Relative averaging operators and trialgebras

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Abstract. In this paper, the relative averaging operator is introduced as a relative generalization of the averaging operator. We explicitly determine all averaging operators on the 2-dimensional complex associative algebra. The results show that not every dialgebra can be derived from an averaging algebra. We then generalize the construction of dialgebras and trialgebras from averaging operators to a construction from relative averaging operators. It is shown that this construction from relative averaging operators gives all dialgebras and trialgebras.
Keywords: averaging operator, relative averaging operator, dialgebra, trialgebra.

1. Introduction

There are two seemingly unrelated objects, namely averaging operators (resp., of weight λ) and dialgebras (resp., trialgebras). This paper shows that there is a close tie between them, generalizing and strengthening a previously established connection from averaging algebras to dialgebras [1, 12, 13].

Let $\mathbf{k}$ be a unitary commutative ring and A a $\mathbf{k}$-algebra. If a $\mathbf{k}$-linear map $P : A \to A$ satisfies the averaging relations:

$$(1) \qquad P(x \cdot P(y)) = P(x) \cdot P(y) = P(P(x) \cdot y), \quad \forall x, y \in A,$$

then P is called an averaging operator and (A, P) is called an averaging algebra.

Averaging operator was implicitly studied in the famous paper of O. Reynolds [15] in connection with the theory of turbulence and explicitly defined by Kolmogoroff and Kampé de Fériet [7]. It later attracted the attentions of other well-known mathematicians including G. Birkhoff [4] and Rota with motivation from quantum physics and combinatorics. It has found diverse applications in

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many areas of pure and applied mathematics, such as the theory of turbulence, probability, function analysis, and information theory [8, 15, 16, 17, 18, 19].

Recently, averaging operators have been studied for many algebraic structures [1, 6, 12, 13]. In [14], we studied the averaging operators from an algebraic point of view and built a connection between averaging operators and large Schröder numbers. We also defined a related new operator, called averaging operator of weight λ in [13]. For a fixed $\lambda \in \mathbf{k}$. An *averaging operator of weight* λ on A is a $\mathbf{k}$-linear map $P : A \longrightarrow A$ such that Eq. (1) holds and

$$P(x) \cdot P(y) \;=\; \lambda P(x \cdot y), \quad \forall x, y \in A. \tag{2}$$

By definition, if P is an averaging operator of weight 1, then λP is an averaging operator of weight λ. We note that an averaging operator of weight zero is not an averaging operator. So we can't give a uniform definition for the averaging operator as in the case of Rota-Baxter operators of weight λ.

On the other hand, motivated by the study of the periodicity in algebraic K-theory, J.-L. Loday [9] introduced the concept of Leibniz algebra thirty years ago as a non-skew-symmetric generalization of Lie algebra. He then defined dialgebra [10] as the enveloping algebra of Leibniz algebra by analogy with associative algebra as the enveloping algebra of Lie algebra.

Definition 1.1. A *dialgebra* is a $\mathbf{k}$-module D with two associative bilinear operations $\dashv$ and $\vdash$ such that

$$x \dashv (y \dashv z) = x \dashv (y \vdash z), \tag{3}$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{4}$$
$$(x \dashv y) \vdash z = (x \vdash y) \vdash z, \tag{5}$$

for all $x, y, z \in D$.

M. Aguiar showed the following connection from averaging algebras to dialgebras.

Theorem 1.1 ([1]). *Let (A, P) be an averaging $\mathbf{k}$-algebra. Define two new operations on A by*

$$x \dashv y = xP(y), \quad x \vdash y = P(x)y, \quad \forall x, y \in A. \tag{6}$$

Then $(A, \dashv, \vdash)$ is a dialgebra.

Theorem 1.1 gives a functor from the category of averaging algebras to the category of dialgebras. The relationship between averaging algebras and dialgebras is generalized in [13] in two directions. In one direction, the relationship is generalized from associative algebras to other algebraic structures. In the other direction, the averaging operator of weight λ is introduced to give trialgebra.

The former studies told us that there is a close tie between averaging algebra (resp., of weight λ) and dialgebra (resp., trialgebra). Then it is natural to ask whether every dialgebra (resp., trialgebra) could be derived from an averaging algebra (resp., of weight λ) by a construction like Eq. (6). As Section 2 shows, the answer is no.

Interestingly, there is an analogous phenomenon that a Rota-Baxter algebra gives a dendriform or tridendriform algebra, depending on the weight. The problem that whether every dendriform algebra and tridendriform algebra could be derived from a Rota-Baxter algebra was solved by C. Bai, L. Guo and X. Ni [3]. They found there is a generalization of the concept of a Rota-Baxter operator that could derive all the dendriform algebras and tridendriform algebras. In this paper, we turn to consider the recovering problem for dialgebras from averaging algebras. Inspired by their observation, we define the concept of relative averaging operator (resp., of weight λ) as a generalization of averaging operator (resp., of weight λ) and show that every dialgebra (resp., trialgebra) can be recovered from a relative averaging operator (resp., of weight λ).

This paper is organized as follows. In the next section, we first determine all averaging operators on the 2-dimensional complex associative algebra and then list the dialgebras induced by these averaging operators. In Section 3 , the definitions of relative averaging operator and relative averaging operator of weight λ are given. Finally, we prove that every dialgebra (resp., trialgebra) can be derived from relative averaging algebra (resp., of weight λ).

2. Averaging operators on the complex 2-dimensional associative algebra

In this section, we determine all averaging operators on 2-dimensional complex associative algebras. Then we find all dialgebras induced by averaging operators on the 2-dimensional complex associative algebra. The results show that not every dialgebra can be derived from an averaging algebra.

There are six associative algebras structures on the 2-dimensional vector space $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ except the trivial one, two of them are non-commutative and the other four are commutative [2, 5]. We list their characteristic matrices in the following and denote the corresponding algebra by $(A_i, \bullet_i)$, $1 \le i \le 6$, respectively:

$\bullet_1$	e_1	e_2
e_1	0	e_1
e_2	0	e_2

$\bullet_2$	e_1	e_2
e_1	0	0
e_2	e_1	e_2

$\bullet_3$	e_1	e_2
e_1	e_1	0
e_2	0	0

$\bullet_4$	e_1	e_2
e_1	e_2	0
e_2	0	0

$\bullet_5$	e_1	e_2
e_1	e_1	0
e_2	0	e_2

$\bullet_6$	e_1	e_2
e_1	0	e_1
e_2	e_1	e_2

.

A linear operator $P : A_i \to A_i$ is determined by

$$\begin{pmatrix} P(e_1) \\ P(e_2) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \tag{7}$$

where $a_{ij} \in \mathbb{C}, 1 \leq i, j \leq 2$. P is an averaging operator on A_i if the above matrix $(a_{ij})_{2\times 2}$ satisfies Eq. (1) for $x, y \in \{e_1, e_2\}$.

In order to show P is an averaging operator, we only need to check

$$P(e_i)P(e_j) = P(e_iP(e_j)) = P(P(e_i)e_j), \quad 1 \leq i, j \leq 2. \tag{8}$$

It is clear that the zero operator is an averaging operator on A_i. Furthermore, it follows from a direct check that P is an averaging operator if and only if λP is an averaging operator for $0 \neq \lambda \in \mathbb{C}$. Thus, the set $AV(A_i)$ of averaging operators on A_i carries an action of $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$ by scalar multiplication. To determine all the averaging operators on A_i, we only need to give a complete set of representatives of $AV(A_i)$ under this action.

We only give the sketch of process for determining averaging operators on A_1 here. The others discussions are the same as A_1.

By direct computation, we have

$$\begin{aligned}
&P(e_1)P(e_1) = a_{11}a_{12}e_1 + a_{12}^2e_2, \quad P(e_1P(e_1)) = a_{11}a_{12}e_1 + a_{12}^2e_2,\\
&P(P(e_1)e_1) = 0, \quad P(e_1)P(e_2) = a_{11}a_{22}e_1,\\
&P(e_1P(e_2)) = a_{11}a_{22}e_1, \quad P(P(e_1)e_2) = a_{11}^2e_1,\\
&P(e_2)P(e_1) = 0, \quad P(e_2P(e_1)) = 0, \quad P(P(e_2)e_1) = 0,\\
&P(e_2)P(e_2) = a_{21}a_{22}e_1 + a_{22}^2e_2, \quad P(e_2P(e_2)) = a_{21}a_{22}e_1 + a_{22}^2e_2,\\
&P(P(e_2)e_2) = (a_{11}a_{21} + a_{21}a_{22})e_1 + a_{22}^2e_2.
\end{aligned}$$

By Eq. (8) and comparing the corresponding coefficients of e_1 and e_2, we have

$$a_{11}a_{12} = 0, \quad a_{12}^2 = 0, \quad a_{11}^2 = a_{11}a_{22}, \quad a_{11}a_{21} = 0.$$

Hence, the averaging operators on A_1 are given by a complete set of representatives of $AV(A_1)$ under the action of $\mathbb{C}^*$ by scalar product consists of the 5 averaging operators whose linear transformation matrices with respect to the basis e_1, e_2 are listed below, where a are non-zero complex numbers:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Theorem 2.1. *1. The non-zero averaging operators on A_1 and A_2 are given by*

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a \neq 0.$$

2. The non-zero averaging operators on A_3 are given by

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & a \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad a \neq 0.$$

The non-zero averaging operators on A_4 are given by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & a \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad a \neq 0.$$

3. The non-zero averaging operators on A_5 are given by, $a \neq 0$,

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ a & a \end{pmatrix}.$$

4. The non-zero averaging operators on A_6 are given by, $a \neq 0$,

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.$$

By Theorem 1.1 and Theorem 2.1, after a direct computation, we have

Corollary 2.1. *Let $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and $(V, \dashv, \vdash)$ be a dialgebra which is induced by the averaging operators on A_1—A_6 and the trivial 2-dimensional complex associative algebra A_0. Then either $(V, \dashv) \cong A_i$, $(V, \vdash) \cong A_i$, $0 \leq i \leq 6$, or one of the following items holds:*

(1) $(V, \dashv) \cong A_0$, $(V, \vdash) \cong A_4$;

(2) $(V, \dashv) \cong A_1$, $(V, \vdash) \cong A_3$;

(3) $(V, \dashv) \cong A_3$, $(V, \vdash) \cong A_2$;

(4) $(V, \dashv) \cong A_1$, $(V, \vdash) \cong A_2$;

(5) $(V, \dashv) \cong A_5$, $(V, \vdash) \cong A_2$.

Remark 2.1. Let $\dashv$ be the zero multiplication and $\vdash = \bullet_i$, $i = 1, 2, 3, 5, 6$. For each i, the multiplications $\dashv$ and $\vdash$ give a dialgebra structure on $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. By Corollary 2.1, the above dialgebras can't be derived from a 2-dimensional complex averaging algebra.

3. Relative averaging operators, dialgebras and trialgebras

In this section we study the relationship between relative averaging operators (resp., of weight λ) and dialgebras (resp., trialgebras) on the domains of these operators. First, we give some related concepts. Then we show that relative averaging operators recover all dialgebras and trialgebras on the domains of the operators.

3.1 A-bimodule k-algebras and relative averaging operators

First, we recall a generalization of the well-known concept of bimodules in [3].

Definition 3.1. Let $(A, *)$ be a $\mathbf{k}$-algebra with multiplication $*$ and $(R, \circ)$ be a $\mathbf{k}$-algebra with multiplication $\circ$. Let $\ell, r : A \longrightarrow End_{\mathbf{k}}(R)$ be two linear maps. We call $(R, \circ, \ell, r)$ or simply R an *A-bimodule $\mathbf{k}$-algebra* if (R, ℓ, r) is an A-bimodule that is compatible with the multiplication $\circ$ on R. More precisely, forall $x, y \in A, v, w \in R$, we have

$$\ell(x * y)v = \ell(x)(\ell(y)v), \quad \ell(x)(v \circ w) = (\ell(x)v) \circ w, \tag{9}$$

$$vr(x * y) = (vr(x))r(y), \quad (v \circ w)r(x) = v \circ (wr(x)), \tag{10}$$

$$(\ell(x)v)r(y) = \ell(x)(vr(y)), \quad (vr(x)) \circ w = v \circ (\ell(x)w). \tag{11}$$

Note that an A-bimodule (V, ℓ, r) becomes an A-bimodule $\mathbf{k}$-algebra if V is regarded as an algebra with the zero multiplication. For a $\mathbf{k}$-algebra $(A, *)$ and $x \in A$, define the left and right actions $L(x) : A \longrightarrow A$, $L(x)y = x * y$; $R(x) : A \longrightarrow A$, $yR(x) = y * x$, $y \in A$. For $x \in A$, define

$$L = L_A : A \longrightarrow End_{\mathbf{k}}(A), x \longmapsto L(x); R = R_A : A \longrightarrow End_{\mathbf{k}}(A), x \longmapsto R(x).$$

Then (A, L, R) is an A-bimodule and $(A, *, L, R)$ is an A-bimodule $\mathbf{k}$-algebra.

Now, we can define our generalization of the averaging operator .

Definition 3.2. Let $(A, *)$ be a $\mathbf{k}$-algebra.

1. Let V be an A-bimodule. A linear map $Q : V \longrightarrow A$ is called a *relative averaging operator* on the module V if Q satisfies

$$Q(u) * Q(v) = Q(\ell(Q(u)v)) = Q(ur(Q(v))), \quad u, v \in V. \tag{12}$$

2. Let $(R, \circ, \ell, r)$ be an A-bimodule $\mathbf{k}$-algebra and $\lambda \in \mathbf{k}$. A linear map $Q : R \longrightarrow A$ is called a *relative averaging operator of weight* λ on the algebra R if Q satisfies

$$Q(u) * Q(v) = Q(\ell(Q(u))v) = Q(ur(Q(v))) = \lambda Q(u \circ v), \quad u, v \in R. \tag{13}$$

When V is taken to be the A-bimodule (A, L, R) associated to the algebra A, a relative averaging operator (resp., of weight λ) on the module is just an averaging operator (resp., of weight λ).

3.2 Averaging algebras, dialgebras and trialgebras

The concept of a trialgebra was introduced by Loday and Ronco as a generalization of a dialgebra.

Definition 3.3 ([11])**.** *A trialgebra is a* **k**-*module* T *with three associative bilinear operations* $\dashv$, $\vdash$ *and* $\perp$ *such that*

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \quad (x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{14}$$
$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \quad (x \dashv y) \dashv z = x \dashv (y \perp z), \tag{15}$$
$$(x \perp y) \dashv z = x \perp (y \dashv z), \quad (x \dashv y) \perp z = x \perp (y \dashv z), \tag{16}$$
$$(x \vdash y) \perp z = x \vdash (y \perp z), \quad (x \perp y) \vdash z = x \vdash (y \vdash z), \tag{17}$$

for all $x, y, z \in T$.

The Corollary 4.9 in [13] generalized Theorem 1.1 and showed that if $(A, \circ, P)$ is an averaging algebra of weight $\lambda \neq 0$, then the multiplications

$$x \dashv_P y := x \circ P(y), \quad x \vdash_P y := P(x) \circ y, \quad x \perp_P y := \lambda x \circ y, \quad \forall x, y \in A, \tag{18}$$

define a trialgebra $(A, \dashv_P, \vdash_P, \perp_P)$.

For a given **k**-module V, define $\mathcal{AV}(V)$ (resp., $\mathcal{AV}_\lambda(V)$) to be the set of all averaging algebras (resp., of weight λ) on V. Let $\mathcal{AD}(V)$ (resp., $\mathcal{AT}(V)$) be the set of all dialgebras (resp., trialgebras) on V.

Then Eqs. (6) and (18) induce two maps

$$\Phi : \mathcal{AV}(V) \longrightarrow \mathcal{AD}(V), \tag{19}$$
$$\Phi_\lambda : \mathcal{AV}_\lambda(V) \longrightarrow \mathcal{AT}(V). \tag{20}$$

Thus deriving all dialgebras (resp., trialgebras) on V from averaging operators (resp., of weight λ) on V amounts to the surjectivity of Φ (resp., Φ_λ). Unfortunately, by Remark 2.1, these maps are not surjective. Next, we will consider the case of relative averaging operators.

3.3 From relative averaging operators to dialgebras and trialgebras

Theorem 3.1. *Let* $(A, *)$ *be an associative algebra.*

(a) *Let* $(R, \circ, \ell, r)$ *be an* A-*bimodule* **k**-*algebra. Let* $Q : R \longrightarrow A$ *be a relative averaging operator of weight* λ *on the algebra* R. *Then the multiplications*

$$u \dashv_Q v := ur(Q(v)), u \vdash_Q v := \ell(Q(u))v, u \perp_Q v := \lambda u \circ v, \quad \forall u, v \in R, \tag{21}$$

define a trialgebra $(R, \dashv_Q, \vdash_Q, \perp_Q)$.

(b) *Let* (V, ℓ, r) *be an* A-*bimodule. Let* $Q : V \longrightarrow A$ *be a relative averaging operator on the module* V. *Then the multiplications*

$$u \dashv_Q v := ur(Q(v)), u \vdash_Q v := \ell(Q(u))v, \quad \forall u, v \in V, \tag{22}$$

define a dialgebra $(V, \dashv_Q, \vdash_Q)$.

Proof. (a) For any $x, y, z \in R$, by the definitions of $\dashv_Q$, $\vdash_Q$ and $\perp_Q$ and A-bimodule $\mathbf{k}$-algebra, we have

$$(x \dashv_Q y) \dashv_Q z = \big(xr(Q(y))\big)r(Q(z)) = xr(Q(y) * Q(z)).$$

Since $Q(y) * Q(z) = Q(\ell(Q(y))z) = Q(yr(Q(z))) = \lambda Q(y \circ z)$, we have

$$(x \dashv_Q y) \dashv_Q z = x \dashv_Q (y \vdash_Q z) = x \dashv_Q (y \dashv_Q z) = x \dashv_Q (y \perp_Q z).$$

It follows from $x \vdash_Q (y \vdash_Q z) = \ell(Q(x))\big(\ell(Q(y))z\big) = \ell(Q(x) * Q(y))z$ and $Q(x) * Q(y) = Q(\ell(Q(x))y) = Q(xr(Q(y))) = \lambda Q(x \circ y)$ that

$$x \vdash_Q (y \vdash_Q z) = (x \vdash_Q y) \vdash_Q z = (x \dashv_Q y) \vdash_Q z = (x \perp_Q y) \vdash_Q z.$$

We also, have

$$\begin{aligned}
&(x \vdash_Q y) \dashv_Q z = \big(\ell(Q(x))y\big)r(Q(z)) = \ell(Q(x))(yr(Q(z))) = x \vdash_Q (y \dashv_Q z),\\
&(x \perp_Q y) \dashv_Q z = (\lambda x \circ y)r(Q(z)) = \lambda x \circ (yr(Q(z))) = x \perp_Q (y \dashv_Q z),\\
&(x \dashv_Q y) \perp_Q z = \lambda(xr(Q(y)) \circ z = x \circ \big(\ell(Q(y))z\big) = x \perp_Q (y \dashv_Q z),\\
&(x \vdash_Q y) \perp_Q z = \lambda(\ell(Q(x))y) \circ z = \ell(Q(x))(\lambda y \circ z) = x \vdash_Q (y \perp_Q z),\\
&(x \perp_Q y) \perp_Q z = \lambda(\lambda x \circ y) \circ z = \lambda(x \circ (\lambda y \circ z)) = x \perp_Q (y \perp_Q z).
\end{aligned}$$

The above relations for $\dashv_Q$, $\vdash_Q$ and $\perp_Q$ coincide with the axioms of trialgebra in Definition 3.3.
(b) By the definitions of $\dashv_Q$, $\vdash_Q$ and bimodule, similar to the proof of (a), $(V, \dashv_Q, \vdash_Q)$ is a dialgebra. □

For a $\mathbf{k}$-algebra A and an A-bimodule $\mathbf{k}$-algebra $(R, \circ)$, denote

$$\begin{aligned}
&\mathcal{RA}_\lambda^{alg}(R, A)\\
&:= \{Q : R \to A | Q \text{ is a relative averaging operator of weight } \lambda \text{ on algebra } R\}.
\end{aligned}$$

By (a) of Theorem 3.1, we obtain a map

$$\Phi_{\lambda,R,A}^{alg} : \mathcal{RA}_\lambda^{alg}(R, A) \longrightarrow \mathcal{AT}(R_{mod}), \tag{23}$$

where R_{mod} denotes the underlying $\mathbf{k}$-module of R.

Now let V be a $\mathbf{k}$-module. Let $\mathcal{AV}_\lambda(V, -)$ be the set of relative averaging operators of weight λ on algebra $(V, \circ)$, where $\circ$ is an associative product on V. In other words,

$$\mathcal{AV}_\lambda(V, -) := \coprod_{R,A} \mathcal{AV}_\lambda^{alg}(R, A), \tag{24}$$

where the disjoint union runs through all pairs (R, A) where A is a $\mathbf{k}$-algebra and R is an A-bimodule $\mathbf{k}$-algebra such that $R_{mod} = V$. Then from the map $\Phi_{\lambda,V,A}^{alg}$ in Eq. (23), we have

$$\Phi_{\lambda,V}^{alg} := \coprod_{R,A} \Phi_{\lambda,V,A}^{alg} : \mathcal{AV}_\lambda^{alg}(V, -) \longrightarrow \mathcal{AT}(V). \tag{25}$$

Similarly, for a **k**-module V and **k**-algebra A, denote

$$\mathcal{RA}^{mod}(V,A)$$
$$:= \{Q : V \to A | Q \text{ is a relative averaging operator on the module } V\},$$

By (b) of Theorem 3.1, we obtain a map

$$\Phi^{alg}_{V,A} : \mathcal{AV}^{mod}(V,A) \longrightarrow \mathcal{AD}(V) \tag{26}$$

Let $\mathcal{AV}^{mod}(V,-)$ be the set of relative averaging operators on the module V. In other words, $\mathcal{AV}^{mod}(V,-) := \coprod_A \mathcal{AV}^{mod}(V,A)$, where A runs through all the **k**-algebras. Then we have

$$\Phi^{mod}_V := \coprod_A \Phi^{mod}_{V,A} : \mathcal{AV}^{mod}(V,-) \longrightarrow \mathcal{AD}(V). \tag{27}$$

Theorem 3.2. *Let V be a* **k***-module. The maps $\Phi^{alg}_{1,V}$ and Φ^{mod}_V are surjective.*

Proof. We first prove the surjectivity of $\Phi^{alg}_{1,V}$. Let $(V,\dashv,\vdash,\perp)$ be a trialgebra. Define two linear maps

$$L_\vdash, R_\dashv : V \longrightarrow End_{\mathbf{k}}(V), L_\vdash(x)(y) = x \vdash y, R_\dashv(x)(y) = y \dashv x, \forall x,y \in V. \tag{28}$$

Let I be the ideal generated by the set $\{u \dashv v - u \vdash v \mid u,v \in V\} \cup \{u \dashv v - u \perp v \mid u,v \in V\}$. Let $\widetilde{V} := V/I$, then we have $\dashv=\vdash=\perp$ in $\widetilde{V}$. Furthermore, $\widetilde{V}$ can be regarded as an associative algebra with an operation $* :=\dashv=\vdash=\perp$.

By comparing the trialgebra axioms and the axioms of $(V,*)$-bimodule **k**-algebra, we have that if we replace the operation $*$ in Eq. (9) and (10), by any of $\dashv,\vdash,\perp$, the equations still hold. Hence, $(V,\perp,L_\vdash,R_\dashv)$ is a $(\widetilde{V},*)$-bimodule **k**-algebra.

Let Q be the natural projection from V to $\widetilde{V}$. Then we have

$$Q(x) = x, \quad Q(x \dashv y) = Q(x \vdash y) = Q(x \perp y) = Q(x) * Q(y).$$

Hence,

$$Q(x) * Q(y) = Q(Q(x) \vdash y) = Q(x \dashv Q(y)) = Q(x \perp y),$$

and then

$$Q(x) * Q(y) = Q(L_\vdash(Q(x))y) = Q(xR_\dashv(Q(y))) = Q(x \perp y).$$

That is Q is a relative averaging operator of weight 1 on the algebra $(V,\perp)$.

To prove the surjective of Φ^{mod}_V, let $(V,\dashv,\vdash)$ be a dialgebra. Let I be the ideal generated by the set $\{u \dashv v - u \vdash v \mid u,v \in V\}$. Define Q be the natural projection from V to V/I. Similar to the proof for $\Phi^{alg}_{1,V}$, we get Q is a relative averaging operator on bimodule $(V,L_\vdash,R_\dashv)$. □

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