents the approximate solution of equation (13), for $n = 3$ and some selective $\alpha$ and $\beta$.

Graphs of approximate solutions for equation (13)




**Example 4.2.** Consider the following integro-differential equation:

$$
\begin{aligned}
&{}_{2}^{c}D_x^{\alpha} y(x) + g_1(x)\ {}_{2}^{c}D_x^{\frac{\alpha}{2}} y(x) + g_2(x)\ {}_{2}^{c}D_x^{\frac{\alpha}{3}} y(x) = \\
&\qquad f(x) + \int_2^4 K(x,t)\ {}_{2}^{c}D_t^{\beta} y(t) dtm,
\end{aligned} \tag{18}
$$

where $g_1(x) = -\Gamma(4-\frac{\alpha}{2})(x-2)^{\frac{\alpha}{2}}$, $g_2(x) = \Gamma(4-\frac{\alpha}{3})(x-2)^{\frac{\alpha}{3}}$, $f(x) = \frac{72(x-2)^{3-\alpha}}{\Gamma(4-\alpha)} + 16(x-2) - 6\alpha(x-2)^2 - x[10 - \frac{6}{2-\beta}]$, $K(x,t) = \frac{\Gamma(2-\beta)}{16} x(t-2)^{\beta}$, $2 < \alpha \leq 3$, $\beta \leq \alpha$ and $2 \leq t, x \leq 4$. Subject to the conditions $y(2) = 0$, $y'(2) = 8$, and $y''(2) = -36$.

By using Bernstein polynomials of degree $n = 5$, we approximate the solution as:

$$
y(x) = \sum_{i=0}^{5} c_i B_{i,5}(x). \tag{19}
$$

From equations (9), (10), we obtain that $c_0 = 0$, $c_1 = 3.2$ and $c_2 = -0.8$.

For a particular case, if we take $\alpha = 3$ and $\beta = 1$, the exact solution of equation (18) is $y(x) = 12x^3 - 90x^2 + 224x - 184$. Applying equation (6) on $y(x)$ and substituting in equation (18), we obtain a system of equations and solving for $c_i$'s we obtain that $c_3 = -2.4$, $c_4 = 8$ and $c_5 = 40$. The approximate solution of equation (18) is

$$
\begin{aligned}
y(x) &\approx 3.2 \times 5(x-2)(4-x)^4 - 0.8 \times 10(x-2)^2(4-x)^3 \\
&-2.4 \times 10(x-2)^3(4-x)^2 + 8 \times 5(x-2)^4(4-x) + 40(x-2)^5.
\end{aligned}
$$

Table 3: Exact and approximate solution of equation (18) when $\alpha = 3$ and $\beta = 1$

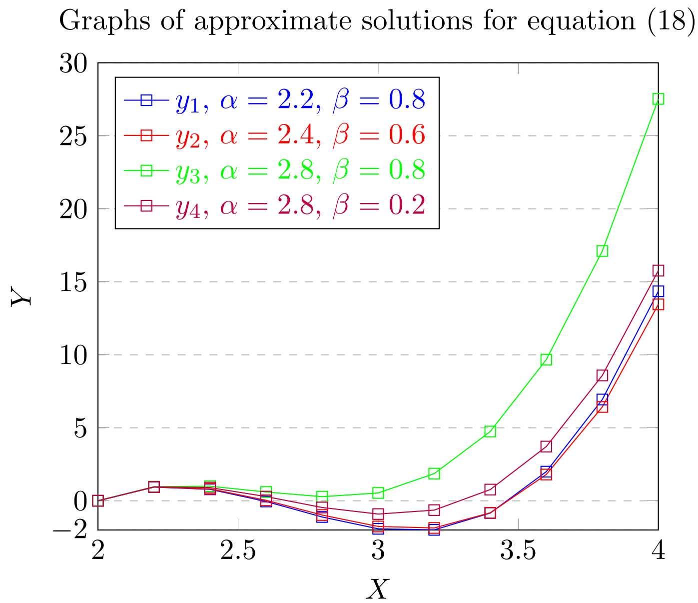
| x | $y_{Exact}$ | $y_{Approx}$ |
|---|---|---|
| 2 | 0 | 0 |
| 2.2 | 0.976 | 0.976 |
| 2.4 | 1.088 | 1.088 |
| 2.6 | 0.912 | 0.912000000000003 |
| 2.8 | 1.024 | 1.024 |
| 3 | 2 | 2.00000000000001 |
| 3.2 | 4.416 | 4.41600000000002 |
| 3.4 | 8.848 | 8.84800000000002 |
| 3.6 | 15.872 | 15.872 |
| 3.8 | 26.064 | 26.064 |
| 4 | 40 | 40 |

Table (3), describes the relation between the exact and approximate solution of some selected values of $x$ when $n = 5$, $\alpha = 3$ and $\beta = 1$.

In Table 4, the approximate solution of equation (18) for some selected values of $n$, $\alpha$ and $\beta$ is given. Where ($y_1$, $y_2$, $y_3$ and $y_4$) represent the approximate solution when $n = 5$, ($\alpha = 2.2$, $\beta = 0.8$), ($\alpha = 2.4$, $\beta = 0.6$), ($\alpha = 2.8$, $\beta = 0.8$) and ($\alpha = 2.8$, $\beta = 0.2$) respectively.

Table 4: Approximate solution of equation (18) when ($n = 5$)

| $x$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|---|
| 2 | 0 | 0 | 0 | 0 |
| 2.2 | 0.935068639 | 0.940240031 | 0.964551823 | 0.952827794 |
| 2.4 | 0.783006127 | 0.815156792 | 0.996094846 | 0.902813899 |
| 2.6 | -0.04904965 | 0.03156834 | 0.600239489 | 0.286683772 |
| 2.8 | -1.109884972 | -0.977248379 | 0.280076751 | -0.461247938 |
| 3 | -1.920849711 | -1.761022338 | 0.535063774 | -0.91120986 |
| 3.2 | -1.992496767 | -1.86195325 | 1.859909408 | -0.640106363 |
| 3.4 | -0.841221493 | -0.824176374 | 4.743459784 | 0.766349975 |
| 3.6 | 1.994098875 | 1.796772669 | 9.667583876 | 3.711505903 |
| 3.8 | 6.93546579 | 6.424493083 | 17.10605907 | 8.58563503 |
| 4 | 14.34911996 | 13.45225408 | 27.52345674 | 15.76380536 |




**Example 4.3.** Consider the integro differential equation

$$ {}_0^cD_x^{\alpha}y(x) - {}_0^cD_x^{\frac{\alpha}{2}}y(x) = f(x) + \int_0^1 e^x \ y(t)dt, \tag{20}$$

where $f(x) = e^x(1-e)$, $1 < \alpha \leq 2$, and $0 \leq t, x \leq 1$.

Subject to the conditions $y(0) = y'(0) = 1$.

By using Bernstein polynomials of degree $n = 5$, we approximate the solution as:

$$y(x) = \sum_{i=0}^{5} c_i B_{i,5}(x). \tag{21}$$

From equations (9), (10), we obtain that $c_0 = 1$ and $c_1 = 1.2$.

For a particular case, if we take $\alpha = 1.5$, the exact solution of equation (20) is $y(x) = e^x$. Applying equation (6) on $y(x)$ and substituting in equation (20), we obtain a system of equations and solving for $c_i$'s we obtain that $c_2 = 1.4499$, $c_3 = 1.766749$, $c_4 = 2.1746$ and $c_5 = 2.71818$. The approximate solution of equation (18) is

$$\begin{gathered} y(x) \approx (1-x)^5 + 1.2 \times 5x(1-x)^4 + 1.4499 \times 10x^2(1-x)^3 \\ +1.766749 \times 10x^3(1-x)^2 + 2.1746 \times 5x^4(1-x) + 2.71818x^5. \end{gathered}$$

Table 5, describes the relation between the exact and approximate solution of some selected values of $x$ when $n = 5$ and $\alpha = 1.5$.

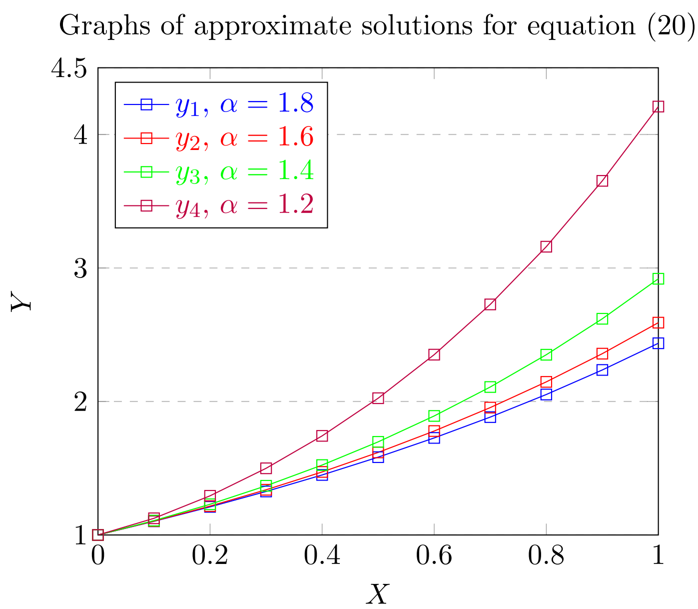
Table 5: Exact and approximate solution of equation (20) when $\alpha = 1.5$

| x | $y_{Exact}$ | $y_{Approx}$ | Error |
|---|---|---|---|
| 0 | 1 | 1 | 0 |
| 0.1 | 1.105170918 | 1.10516730537358 | 0.36127E-07 |
| 0.2 | 1.221402758 | 1.22139337801439 | 0.938015E-07 |
| 0.3 | 1.349858808 | 1.34984418528478 | 0.146223E-06 |
| 0.4 | 1.491824698 | 1.49180461512623 | 0.200825E-06 |
| 0.5 | 1.648721271 | 1.64869423914596 | 0.270316E-06 |
| 0.6 | 1.8221188 | 1.82208307570373 | 0.357247E-06 |
| 0.7 | 2.013752707 | 2.01370735299845 | 0.453545E-06 |
| 0.8 | 2.225540928 | 2.22548527215494 | 0.556563E-06 |
| 0.9 | 2.459603111 | 2.45953277031058 | 0.703408E-06 |
| 1 | 2.718281828459050 | 2.71817928370205 | 0.102545E-03 |

The following table describes the approximate solution of equation (20) when $n = 5$ and for some selected values of $\alpha$. Where $y_1$, $y_2$, $y_3$ and $y_4$ represent the approximate solution when ($\alpha = 1.8$), ($\alpha = 1.6$), ($\alpha = 1.4$) and ($\alpha = 1.2$) respectively.

Table 6: Approximate solution of equation (20) when ($n = 5$)

| $x$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 0.1 | 1.10240044295794 | 1.1038346323492 | 1.10747314761792 | 1.12496938543169 |
| 0.2 | 1.21058024191943 | 1.21626456566409 | 1.2300657682098 | 1.29333323551814 |
| 0.3 | 1.32590023707549 | 1.33863237800476 | 1.36843578414082 | 1.49945853949894 |
| 0.4 | 1.44961437376952 | 1.47225158750577 | 1.52372197489588 | 1.74235948212542 |
| 0.5 | 1.58291419217209 | 1.61844778803726 | 1.69748145554922 | 2.02453402379607 |
| 0.6 | 1.72697331695556 | 1.77859978486614 | 1.891627155234 | 2.35080048069196 |
| 0.7 | 1.88299194696886 | 1.95418073031716 | 2.10836529561194 | 2.72713410491211 |
| 0.8 | 2.05224134491214 | 2.14679925943408 | 2.35013286934289 | 3.15950366460895 |
| 0.9 | 2.23610832701151 | 2.35824062564082 | 2.61953511855447 | 3.65270802412371 |
| 1 | 2.43613975269372 | 2.59050783640254 | 2.91928301331163 | 4.20921272412183 |




**Example 4.4.** Consider the integro-differential equation:

$$\begin{aligned} &{}^c_2D^{\alpha}_x y(x) + \frac{1}{6}\sum_{k=2}^{n} g_k(x)\ {}^c_2D^{\frac{\alpha}{k}}_x y(x) + g_0(x)\ \ y(x) \\ &= f(x) + \frac{1}{64}\int_2^6 \sum_{m=1}^{2} K_m(x,t)\ {}^c_2D^{\frac{\beta}{m}}_t y(t)dt, \end{aligned} \tag{22}$$

where $g_0(x) = -5$, $g_k(x) = \Gamma(4 - \frac{\alpha}{k})(x-2)^{\frac{\alpha}{k}}$, $k = 2, 3, 4, 5, 6$,

$$K_m(x,t) = 6\Gamma\left(4 - \frac{\beta}{m}\right)(x-2)^2(t-2)^{\frac{\beta}{m}}, m = 1, 2,$$

$$f(x) = \left(6 - 12\beta + \frac{57\alpha}{30}\right)(x-2)^2 + \left(\frac{(12-\alpha)(18-\alpha)}{24} - 45\right)(x-2) - 10,$$

$5 < \alpha \leq 6$, $\beta \leq \alpha$ and $2 \leq t, x \leq 6$.

Subject to the conditions $y(2) = 2$, $y'(2) = 9$, $y''(2) = -12$, $y'''(2) = 6$, $y^{(4)}(2) = y^{(5)}(2) = 0$.

By using Bernstein polynomials of degree $n = 8$, the approximate solution is:

$$y(x) = \sum_{i=0}^{8} c_i B_{i,8}(x). \tag{23}$$

From equations (9), (10), we obtain that $c_0 = 2$, $c_1 = 6.5$, $c_2 = 7.571428571$, $c_3 = 6.357142857$, $c_4 = 4$, $c_5 = 1.642857143$.

Applying equation (6) on $y(x)$ and substituting in equation (22). For a particular case, if we take $\alpha = 6$ and $\beta = 3$, then the exact solution is $y(x) = x^3 - 12x^2 + 45x - 48$. After simplifying, we obtain a system of equations and solving for $c_i's$ we obtain that $c_6 = 0.428571429$, $c_7 = 1.5$ and $c_8 = 6$.

In the following table, we clarify the relation between the exact and approximate solution of some selected values of $x$ when $n = 6$, $\alpha = 6$ and $\beta = 3$.

Table 7: Exact and approximate solution of equation (22) when $n = 6$, $\alpha = 6$ and $\beta = 3$

| x | $y_{Exact}$ | $y_{Approx}$ |
|---|---|---|
| 2 | 2 | 2 |
| 2.2 | 3.568 | 3.568 |
| 2.4 | 4.704 | 4.704 |
| 2.6 | 5.456 | 5.45599999999999 |
| 2.8 | 5.872 | 5.872 |
| 3.2 | 5.888 | 5.88799999999999 |
| 3.6 | 5.136 | 5.13599999999999 |
| 3.8 | 4.592 | 4.59199999999999 |
| 4.2 | 3.408 | 3.40799999999999 |
| 4.6 | 2.416 | 2.416 |
| 4.8 | 2.112 | 2.112 |
| 5 | 2 | 2 |
| 5.2 | 2.128 | 2.128 |
| 5.4 | 2.544 | 2.544 |
| 5.6 | 3.296 | 3.296 |
| 5.8 | 4.432 | 4.43200000000001 |
| 6 | 6 | 6.00000000000002 |

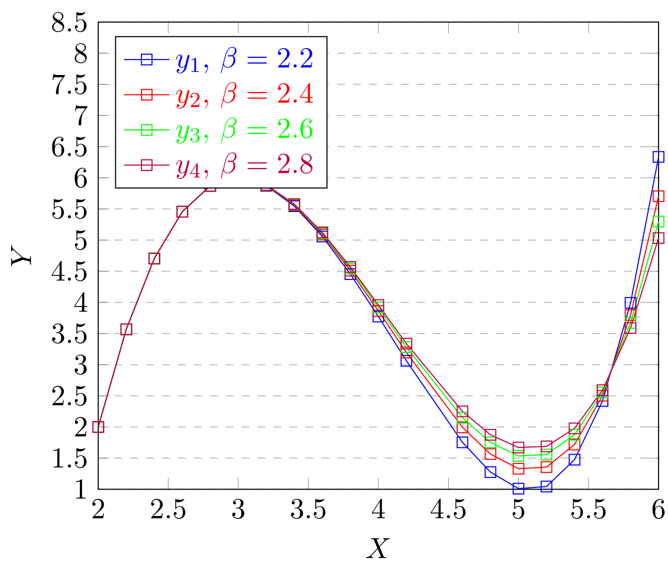
Table 8 describes the approximate solution of equation (22) for some selected values of $n$, $\alpha$ and $\beta$. $y_1$, $y_2$, $y_3$ and $y_4$ represent the approximate solution when $n = 8$, ($\alpha = 5.2$, $\beta = 2.2$), ($\alpha = 5.2$, $\beta = 2.4$), ($\alpha = 5.2$, $\beta = 0.6$) and ($\alpha = 5.2$, $\beta = 2.8$) respectively.

Table 8: Approximate solution for equation (22) when ($n = 8$)

| $x$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|---|
| 2.2 | 3.567999399 | 3.567999659 | 3.567999826 | 3.567999935 |
| 2.6 | 5.455632344 | 5.455789229 | 5.455890186 | 5.455956281 |
| 2.8 | 5.870120127 | 5.870916513 | 5.871428998 | 5.871764511 |
| 3.2 | 5.870551725 | 5.877816333 | 5.882491207 | 5.885551739 |
| 3.6 | 5.058221496 | 5.08987522 | 5.110244822 | 5.12358033 |
| 3.8 | 4.45331972 | 4.50895321 | 4.54475412 | 4.568192147 |
| 4 | 3.772873431 | 3.862434539 | 3.920068334 | 3.957799846 |
| 4.2 | 3.06214361 | 3.195683913 | 3.28161892 | 3.33787858 |
| 4.8 | 1.275625972 | 1.565429718 | 1.751922382 | 1.874014815 |
| 5.2 | 1.044082921 | 1.356223109 | 1.557089579 | 1.688592209 |
| 5.4 | 1.475720031 | 1.718932302 | 1.875442716 | 1.977906463 |
| 5.8 | 3.992755454 | 3.797248767 | 3.671437547 | 3.589071852 |
| 6 | 6.335131668 | 5.704188236 | 5.298167531 | 5.032355169 |

In the following graphs, the approximate solution of equation (22) is drawn with distinct given $\beta$.

Graphs of approximate solutions for equation (22) when $n = 8$ and $\alpha = 5.2$




## 5. Conclusion

In this paper, an approximate solution of certain types of Fredholm integro-differential equations of fractional order $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ is given by using the general form of Bernstein polynomials of various degrees. It is noted that the approximate solution of such equations is very close to the exact one.

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