A note on k-zero-divisor hypergraphs of some commutative rings

Elham Mehdi-Nezhad
Department of Mathematics and Applied Mathematics
University of the Western Cape, Private Bag X17,
Bellville 7535, Cape Town
South Africa
emehdinezhad@uwc.ac.za

Amir M. Rahimi*
School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
P.O. Box 19395-5746, Tehran
Iran
amrahimi@ipm.ir

Abstract. The main object of this paper is to study and characterize the connectedness, diameter, dominating sets and domination number of the k-zero-divisor hypergraph $H_k(R)$ of a finite direct product of integral domains and a class of commutative Artinian rings R, respectively. We will show that the k-zero-divisor hypergraph associated to the direct product of $k \geq 3$ integral domains (resp., commutative Artinian rings which are the direct product of $k \geq 3$ local rings) are connected with diameter at most 3 and domination number at most 2 (resp., connected with diameter at most 4 and domination number at most $2k$). We will also provide some examples related to these results.

Keywords: k-uniform hypergraph, k-zero-divisor, dominating set, domination number, connectedness, diameter, Artinian ring.

1. Introduction and definitions

The main goal of this paper is to study and characterize *the connectedness, diameter, dominating sets and domination number* of the k-zero-divisor hypergraphs $H_k(R)$ of two well-known classes of commutative rings R; namely, *a finite direct product of integral domains and a class of commutative Artinian rings*, respectively. Through out this work, all rings are *commutative with identity* $1 \neq 0$, $J(R)$ denotes the Jacobson radical of R, and a *local ring* is a ring with only one maximal ideal.

In this section we recall some definitions together with some references and will discuss the main results in the next section. We will show (Theorem 2.1) that the k-zero-divisor hypergraph associated to $k \geq 3$ direct product of integral

*. Corresponding author

domains (e.g., $R/J(R)$ of a semilocal ring (Corollary 2.1), finite reduced rings (Corollary 2.2)), respectively commutative Artinian rings which are the direct product of $k \geq 3$ local rings (Theorem 2.2), e.g., ring of integers modulo n Corollary 2.3 are connected with diameter at most 3 and domination number at most 2 (resp., connected with diameter at most 4 and domination number at most $2k$).

The concept of the *zero-divisor graph* of a commutative ring has been studied extensively by many authors, and the *k-zero-divisor hypergraph* of a commutative ring R, denoted by $H_k(R)$, is a nice abstraction of this concept which was first introduced by Eslahchi and Rahimi [6]. In their work, they studied some ring-theoretic properties of the k-zero-divisors of R and graph-theoretic properties of $H_k(R)$ and investigated the interplay between the ring-theoretic properties of R and the graph-theoretic properties of its associated *k-uniform hypergraph* $H_k(R)$. Specially, in Section 3, they discussed the connectedness and completeness of $H_3(R)$ and showed that its (diameter, girth) is bounded above by (4, 9) and also found a lower bound for its clique number. Furthermore, the research on this subject continued and extended by other authors as well (e.g., [14], [15], [16]).

We now define the zero-divisor graph of a commutative ring.

The *zero-divisor graph* of a commutative ring R, denoted $\Gamma(R)$, is an undirected graph whose vertices are the nonzero zero-divisors of R and two distinct vertices x and y are adjacent if and only if $xy = 0$. Thus $\Gamma(R)$ is an empty graph if and only if R is an integral domain. Beck in [4] introduced the concept of a zero-divisor graph of a commutative ring, but this work was mostly connected with colorings of zero-divisor of rings. The above definition first appeared in the work of D.F. Anderson and Livingston [2], which contains several fundamental results concerning $\Gamma(R)$. This definition, unlike the earlier work of D.D. Anderson and Naseer [1] and Beck [4], does not take zero to be a vertex of $\Gamma(R)$.

We now recall the following two definitions, i.e., the k-zero-divisor and k-zero-divisor hypergraph of a ring, respectively from [6].

Definition 1.1. *Let R be a commutative ring and $k \geq 2$ a fixed integer. A nonzero non unit element a_1 in R is said to be a k-zero-divisor in R if there exist $k-1$ distinct non unit elements $a_2, a_3, \ldots, a_k$ in R different from a_1 such that $a_1a_2a_3 \cdots a_k = 0$ and the product of no elements of any proper non-singleton subset of $A = \{a_1, a_2, \ldots, a_k\}$ is zero.*

Definition 1.2. *Let R be a commutative ring (with $1 \neq 0$) and let $Z(R, k)$ be the set of all k-zero-divisors in R. We associate a k-uniform hypergraph $H_k(R)$ to R with vertex set $Z(R, k)$, and for distinct elements $x_1, x_2, \ldots, x_k$ in $Z(R, k)$, the set $\{x_1, x_2, \ldots, x_k\}$ is an edge of $H_k(R)$ if and only if $x_1x_2 \cdots x_k = 0$ and the product of elements of no $(k-1)$-subset of $\{x_1, x_2, \ldots, x_k\}$ is zero.*

Remark 1.1. It is not difficult to show that the statement "the product of no elements of any proper (nonsingleton) subset of A is zero" or the statement "the product of no elements of any $(k-1)$-subset of A is zero" can be used in Definition 1.1 equivalently. Clearly, from Definition 1.1, every element of the set $\{a_2, a_3, \ldots, a_k\}$ is a k-zero-divisor in R. It is clear that every k-zero-divisor in R is also a zero-divisor in R, but, the converse is not true in general. For example, the element 2 is a zero-divisor, but not a 3-zero-divisor in $\mathbb{Z}_{10}$ and 2 in $\mathbb{Z}_4$ is a zero-divisor but not a 2-zero-divisor.

We now review some basic graph-theoretic definitions and notions used throughout to keep this paper as self contained as possible; and for the necessary definitions and notations of graphs and hypergraphs, we refer the reader to standard texts of graph theory such as [17] and [5].

A hypergraph is a pair (V, E) of disjoint sets, where the elements of E are non-empty subsets (of any cardinality) of V. The elements of V are the vertices, and the elements of E are the edges of the hypergraph. The hypergraph $H = (V, E)$ is called k-uniform whenever every edge e of H consists of k vertices. A k-uniform hypergraph H is called complete if every k-subset of the vertices is an edge of H. An r-coloring of a hypergraph $H = (V, E)$ is a map $c : V \to \{1, 2, \cdots, r\}$ such that for every edge e of H, there exist at least two vertices x and y in e with $c(x) \neq c(y)$. The smallest integer r such that H has an r-coloring is called the chromatic number of H and is denoted by $\chi(H)$. A path in a hypergraph H is an alternating sequence of distinct vertices and edges of the form $v_1, e_1, v_2, e_2, \ldots, v_k$ such that v_i, v_{i+1} is in e_i for all $1 \leq i \leq k-1$. The number of edges of a path is its length. The distance between two vertices x and y of H, denoted by $d_H(x, y)$, is the length of the shortest path from x to y. If no such path between x and y exists, we set $d_H(x, y) = \infty$. The greatest distance between any two vertices in H is called the diameter of H and is denoted by $\mathrm{diam}(H)$. The hypergraph H is said to be connected whenever $\mathrm{diam}(H) < \infty$. A cycle in a hypergraph H is an alternating sequence of distinct vertices and edges of the form $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1$ such that v_i, v_{i+1} is in e_i for all $1 \leq i \leq k-1$ with $v_k, v_1 \in e_k$. The girth of a hypergraph H containing a cycle, denoted by $\mathrm{gr}(H)$, is the smallest size of the length of cycles of H.

We now define the notion of the dominating set and domination number of a hypergraph and for a detailed study of the dominating sets and domination number of the zero-divisor graph of a commutative ring (resp., with respect to an ideal), see [13] and [10], respectively (see, also, [7]).

Definition 1.3. *Let $H = (V, E)$ be a hypergraph with vertex set$V$ and edge set E. A nonempty set $S \subseteq V$ is a dominating set of H if every vertex in V is either in S or is adjacent to a vertex in S. That is, for every $v \in V \setminus S$, there exists an edge $e \in E$ such that $v \in e$ and the intersection of e and S is nonempty. The domination number of H, denoted by $\gamma(H)$, is the minimum cardinality among all dominating sets of H.*

We end this section with a brief general overview related to graphs associated to some algebras.

The area of research on assigning a graph to an algebra (algebraic structure) has been very active (specially) since last two decades and there are many papers which apply combinatorial methods (using graph-theoretic properties and parameters such as *connectedness, planarity, clique number, chromatic number, independence number, domination number*, and so on) to obtain algebraic results and vice versa. For instance, there are many papers on this interdisciplinary subject and for a short list of them, see for example [11] and [12] (covering many different cases using *commutator theory*) and also see the work of Mehdi-Nezhad and Rahimi in [9] for some other references and a *brief historical note* on some graphs associated to some algebraic structures.

2. Main results

We begin this section with a lemma using for Theorem 2.1 and provide some examples and corollaries as an application to this theorem (e.g., $R/J(R)$ of a semilocal ring (Corollary 2.1), finite reduced rings (Corollary 2.2) to show that their corresponding k-zero-divisor hypergraphs are connected with diameter (resp., domination number) at most 3 (resp., 2). Then, we continue to show that $H_k(R)$ is connected with diameter at most 4 and domination number at most $2k$ (Theorem 2.2), where R is an Artinian ring which is the direct product of $k \geq 3$ local rings (see also Corollary 2.3 as an application to this theorem).

Lemma 2.1. *Let $k \geq 3$ be a fixed integer and $R = R_1 \times R_2 \times \cdots \times R_k$ the direct product of k integral domains. Then, $(a_1, a_2, \ldots, a_k) \in R$ is a vertex in $H_k(R)$ if and only if exactly one of its components is zero. That is,*

$$Z(R, k) = \{(a_1, a_2, \ldots, a_k) \in R \mid exactly\ one\ of\ the\ a_i{}'s\ is\ zero\ for\ 1 \leq i \leq k\}.$$

Proof. The sufficient part follows directly from definition. For example, let $x_1 = (a_1, a_2, \ldots, a_k) \in R$ such that exactly one and only one of the components is zero. Without loss of generality, assume that $a_1 = 0$. Let $x_i = (1, 1, \ldots, 1, 0, 1, 1, \ldots, 1)$, where the ith component is the only zero component of x_i for each $2 \leq i \leq k$. Now, it is obvious that $\{x_1, x_2, \ldots, x_k\} \in E(H_k(R))$.

For the necessary part, it is obvious that any k-zero-divisor of R must have at least one zero component. Now, let $x_1 = (a_{11}, a_{12}, \ldots, a_{1k})$ be a k-zero-divisor (vertex in $H_k(R)$) with at least two zero components. Without loss of generality, assume that $a_{11} = a_{12} = 0$. Consequently, there exist $x_2, x_3, \ldots, x_k \in V(H_k(R))$ such that $\{x_1, x_2, \ldots, x_k\} \in E(H_k(R))$, where $x_i = (a_{i1}, a_{i2}, \ldots, a_{ik})$ for all $1 \leq i \leq k$. Thus, $\prod_{i \geq 1} a_{ij} = 0$ for each $j \geq 3$. Now, since R_j is an integral domain, then for each fixed $j \geq 3$, there exists at least one i_j with $1 \leq i \leq k$ such that $a_{i_j j} = 0$. Let I be the set of all i_j's such that $a_{i_j j} = 0$ for the smallest i in the set $\{1, 2, \ldots, k\}$. Thus, we have $x_1 \prod_{i \in I} x_i = 0$ and since $|I| \leq k - 2$, we have a contradiction and the proof is complete. $\square$

Theorem 2.1. *For any fixed integer $k \geq 3$, there exists a ring R whose k-zero-divisor hypergraph is connected with diameter at most 3 and domination number at most 2.*

Proof. Let $R = R_1 \times R_2 \times \cdots \times R_k$ be the direct product of k integral domains. Now, the proof is straight forward by using the above lemma. For instance, $D = \{x_1, x_2\}$ is a dominating set in $H_k(R)$, where $x_1 = (0, 1, 1, \ldots, 1)$ and $x_2 = (1, 0, 1, \ldots, 1)$. Note that $e = \{x_1, x_2, \ldots, x_k\}$ is an edge in $H_k(R)$, where x_i is a k-tuple with ith component 0 and jth component 1 for each $1 \leq i \neq j \leq k$. □

We now provide some examples as an application to the above theorem.

Example 2.1. For any fixed integer $k \geq 3$, we have the following:

(a) Let R be the direct product of k factors of the ring $\mathbb{Z}_2$. Clearly, $H_k(R)$ has only one edge and hence is connected and its domination number is 1 since the singleton set of each vertex is a dominating set. Note that the chromatic number of this hypergraph is 2.

(b) Let R be the direct product of k factors of the ring $\mathbb{Z}_p$ for some prime $p \geq 2$. Then, by the above theorem, $H_k(R)$ is a connected k-zero-divisor hypergraph with diameter at most 3 and domination number at most 2.

(c) let $n = p_1 \cdots p_k$ for distinct primes $p_1, \ldots, p_k$. Then, $H_k(\mathbb{Z}_n)$ is a connected k-zero-divisor hypergraph with diameter at most 3 and domination number at most 2. The proof follows directly from the above theorem and the fact that $\mathbb{Z}_n \cong \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_k}$.

We now apply the above theorem to a semilocal ring.

Corollary 2.1. *For a semilocal ring R with $k \geq 3$ maximal ideals $M_1, M_2, \ldots, M_k$, there exists a connected k-zero-divisor hypergraph associated to $R/J(R)$ whose diameter is bounded above by 3 and its domination number is at most 2, where $J(R)$ is the Jacobson radical of R.*

Proof. The proof is an immediate consequence of the above theorem since (by Chinese Remainder Theorem) $R/J(R) \cong F_1 \times F_2 \times \ldots \times F_k$, where $F_i = R/M_i$ for each $1 \leq i \leq k$. □

We next apply the above theorem when R is a reduced or finite reduced ring, i.e., a direct product of finitely many finite fields.

Corollary 2.2. *Let R be a reduced* (*resp., finite reduced*) *commutative ring* (*which is not an integral domain*) *with at least $k \geq 3$ minimal prime ideals and nil(R) the ideal of nilpotent elements of R. Then, there exists a ring whose k-zero-divisor hypergraph is connected with diameter at most 3 and domination number at most 2* (*resp., $H_k(R)$ satisfies the mentioned properties*).

Proof. Let $P_1, \ldots, P_k$ be the minimal prime ideals of R. Then, $P_1 \cap \cdots \cap P_k = \text{nil}(R) = \{0\}$ since R is reduced. Thus there is a monomorphism from R to $T = R/P_1 \times \cdots \times R/P_k$. Now, the proof follows from the above theorem and for the finite case, R is isomorphic to T, by Chinese Remainder Theorem, since prime ideals are maximal in a finite ring. □

We next discuss the results of Theorem 2.1 for commutative Artinian rings which are the direct product of $k \geq 3$ local rings. Recall that any commutative Artinian ring is a finite direct product of Artinian local rings ([3, Theorem 8.7]).

Theorem 2.2. *Let R be a commutative Artinian ring (in particular, R could be a finite commutative ring) which is the direct product of $k \geq 3$ Artinian local rings, where k is a fixed integer. Then, $H_k(R)$, the k-zero-divisor hypergraph of R, is connected with diameter at most 4 and domination number at most $2k$.*

Proof. Let $R = R_1 \times R_2 \times \cdots \times R_k$, where R_i is an Artinian local ring with maximal ideal M_i and assume $M_i \neq 0$ for each $i = 1, 2, \ldots, k$. By [8, Theorem 82], suppose $M_i = \text{ann}(m_i)$ for some nonzero $m_i \in M_i$ and each $i = 1, 2, \ldots, k$. We now construct a dominating set S of size $2k$ for $H_k(R)$. Let $S = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$, where for each $1 \leq i \leq k$, y_i is a k-tuple whose ith component is 0 and other components are 1's; and for each $1 \leq i \leq (k-1)$, x_i is a k-tuple whose ith component is m_i, its kth component is 0, and the other components are all 1's, and $x_k = (1, 1, \ldots, 1, 0, m_k)$. Further, we take $m_i = 1$ whenever $M_i = (0)$ for any $1 \leq i \leq k$. Note that a nonzero element $(a_1, a_2, \ldots, a_k)$ is a vertex in $H_k(R)$ (k-zero-divisor in R) provided that at most one of its components can be 0 and at least one of its components; must belong to its corresponding maximal ideal. □

We now end the paper by applying the above theorem to $\mathbb{Z}_n$, the ring of integers modulo n.

Corollary 2.3. *For any fixed integer $k \geq 3$, let $n = p_1^{t_1} \cdots p_k^{t_k}$ for distinct primes $p_1, \ldots, p_k$ and positive integers $t_1, \ldots, t_k$. Then, $H_k(\mathbb{Z}_n)$ is a connected k-zero-divisor hypergraph with diameter at most 4 and domination number at most $2k$.*

Proof. The proof follows directly from Theorem 2.2 and the fact that $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{t_1}} \times \cdots \times \mathbb{Z}_{p_k^{t_k}}$. Note that for any prime $p \geq 2$ and integer $t \geq 2$, $\mathbb{Z}_p^t$ is a local ring. □

Acknowledgement

The research of the second author was in part supported by grant no. 1400130011 from IPM.

References

[1] D.D. Anderson, M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra, 159 (1993), 500-514.

[2] D.F. Anderson, P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, 217 (1999), 434-447.

[3] M.F. Atiyah, I.G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, MA, 1969.

[4] I. Beck, *Coloring of commutative rings*, J. Algebra, 116 (1988), 208-226.

[5] C. Berge, *Graphs and hypergraphs*, North-Holland Publishing Company, London, 2003.

[6] Ch. Eslahchi, A.M. Rahimi, *The k-zero-divisor hypergraph of a commutative ring*, Int. J. Math. Math. Sci. 2007, Art. ID 50875, 15 pp.

[7] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, New York, 1998.

[8] I. Kaplansky, *Commutative rings*, The University of Chicago Press, Chicago, Ill.-London, 1974.

[9] E. Mehdi-Nezhad, A. M Rahimi, *Comaximal submodule graphs of unitary modules*, Lib. Math., 38 (2018), 75-98.

[10] E. Mehdi-Nezhad, A. M Rahimi, *Dominating sets of the comaximal and ideal-based zero-divisor graphs of commutative rings*, Quaestiones Mathematicae, 38 (2015), 17pages.

[11] E. Mehdi-Nezhad, A. M Rahimi, *The annihilation graphs of commutator posets and lattices with respect to an element*, Journal of Algebra and Its Applications, Volume 16, Issue 06, (2017), 20 pages.

[12] E. Mehdi-Nezhad, A. M Rahimi, *The annihilation graphs of commutator posets and lattices with respect to an ideal*, Journal of Algebra and its applications, J. Algebra Appl. 17, Issue 06, (2018), 23 pages.

[13] D. A. Mojdeh, A. M. Rahimi, *Dominating sets of some graphs associated to commutative rings*, Communications in Algebra, 40 (2012), 3389-3396

[14] K. Selvakumar, V. Ramanathan, *On the genus of the k-annihilating-ideal hypergraph of commutative rings*, Indian Journal of Pure and Applied Mathematics, (2019), 461-475

[15] K. Selvakumar, V. Ramanathan, *Classification of nonlocal rings with genus one 3-zero-divisor hypergraphs*, Comm. Algebra, 45 (2016), 275-284.

[16] T. Tamizh Chelvam, K. Selvakumar, V. Ramanathan, *On the planarity of the k-zero-divisor hypergraphs*, AKCE Inter. J. Graphs and Combin., 12 (2015), 169-179.

[17] D.B. West, *Introduction to graph theory*, (Second Edition), Prentice Hall, USA, 2001.

Accepted: March 18, 2022