Generalized hesitant fuzzy N-soft sets and their applications

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Abstract. The N-soft Set as a generalization of the Soft Sets was introduced in 2018 by Fatimah et al. The concept of the N-soft Sets combined with the hesitant fuzzy sets is called hesitant fuzzy N-soft sets. On the other hand, the concept of fuzzy soft sets as a combination of soft sets and fuzzy sets was generalized by Majumdar and Samanta in 2010, called Generalized fuzzy soft sets, where many scholars have studied their properties and characteristics. This paper aims to extend the hesitant fuzzy N-soft set to a generalized hesitant fuzzy N-soft set that incorporates some characteristics of generalized fuzzy soft sets. Definition of the generalized hesitant fuzzy N-soft set, complements, and some of their operations are defined. Moreover, some of their properties, such as associative and distributive related to binary operations, are studied. Finally, we propose two algorithms for decision-making problems by extending the TOPSIS method to apply under generalized hesitant fuzzy N-soft set information.
Keywords: N-soft sets, hesitant fuzzy N-soft sets, generalized hesitant fuzzy soft sets, TOPSIS method.

1. Introduction

In real life, many uncertainty or ambiguity problems cannot be expressed by a crisp set, while decision-making is needed to obtain a possible result on a problem. In 1965, Zadeh [23] introduced a theory to solve this problem called the fuzzy set (FS). The FS theory is usually used to facilitate decision-making on uncertain or unclear problems by defining the degree of each object under

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consideration, called the membership value, in the interval [0,1]. In an FS, only one parameter is considered. In 1999 Molodtsov [16] introduced soft sets that associate objects with more than one parameter. A soft set (SS) is a set of ordered pairs of each parameter or attribute with related objects. Studies on Soft Sets have developed rapidly. Mostafa et al. [17], constructed codes by soft sets PU-valued functions. Zhan and Alcantud [24] reviewed some different algorithms of parameter reduction based on some types of (fuzzy) soft sets and compared these algorithms to emphasize their respective advantages and disadvantages. The methodologies and applications of soft set theory in Multi-attribute decision-making (MADM) have been studied by Khameneh and Kilicman [12] from 71 research papers published in 30 academic journals.

Based on the definition of the fuzzy set and the soft set, researchers have introduced hybrid models, their generalization, and their decision-making applications. Maji et al. [14] defined fuzzy soft sets (FSSs). Then, Roy and Maji [18] studied FSSs in a theoretic approach to decision making-problems. Majumdar and Samanta [15] have further generalized the concept of fuzzy soft sets introduced by Maji et al. [14] and have shown their application in decision-making and medical diagnosis problems. Wang et al. [20] extended the classical soft sets to hesitant fuzzy soft sets (HFSS) which are combined by the soft sets and hesitant fuzzy sets. In 2019, Wang and Qin [22] proposed an algorithm of fuzzy soft sets based on decision-making problems under incomplete information. Li et al. [13] proposed generalized hesitant fuzzy soft sets (GHFSS) by integrating generalized fuzzy soft sets with hesitant fuzzy sets and provided an effective approach to decision making. Recently, Karaaslan and Karamaz [11] defined the concept of hesitant fuzzy parameterized hesitant fuzzy soft (HFPHFSs) sets and set-theoretical operations of them and then developed two decision-making algorithms based on the proposed distance measure methods. An FSS is the collection of pairs between a parameter with an FS. However, a generalized fuzzy soft sets (GFSS) is an FSS, along with the degree of importance of each parameter. An HFSS is similar to the FSS, but the membership value of each object is some values in [0,1].

The definition of the SS was generalized to a new set called the N-soft set (NSS), which was introduced by Fatimah et al. [7]. In the same year, Akram et al. [1] introduced the fuzzy N-soft set (FNSS), and then in 2019, Akram et al. [2] generalized the definition of NSS or FNSS into a Hesitant fuzzy N-soft set (HFNSS) and developed new approaches to decision-making such as TOPSIS, choose value, L-choose value, etc. The Research related to decision-making using the approach of the N-soft set continues to grow. Akram et al. [3] extended the notion of parameter reduction to N-soft set theory and developed its application. On the other hand, Alcantud et al. [5] offered a fresh insight into rough set theory from the perspective of N-soft sets, and the applicability of the theoretical results is highlighted with a case study using real data regarding hotel rating. Fatimah and Alcantud [8] introduced a novel hybrid model called a multi-fuzzy N-soft set and designed an adjustable decision-making methodology for solving problems. Kamaci and Petchimuthu [10] proposed a bipolar extension of the N-soft set and set forth two outstanding algorithms to handle the decision-making problems under bipolar N-soft set environments. In 2021, Akram et al. [4] presented a new framework called bipolar fuzzy N-soft set as an extended model of [10] and proposed three algorithms to handle MADM problems. Newly, Alcantud [6] presented the first detailed analysis of the semantics of N-soft sets and designed three-way decision models with both a qualitative and a quantitative character. Another sophisticated hybrid model proposed recently is defined by Zhang et al. [25], where they proposed a q-rung orthopair fuzzy N-soft set (q-ROFNSS) and established two kinds of multiple-attribute group decision-making (MAGDM) methods.

In real life, a decision-maker sometimes needs to consider that the degree or contribution of each parameter in a decision-making problem is not necessarily the same. However, this problem cannot be solved using the HFNSS concept [2]. Therefore, it should be considered a new model in which the degree of each parameter is not the same. This degree is called the degree of preference.

This article constructs a new definition to generalize an HFNSS, called the generalized hesitant fuzzy N-soft set (GHFNSS). On the other hand, the GHFNSS is also a new hybrid model between the generalized fuzzy soft set (GFSS), HFSS and NSS. With this definition, the GHFNSS does not consider only the membership degrees (not necessarily unique for each object) and grades of objects but also the preference degree (the importance degree) of parameters. Furthermore, we can define some operations on GHFNSSs and prove the related properties. Finally, we apply the new TOPSIS algorithms for decision-making problems based on GHFNSS information.

We organize this paper as follows. Section 2 recalls the definitions and operations of SSs, FSs, FSSs, GFSSs, HFSSs, NSSs, FNSSs, and HFNSSs. Section 3 introduces a new model GHFNSS, some of its complements and examples. Then we propose some operations on GHFNSSs, and related to the operations, we derive some properties, such as associative and distributive, in Section 4. Section 5 proposes two algorithms by extending the TOPSIS method to apply under GHFNSS information and give a numerical example. Section 6 concludes the paper.

2. Preliminaries

In this section, the definitions introduced by previous scholars, such as soft sets, fuzzy sets, hesitant fuzzy sets, fuzzy soft sets, hesitant fuzzy soft sets, and N-soft sets, are recalled.

Definition 2.1 ([16]). *Suppose that U is a set of objects, $P(U)$ is the power set of U, and E is the set of parameters, $A \subseteq E$. A soft set (SS) F_A over U is a set, defined by a function f_A, that is represented as*

$$F_A = \{(e, f_A(e)) \mid e \in A, f_A(e) \in P(U)\}$$

Table 1: The soft set F_A

U \ A	e_2	e_3	e_4
u_1	0	1	1
u_2	1	1	0
u_3	0	1	0
u_4	1	1	0
u_5	1	0	1
u_6	0	0	1

where $f_A : A \to P(U)$.

Example 2.1. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be a set of job applicants and $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters. Given the parameters "appearance" (e_1), "courtesy" (e_2), "public speaking" (e_3), "innovative" (e_4) and $A = \{e_2, e_3, e_4\}$. By a decision-maker, based on his/her monitoring, a relation between each parameter with objects is represented as $f(e_2) = \{u_2, u_4, u_5\}$, $f(e_3) = \{u_1, u_2, u_3, u_4\}$ and $f(e_4) = \{u_1, u_5, u_6\}$. By definition, it is obtained an SS F_A as follows.

$F_A = \{(e_2, \{u_2, u_4, u_5\}), (e_3, \{u_1, u_2, u_3, u_4\}), (e_4, \{u_1, u_5, u_6\})\}$.

The SS F_A can be represented as in Table 1.

Definition 2.2 ([23]). *Suppose that U is a set of objects. A Fuzzy Set (FS) F over U is defined as*

$$F = \{(u, \mu(u)) \mid u \in U\}$$

where $\mu : U \to [0, 1]$ is called the membership function of F over U and $\mu(u)$ is the membership value of u.

A membership value of u represents the degree of the trust of an object u over a valuation of a decision-maker. Membership values of objects in an FS over U represent membership in a vaguely defined set.

Definition 2.3 ([19]). *Suppose that U is a set of objects. A Hesitant Fuzzy Set (HFS) H over U is defined as*

$$H = \{(u, \mu(u)) \mid u \in U\},$$

where $\mu : U \to int[0, 1]$ is called the membership function of H over U and $\mu(u)$ is the set of membership values of u. Here, $int[0, 1]$ is the collection of all subsets of $[0, 1]$.

The concept of the HFS is almost the same as the FS, but an object u may have more than one membership value. This happens because a decision-maker hesitates to valuation for an object or more than one decision-maker evaluates objects.

Definition 2.4 ([18]). *Suppose that U is a set of objects, E is the set of parameters and $A \subseteq E$. A Fuzzy Soft Set (FSS) G_A over U is a set*

$$G_A = \{(e, g_A(e)) \mid e \in A, g_A(e) \in I^U\},$$

where $g_A : A \to I^U$ and I^U is the collection of all FSs over U.

An FSS is the ordered pair of each parameter or attribute with an FS over U. This set provides more explanation than FSs and SSs to give more meaning to the assessment of objects.

Definition 2.5 ([15]). *Suppose that U is a set of objects, E is the set of parameters and I^U is the collection of all FSs over U. A Generalized Fuzzy Soft Set (GFSS) F_μ over U is defined as*

$$F_\mu = \{(e, F_\mu(e)) \mid e \in E\} = \{(e, (F(e), \mu(e))) \mid e \in E\}$$

where $F_\mu : E \longrightarrow I^U \times [0,1]$, $F\colon E \longrightarrow I^U$ is an FSS over U, $\mu\colon E \longrightarrow [0,1]$ is an FS over E, and $\mu(e)$ is called the degree of preference of $e \in E$ in F_μ .

Example 2.2. Suppose that a decision-maker interviews three candidates for agricultural extension workers, which are expressed in the set of objects $U = \{c_1, c_2, c_3\}$. Competencies (parameters) interviewed are e_1=Development of Farmer Participation and e_2=Development of Extension Programs. The candidate's ability to explain all the competencies tested will be assessed from the interview test. The results of this assessment are expressed as real numbers in [0,1], which are the membership values of each candidate for each parameter. Assume that a decision-maker defines the degrees of importance for each parameter as follows.

$$\mu(e_1) = 0.6; \mu(e_2) = 0.4.$$

Following are the results of the assessment of all candidates, which can be stated in the GFSS F_μ.

$$\begin{aligned} F_\mu &= \{(e_1, (F(e_1), \mu(e_1))), (e_2, (F(e_2), \mu(e_2)))\} \\ &= \{(e_1, (\{(c_1, 0.4), (c_2, 0.5), (c_3, 0.8)\}, 0.6)), \\ &\quad (e_2, (\{(c_1, 0.6), (c_2, 0.5), (c_3, 0.4)\}, 0.4))\}. \end{aligned}$$

Definition 2.6 ([21]). *Suppose that U is a set of objects, E is the set of parameters and $A \subseteq E$. A Hesitant Fuzzy Soft Set (HFSS) H_A over U is defined as*

$$H_A = \{(e, h_A(e)) \mid e \in A\},$$

where $h_A : A \to H^U$ and H^U is the collection of all HFSs over U.

As illustrated in Example 2.1, the SS can be represented as a matrix; their entries consist of 0 or 1. Fatimah et al. [7] generalized the concept of SSs called N-soft set as in the following definition.

Definition 2.7 ([7]). *Suppose that U is a set of objects, E is the set of parameters or attributes, and $A \subseteq E$. $R = \{0, 1, 2, ..., N-1\}$ is the set of grades where $N \in \{2, 3, ...\}$. An N-soft set (NSS) (F, A, N) over U is defined as*

$$(F, A, N) = \{(a, F(a)) \mid a \in A\},$$

where $F : A \to 2^{U \times R}$ such that $F(a) = \{(u, r_{au}) \mid u \in U, r_{au} \in R\}$. Here we also write $r_{au} = F(u)(a)$ as the grade of the object u related to the parameter a, and for each $a \in A$ and $u \in U$, there exists a unique $r_{au} \in R$.

Example 2.3. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be a set of cinemas and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ be the set of medias that making valuation. Suppose that $A = \{e_1, e_3, e_5\}$. For $N = 4$, $R = \{0, 1, 2, 3\}$, suppose that

$$\begin{aligned} F(e_1) &= \{(u_1, 3), (u_2, 1), (u_3, 0), (u_4, 2), (u_5, 2)\} \\ F(e_3) &= \{(u_1, 2), (u_2, 1), (u_3, 3), (u_4, 1), (u_5, 3)\} \\ F(e_5) &= \{(u_1, 0), (u_2, 3), (u_3, 1), (u_4, 2), (u_5, 3)\}. \end{aligned}$$

Then, by definition, we obtain the NSS (F, A, N) as follows.

$$\begin{aligned} (F, A, N) = \{ &(e_1, \{(u_1, 3), (u_2, 1), (u_3, 0), (u_4, 2), (u_5, 2)\}), \\ &(e_3, \{(u_1, 2), (u_2, 1), (u_3, 3), (u_4, 1), (u_5, 3)\}), \\ &(e_5, \{(u_1, 0), (u_2, 3), (u_3, 1), (u_4, 2), (u_5, 3)\})\}. \end{aligned}$$

The NSS (F, A, N) can be represented as in Table 2.

Table 2: The Representation Table of the NSS (F, A, N)

U \ A	e_1	e_3	e_5
u_1	3	2	0
u_2	1	1	3
u_3	0	3	1
u_4	2	1	2
u_5	2	3	3

Definition 2.8 ([7]). *Suppose that U is a set of objects, E is the set of parameters or attributes, $A \subseteq E$, $B \subseteq E$ and $A \cap B \neq \emptyset$. Let $R_1 = \{0, 1, 2, ..., N_1 - 1\}$ and $R_2 = \{0, 1, 2, ..., N_2 - 1\}$ be the sets of grades where $N_1, N_2 \in \{2, 3, ...\}$. The restricted intersection of (F, A, N_1) and (G, B, N_2) is defined as*

$$(F, A, N_1) \cap_{\Re} (G, B, N_2) = (J, A \cap B, \min(N_1, N_2))$$

where, for $e \in A \cap B$, $u \in U$, $(u, r_{eu}) \in J(e)$ *if and only if* $r_{eu} = \min(r_{eu}^{(1)}, r_{eu}^{(2)})$, *with* $(u, r_{eu}^{(1)}) \in F(e)$ *and* $(u, r_{eu}^{(2)}) \in G(e)$.

Definition 2.9 ([7]). *Suppose that* U *is a set of objects,* E *is the set of parameters or attributes,* $A \subseteq E$ *and* $B \subseteq E$. *Let* $R_1 = \{0, 1, 2, ..., N_1 - 1\}$ *and* $R_2 = \{0, 1, 2, ..., N_2 - 1\}$ *be the sets of grades where* $N_1, N_2 \in \{2, 3, ...\}$. *The extended intersection of* (F, A, N_1) *and* (G, B, N_2) *is defined as*

$$(F, A, N_1) \cap_{\mathcal{E}} (G, B, N_2) = (J, A \cup B, \max(N_1, N_2))$$

where, for $e \in A \cup B$, $u \in U$,

$$J(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ \{(u, r_{eu}) \mid u \in U\}, & \text{if } e \in A \cap B, \end{cases}$$

with $r_{eu} = \min(r_{eu}^{(1)}, r_{eu}^{(2)})$, *for* $(u, r_{eu}^{(1)}) \in F(e)$ *and* $(u, r_{eu}^{(2)}) \in G(e)$.

Definition 2.10 ([7]). *Suppose that* U *is a set of objects,* E *is the set of parameters or attributes,* $A \subseteq E$, $B \subseteq E$ *and* $A \cap B \neq \emptyset$. *Let* $R_1 = \{0, 1, 2, ..., N_1 - 1\}$ *and* $R_2 = \{0, 1, 2, ..., N_2 - 1\}$ *be the sets of grades where* $N_1, N_2 \in \{2, 3, ...\}$. *The restricted union of* (F, A, N_1) *and* (G, B, N_2) *is defined as*

$$(F, A, N_1) \cup_{\Re} (G, B, N_2) = (J, A \cap B, \max(N_1, N_2))$$

where, for $e \in A \cap B$, $u \in U$, $(u, r_{eu}) \in J(e)$ *if and only if* $r_{eu} = \max(r_{eu}^{(1)}, r_{eu}^{(2)})$, *with* $(u, r_{eu}^{(1)}) \in F(e)$ *and* $(u, r_{eu}^{(2)}) \in G(e)$.

Definition 2.11 ([7]). *Suppose that* U *is a set of objects,* E *is the set of parameters or attributes,* $A \subseteq E$ *and* $B \subseteq E$. *Let* $R_1 = \{0, 1, 2, ..., N_1 - 1\}$ *and* $R_2 = \{0, 1, 2, ..., N_2 - 1\}$ *be the sets of grades where* $N_1, N_2 \in \{2, 3, ...\}$. *The extended union of* (F, A, N_1) *and* (G, B, N_2) *is defined as*

$$(F, A, N_1) \cup_{\mathcal{E}} (G, B, N_2) = (J, A \cup B, \max(N_1, N_2))$$

where, for $e \in A \cup B$, $u \in U$,

$$J(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ \{(u, r_{eu}) \mid u \in U\}, & \text{if } e \in A \cap B, \end{cases}$$

with $r_{eu} = \max(r_{eu}^{(1)}, r_{eu}^{(2)})$ *for* $(u, r_{eu}^{(1)}) \in F(e)$ *and* $(u, r_{eu}^{(2)}) \in G(e)$.

Akram et al. [1] constructed a new hybrid model called fuzzy N-soft set as a suitable combination of FS theory with NSS.

Definition 2.12 ([1]). *Suppose that U is a set of objects, E is the set of parameters or attributes, $A \subseteq E$. A pair (μ, K), called a fuzzy N-soft set (FNSS) over U, with $K = (F, A, N)$ is an NSS over U, is defined as*

$$(\mu, K) = \{(a, \mu(a)) \mid a \in A\} = \left\{ \left(a, \left\{ \frac{(u, r_{au})}{m_{au}} \middle| u \in U \right\} \right) \middle| a \in A \right\},$$

where $\mu : A \to \bigcup_{a \in A} \mathcal{F}(F(a))$ with $\mathcal{F}(F(a))$ is the collection of all fuzzy sets over $F(a)$, $(u, r_{au}) \in F(a)$ and $m_{au} \in [0, 1]$ is the membership value of (u, r_{au}).

In 2019, Akram et al. [2] again introduced a novel model called hesitant fuzzy N-soft set as a hybrid of HFS and NSS.

Definition 2.13 ([2]). *Suppose that U is a set of objects, E is the set of parameters or attributes, $A \subseteq E$ and $N \in \{2, 3, ...\}$. A hesitant fuzzy N-soft set (HFNSS) $(\tilde{h}_f, A, N)$ over U is defined as*

$$(\tilde{h}_f, A, N) = \{((u, a), \tilde{h}_f(u, a)) \mid a \in A, u \in U\},$$

where $\tilde{h}_f : U \times A \to R \times \mathcal{P}^*([0, 1])$, with $\mathcal{P}^*([0, 1])$ denotes the set of non-empty subsets of real numbers in [0, 1]. Here $\tilde{h}_f(u, a) = (r_{au}, m_{au})$ with m_{au} and r_{au} denote the possible membership degrees and the grade of the element u related to parameter a, respectively, and for each $a \in A$ and $u \in U$, there exists a unique $r_{au} \in R$.

The HFNSS over U can also be represented by

$$(1) \quad (\hbar_f, A, N) = \{(a, \hbar_f(a)) \mid a \in A\} = \left\{ \left(a, \left\{ \frac{(u, r_{au})}{m_{au}} \middle| u \in U \right\} \right) \middle| a \in A \right\},$$

with $\hbar_f : A \longrightarrow \bigcup_{a \in A} \mathcal{H}(F(a))$, where $\mathcal{H}(F(a))$ is the collection of all HFSs over $F(a)$. Related to m_{au}, we defined $m^+_{au} = \max\{\gamma \mid \gamma \in m_{au}\}$ and $m^-_{au} = \min\{\gamma \mid \gamma \in m_{au}\}$.

The following definitions (Definitions 2.14-2.16) recall some complements of an HFNSS.

Definition 2.14 ([2]). *Given an HFNSS $(\hbar_f, A, N)$ over U as in the equation (1). The Hesitant Fuzzy Complement of $(\hbar_f, A, N)$ is defined as*

$$(\hbar^c_f, A, N) = \left\{ \left(a, \left\{ \frac{(u, r_{au})}{m^c_{au}} \middle| u \in U \right\} \right) \middle| a \in A \right\},$$

with

$$m^c_{au} = \bigcup_{\lambda \in m_{au}} \{1 - \lambda\}.$$

Definition 2.15 ([2]). *Given an HFNSS $(\tilde{h}_f, A, N)$ over U. The Top Weak Hesitant Fuzzy Complement $(\tilde{h}_f^T, A, N)$ of $(\tilde{h}_f, A, N)$ is defined as*

$$\tilde{h}_f^T(u,a) = \begin{cases} (N-1, \bigcup_{\lambda \in m_{au}} \{1-\lambda\}), & \text{if } r_{au} < N-1, \\ (0, \bigcup_{\lambda \in m_{au}} \{1-\lambda\}), & \text{if } r_{au} = N-1, \end{cases}$$

where $\tilde{h}_f(u,a) = (r_{au}, m_{au})$.

Definition 2.16 ([2]). *Given an HFNSS $(\tilde{h}_f, A, N)$ over U. The Bottom Weak Hesitant Fuzzy Complement $(\tilde{h}_f^B, A, N)$ of $(\tilde{h}_f, A, N)$ is defined as*

$$\tilde{h}_f^B(u,a) = \begin{cases} (0, \bigcup_{\lambda \in m_{au}} \{1-\lambda\}), & \text{if } r_{au} > 0, \\ (N-1, \bigcup_{\lambda \in m_{au}} \{1-\lambda\}), & \text{if } r_{au} = 0, \end{cases}$$

where $\tilde{h}_f(u,a) = (r_{au}, m_{au})$.

Now, we review the fundamental set-theoretic operations on HFNSSs.

Definition 2.17 ([2]). *Given two HFNSSs over U $(\tilde{h}_{f_1}, A, N_1)$ and $(\tilde{h}_{f_2}, B, N_2)$. The restricted intersection $(\tilde{h}_f, C, N)$ of them is defined as*

$$(\tilde{h}_f, C, N) = (\tilde{h}_{f_1}, A, N_1) \cap_{\Re} (\tilde{h}_{f_2}, B, N_2) = (\tilde{h}_f, A \cap B, \min(N_1, N_2))$$

where, for $c \in A \cap B \neq \emptyset$ and $u \in U$, $(r_{cu}, m_{cu}) = \tilde{h}_f(u,c)$ if and only if $r_{cu} = \min(r_{cu}^{(1)}, r_{cu}^{(2)})$ and $m_{cu} = \{\lambda \in m_{cu}^{(1)} \cup m_{cu}^{(2)} \mid \lambda \leq \min({m_{cu}^{(1)}}^+, {m_{cu}^{(2)}}^+\}$ with $(r_{cu}^{(1)}, m_{cu}^{(1)}) = \tilde{h}_{f_1}(u,c)$, $(r_{cu}^{(2)}, m_{cu}^{(2)}) = \tilde{h}_{f_2}(u,c)$.

Definition 2.18 ([2]). *Given two HFNSSs over U $(\hbar_{f_1}, A, N_1)$ and $(\hbar_{f_2}, B, N_2)$. The extended intersection $(\hbar_f, C, N)$ of them is defined as*

$$(\hbar_f, C, N) = (\hbar_{f_1}, A, N_1) \cap_{\mathcal{E}} (\hbar_{f_2}, B, N_2) = (\hbar_f, A \cup B, \max(N_1, N_2))$$

where, for $c \in A \cup B$

$$\hbar_f(c) = \begin{cases} \hbar_{f_1}(c), & \text{if } c \in A - B, \\ \hbar_{f_2}(c), & \text{if } c \in B - A, \\ \left\{ \frac{(u, r_{cu})}{m_{cu}} \middle| u \in U \right\}, & \text{if } c \in A \cap B, \end{cases}$$

with $r_{cu} = \min(r_{cu}^{(1)}, r_{cu}^{(2)})$, $m_{cu} = \{\lambda \in m_{cu}^{(1)} \cup m_{cu}^{(2)} \mid \lambda \leq \min({m_{cu}^{(1)}}^+, {m_{cu}^{(2)}}^+)\}$, $\frac{(u, r_{cu}^{(1)})}{m_{cu}^{(1)}} \in \hbar_{f_1}(c)$ and $\frac{(u, r_{cu}^{(2)})}{m_{cu}^{(2)}} \in \hbar_{f_2}(c)$.

Definition 2.19 ([2]). *Let U be a set of objects. Suppose that $(\tilde{h}_{f_1}, A, N_1)$ and $(\tilde{h}_{f_2}, B, N_2)$ are two HFNSSs over U. The restricted union $(\tilde{h}_f, C, N)$ of them is defined as*

$$(\tilde{h}_f, C, N) = (\tilde{h}_{f_1}, A, N_1) \cup_{\Re} (\tilde{h}_{f_2}, B, N_2) = (\tilde{h}_f, A \cap B, \max(N_1, N_2))$$

where, for $c \in A \cap B \neq \emptyset$ and $u \in U$, $(r_{cu}, m_{cu}) = \tilde{h}_f(u,c)$ if and only if $r_{cu} = \max(r_{cu}^{(1)}, r_{cu}^{(2)})$ and $m_{cu} = \{\lambda \in m_{cu}^{(1)} \cup m_{cu}^{(2)} \mid \lambda \geq \max({m_{cu}^{(1)}}^{-}, {m_{cu}^{(2)}}^{-})\}$ with $(r_{cu}^{(1)}, m_{au}^{(1)}) = \tilde{h}_{f_1}(u,c)$, $(r_{cu}^{(2)}, m_{cu}^{(2)}) = \tilde{h}_{f_2}(u,c)$.

Definition 2.20 ([2])**.** *Let U be a set of objects. Suppose that $(\hbar_{f_1}, A, N_1)$ and $(\hbar_{f_2}, B, N_2)$ are two HFNSSs over U. The extended union $(\hbar_f, C, N)$ of them is defined as*

$$(\hbar_f, C, N) = (\hbar_{f_1}, A, N_1) \cup_{\mathcal{E}} (\hbar_{f_2}, B, N_2) = (\hbar_f, A \cup B, \max(N_1, N_2))$$

where, for $c \in A \cup B$

$$\hbar_f(c) = \begin{cases} \hbar_{f_1}(c), & \text{if } c \in A - B, \\ \hbar_{f_2}(c), & \text{if } c \in B - A, \\ \left\{ \frac{(u, r_{cu})}{m_{cu}} \middle| u \in U \right\}, & \text{if } c \in A \cap B, \end{cases}$$

with $r_{cu} = \max(r_{cu}^{(1)}, r_{cu}^{(2)})$, $m_{cu} = \{\lambda \in m_{cu}^{(1)} \cup m_{cu}^{(2)} \mid \lambda \geq \max({m_{cu}^{(1)}}^{-}, {m_{cu}^{(2)}}^{-})\}$, $\frac{(u, r_{cu}^{(1)})}{m_{cu}^{(1)}} \in \hbar_{f_1}(c)$ and $\frac{(u, r_{cu}^{(2)})}{m_{cu}^{(2)}} \in \hbar_{f_2}(c)$.

3. Generalized Hesitant Fuzzy N-Soft Sets, their complements and some further set-theoretic operations

This section will introduce a novel hybrid model called generalized hesitant fuzzy N-soft set as a hybrid model of HFNSS and GFSS. Furthermore, we construct some complements and operations related to the new model.

Definition 3.1. *Suppose that U is a set of objects and E is the set of parameters. Let $A \subseteq E$, $N \in \{2, 3, ...\}$ and $R = \{0, 1, 2, \ldots, N-1\}$. Let $\mathcal{H} = (\hbar_f, A, N)$ be an HFNSS over U. A Generalized Hesitant Fuzzy N-Soft Set (GHFNSS) $(\mathcal{H}, \mu)$ over U is defined by*

$$\begin{aligned} (\mathcal{H}, \mu) &:= ((\hbar_f, A, N), \mu) = \{(a, \hbar_f(a), \mu(a)) \mid a \in A\} \\ &= \left\{ \left(a, \left\{ \left(\frac{(u, r_{au})}{m_{au}} \right) \middle| u \in U \right\}, \mu(a) \right) \middle| a \in A \right\}, \end{aligned} \tag{2}$$

where $\hbar_f : A \to \bigcup_{a \in A} \mathcal{H}(F(a))$ and $\mu : A \to [0,1]$. For all $a \in A, u \in U$, $r_{au} \in R$, m_{au} is a set of some values in [0,1] and $\mu(a)$ is a degree of preference of the parameter $a \in A$.

A GHFNSS over U can be represented by a representation form.

Definition 3.2. *Suppose that U is a set of objects and E is the set of parameters. Let $A \subseteq E$, $N \in \{2, 3, ...\}$ and $R = \{0, 1, 2, \ldots, N-1\}$. A representation form of a GHFNSS $(\mathcal{H}, \mu)$ over U is defined by*

$$(\mathcal{H}, \mu) = \{((u_i, a_j), \tilde{h}_f(u_i, a_j)) \mid a_j \in A, u_i \in U\}, \tag{3}$$

where $\tilde{h}_f : U \times A \longrightarrow R \times \mathcal{P}^*[0,1] \times [0,1]$ *with* $\tilde{h}_f(u_i, a_j) := (r_{a_j u_i}, m_{a_j u_i}, \mu(a_j))$. *To simplify, we may write* $\tilde{h}_f(u_i, a_j) := (\frac{r_{a_j u_i}}{m_{a_j u_i}}, \mu(a_j))$.

The representation form of a GHFNSS can be presented by a table as in Table 3. Here $r_{ij} = F(u_i)(a_j) = r_{a_j u_i}$, and $m_{ij} = m_{a_j u_i}$.

Table 3: The table of a representation form of a GHFNSS $(\mathcal{H}, \mu)$ over U.

$u_i \backslash a_j$	a_1	$\ldots$	a_j	$\ldots$	a_m
u_1	$(r_{11}, m_{11}, \mu(a_1))$	$\ldots$	$(r_{1j}, m_{1j}, \mu(a_j))$	$\ldots$	$(r_{1m}, m_{1m}, \mu(a_m))$
u_2	$(r_{21}, m_{21}, \mu(a_1))$	$\ldots$	$(r_{2j}, m_{2j}, \mu(a_j))$	$\ldots$	$(r_{2m}, m_{2m}, \mu(a_m))$
u_3	$(r_{31}, m_{31}, \mu(a_1))$	$\ldots$	$(r_{3j}, m_{3j}, \mu(a_j))$	$\ldots$	$(r_{3m}, m_{3m}, \mu(a_m))$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_i	$(r_{i1}, m_{i1}, \mu(a_1))$	$\ldots$	$(r_{ij}, m_{ij}, \mu(a_j))$	$\ldots$	$(r_{im}, m_{im}, \mu(a_m))$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_n	$(r_{n1}, m_{n1}, \mu(a_1))$	$\ldots$	$(r_{nj}, m_{nj}, \mu(a_j))$	$\ldots$	$(r_{nm}, m_{nm}, \mu(a_m))$

Example 3.1. Suppose that $U = \{u_1, u_2, u_3\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and the degrees of preference of parameters in A, $\mu(e_1) = 0.5, \mu(e_2) = 0.6, \mu(e_3) = 0.5, \mu(e_4) = 0.7, \mu(e_5) = 0.6$ and $\mu(e_6) = 0.8$. Suppose that $A, B, C \subseteq E$ with $A = \{e_1, e_2, e_4\}$, $B = \{e_2, e_4, e_5\}$ and $C = \{e_1, e_5, e_6\}$. Given three GHFNSSs over U, $(\mathcal{H}_1, \mu) = ((\hbar_{f_1}, A, 5), \mu)$, $(\mathcal{H}_2, \mu) = ((\hbar_{f_2}, B, 4), \mu)$ and $(\mathcal{H}_3, \mu) = ((\hbar_{f_3}, C, 6), \mu)$ as follows

$$\text{a. } (\mathcal{H}_1, \mu) = \Bigg\{ \left(e_1, \left\{\frac{(u_1,4)}{\{0.7,0.8,0.85\}}, \frac{(u_2,2)}{\{0.4,0.55,0.6\}}, \frac{(u_3,3)}{\{0.5,0.55,0.65\}}\right\}, 0.5\right), \left(e_2, \left\{\frac{(u_1,1)}{\{0.3,0.4,0.45\}}, \frac{(u_2,2)}{\{0.5,0.55,0.65\}}, \frac{(u_3,3)}{\{0.5,0.6,0.65\}}\right\}, 0.6\right), \left(e_4, \left\{\frac{(u_1,2)}{\{0.55,0.6\}}, \frac{(u_2,4)}{\{0.75,0.8,0.85\}}, \frac{(u_3,2)}{\{0.45,0.5,0.6\}}\right\}, 0.7\right) \Bigg\}$$

$$\text{b. } (\mathcal{H}_2, \mu) = \Bigg\{ \left(e_2, \left\{\frac{(u_1,1)}{\{0.3,0.35,0.45\}}, \frac{(u_2,3)}{\{0.5,0.6,0.65\}}, \frac{(u_3,2)}{\{0.45,0.5,0.6\}}\right\}, 0.6\right), \left(e_4, \left\{\frac{(u_1,3)}{\{0.6,0.65,0.7\}}, \frac{(u_2,2)}{\{0.5,0.6,0.75\}}, \frac{(u_3,2)}{\{0.45,0.5,0.55\}}\right\}, 0.7\right), \left(e_5, \left\{\frac{(u_1,2)}{\{0.55,0.6\}}, \frac{(u_2,3)}{\{0.65,0.7,0.75\}}, \frac{(u_3,3)}{\{0.7,0.75,0.8\}}\right\}, 0.6\right) \Bigg\}$$

$$\text{c. } (\mathcal{H}_3, \mu) = \Bigg\{ \left(e_1, \left\{\frac{(u_1,4)}{\{0.6,0.65\}}, \frac{(u_2,3)}{\{0.5,0.55,0.6\}}, \frac{(u_3,5)}{\{0.7,0.75,0.8\}}\right\}, 0.5\right), \left(e_5, \left\{\frac{(u_1,4)}{\{0.65,0.7,0.75\}}, \frac{(u_2,5)}{\{0.8,0.85\}}, \frac{(u_3,3)}{\{0.6,0.65,0.75\}}\right\}, 0.6\right), \left(e_6, \left\{\frac{(u_1,3)}{\{0.55,0.6,0.7\}}, \frac{(u_2,2)}{\{0.45,0.55,0.65\}}, \frac{(u_3,4)}{\{0.6,0.7,0.75\}}\right\}, 0.8\right) \Bigg\}.$$

The GHFNSSs as in Example 3.1 can be presented as representation forms as in Table 4, Table 5 and Table 6 respectively.

Table 4: The repesentation form of the GHFNSS $(\mathcal{H}_1, \mu)$ over U

$u_i \setminus a_j$	e_1	e_2	e_4
u_1	$(4, \{0.7, 0.8, 0.85\}, 0.5)$	$(1, \{0.3, 0.4, 0.45\}, 0.6)$	$(2, \{0.55, 0.6\}, 0.7)$
u_2	$(2, \{0.4, 0.55, 0.6\}, 0.5)$	$(2, \{0.5, 0.55, 0.65\}, 0.6)$	$(4, \{0.75, 0.8, 0.85\}, 0.7)$
u_3	$(3, \{0.5, 0.55, 0.65\}, 0.5)$	$(3, \{0.5, 0.6, 0.65\}, 0.6)$	$(2, \{0.45, 0.5, 0.6\}, 0.7)$

Table 5: The repesentation form of the GHFNSS $(\mathcal{H}_2, \mu)$ over U

$u_i \setminus a_j$	e_2	e_4	e_5
u_1	$(1, \{0.3, 0.35, 0.45\}, 0.6)$	$(3, \{0.6, 0.65, 0.7\}, 0.7)$	$(2, \{0.55, 0.6\}, 0.6)$
u_2	$(3, \{0.5, 0.6, 0.65\}, 0.6)$	$(2, \{0.5, 0.6, 0.75\}, 0.7)$	$(3, \{0.65, 0.7, 0.75\}, 0.6)$
u_3	$(2, \{0.45, 0.5, 0.6\}, 0.6)$	$(2, \{0.45, 0.5, 0.55\}, 0.7)$	$(3, \{0.7, 0.75, 0.8\}, 0.6)$

Table 6: The repesentation form of the GHFNSS $(\mathcal{H}_3, \mu)$ over U

$u_i \setminus a_j$	e_1	e_5	e_6
u_1	$(4, \{0.6, 0.65\}, 0.5)$	$(4, \{0.65, 0.7, 0.75\}, 0.6)$	$(3, \{0.55, 0.6, 0.7\}, 0.8)$
u_2	$(3, \{0.5, 0.55, 0.6\}, 0.5)$	$(5, \{0.8, 0.85\}, 0.6)$	$(2, \{0.45, 0.55, 0.65\}, 0.8)$
u_3	$(5, \{0.7, 0.75, 0.8\}, 0.5)$	$(3, \{0.6, 0.65, 0.75\}, 0.6)$	$(4, \{0.6, 0.7, 0.75\}, 0.8)$

Now, we introduce some definitions of the complement of a GHFNSS (Definitions 3.3-3.5).

Definition 3.3. *Suppose that $(\mathcal{H}, \mu)$ is a GHFNSS over U. We define some complements of such $(\mathcal{H}, \mu)$ as follows.*

a. A Weak Complement of $(\mathcal{H}, \mu)$ is

$$(\mathcal{H}^w, \mu) = \left\{ \left(a, \left\{ \left(\frac{(u, r^c_{au})}{m_{au}} \right) \middle| u \in U \right\}, \mu(a) \right) \middle| a \in A \right\}, \tag{4}$$

where $r^c_{au} \neq r_{au}$.

b. The Hesitant Fuzzy Complement of $(\mathcal{H}, \mu)$ is

$$(\mathcal{H}^f, \mu) = \left\{ \left(a, \left\{ \left(\frac{(u, r_{au})}{m^c_{au}} \right) \middle| u \in U \right\}, \mu(a) \right) \middle| a \in A \right\}, \tag{5}$$

where $m^c_{au} = \bigcup_{\lambda \in m_{au}} \{1 - \lambda\}$.

c. *The Preference Complement of* $(\mathcal{H},\mu)$ *is*

$$(\mathcal{H},\mu^c)=\left\{\left(a,\left\{\left(\frac{(u,r_{au})}{m_{au}}\right)\middle|u\in U\right\},\mu^c(a)\right)\middle|a\in A\right\}, \tag{6}$$

where $\mu^c(a)=1-\mu(a)$.

d. *A Weak Hesitant Fuzzy Complement of* $(\mathcal{H},\mu)$ *is*

$$(\mathcal{H}^c,\mu)=\left\{\left(a,\left\{\left(\frac{(u,r^c_{au})}{m^c_{au}}\right)\middle|u\in U\right\},\mu(a)\right)\middle|a\in A\right\},$$

where r^c_{au} *and* m^c_{au} *are in equations (4) and (5) respectively.*

e. *A Weak Preference Complement of* $(\mathcal{H},\mu)$ *is*

$$(\mathcal{H}^w,\mu^c)=\left\{\left(a,\left\{\left(\frac{(u,r^c_{au})}{m_{au}}\right)\middle|u\in U\right\},\mu^c(a)\right)\middle|a\in A\right\},$$

where r^c_{au} *and* $\mu^c(a)$ *are in equations (4) and (6) respectively.*

f. *The Hesitant Preference Fuzzy Complement of* $(\mathcal{H},\mu)$ *is*

$$(\mathcal{H}^f,\mu^c)=\left\{\left(a,\left\{\left(\frac{(u,r_{au})}{m^c_{au}}\right)\middle|u\in U\right\},\mu^c(a)\right)\middle|a\in A\right\},$$

where m^c_{au} *and* $\mu^c(a)$ *are in equations (5) and (6) respectively.*

g. *A Weak Generalized Hesitant Fuzzy Complement of* $(\mathcal{H},\mu)$ *is*

$$(\mathcal{H}^c,\mu^c)=\left\{\left(a,\left\{\left(\frac{(u,r^c_{au})}{m^c_{au}}\right)\middle|u\in U\right\},\mu^c(a)\right)\middle|a\in A\right\},$$

where r^c_{au}, m^c_{au} *and* $\mu^c(a)$ *are in equations* (4), (5) *and* (6) *respectively.*

It is clear that the complements b., c. and f. above are unique respectively, because of the definition of m^c_{ac} and $\mu^c(a)$.

Example 3.2. Based on Example 3.1, the Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H}_3,\mu)$ is presented in Table 7.

Table 7: The representation form of the Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H}_3,\mu)$.

$({\mathcal{H}_3}^c,\mu^c)$	e_1	e_5	e_6
u_1	$(3,\{0.35,0.4\},0.5)$	$(3,\{0.25,0.3,0.35\},0.4)$	$(2,\{0.3,0.4,0.45\},0.2)$
u_2	$(2,\{0.45,0.4,0.5\},0.5)$	$(4,\{0.15,0.2\},0.4)$	$(1,\{0.35,0.45,0.55\},0.2)$
u_3	$(4,\{0.2,0.25,0.3\},0.5)$	$(2,\{0.25,0.35,0.4\},0.4)$	$(3,\{0.25,0.3,0.4\},0.2)$

Definition 3.4. *Given a GHFNSS $(\mathcal{H}, \mu)$ over U. The following defines some special complements of such $(\mathcal{H}, \mu)$.*

a. *The Top Weak Complement of $(\mathcal{H}, \mu)$ is defined by*

$$(\mathcal{H}^T, \mu) = \{(a, \hbar_f(a)^T, \mu(a)) \mid a \in A\}, \quad \textit{with}$$

$$\hbar_f(a)^T = \begin{cases} \left\{ \frac{(u, N-1)}{m_{au}} \middle| u \in U \right\}, & \textit{if } r_{au} < N-1 \\ \left\{ \frac{(u, 0)}{m_{au}} \middle| u \in U \right\}, & \textit{if } r_{au} = N-1. \end{cases} \tag{7}$$

b. *The Top Weak Hesitant Fuzzy Complement of $(\mathcal{H}, \mu)$ is defined by*

$$(\mathcal{H}^{T^c}, \mu) = \{(a, \hbar_f(a)^{T^c}, \mu(a)) \mid a \in A\}, \quad \textit{with}$$

$$\hbar_f(a)^{T^c} = \begin{cases} \left\{ \frac{(u, N-1)}{\bigcup_{\lambda \in m_{au}} \{1-\lambda\}} \middle| u \in U \right\}, & \textit{if } r_{au} < N-1, \\ \left\{ \frac{(u, 0)}{\bigcup_{\lambda \in m_{au}} \{1-\lambda\}} \middle| u \in U \right\}, & \textit{if } r_{au} = N-1. \end{cases} \tag{8}$$

c. *The Top Weak Preference Complement of $(\mathcal{H}, \mu)$ is defined by*

$$(\mathcal{H}^T, \mu^c) = \{a, \hbar_f(a)^T, \mu^c(a) \mid a \in A\},$$

where $\hbar_f(a)^T$ is in equation (7), and $\mu^c(a) = 1 - \mu(a)$.

d. *The Top Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H}, \mu)$ is defined by*

$$(\mathcal{H}^{T^c}, \mu^c) = \{a, \hbar_f(a)^{T^c}, \mu^c(a) \mid a \in A\}.$$

where $\hbar_f(a)^{T^c}$ is in equation (8).

Example 3.3. Based on Example 3.1, the Top Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H}_3, \mu)$ is in Table 8.

Table 8: The representation form of the Top Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H}_3, \mu)$.

$({\mathcal{H}_3}^{T^c}, {\mu_3}^c)$	e_1	e_5	e_6
u_1	$(5, \{0.35, 0.4\}, 0.5)$	$(5, \{0.25, 0.3, 0.35\}, 0.4)$	$(5, \{0.3, 0.4, 0.45\}, 0.2)$
u_2	$(5, \{0.45, 0.4, 0.5\}, 0.5)$	$(0, \{0.15, 0.2\}, 0.4)$	$(5, \{0.35, 0.45, 0.55\}, 0.2)$
u_3	$(0, \{0.2, 0.25, 0.3\}, 0.5)$	$(5, \{0.25, 0.35, 0.4\}, 0.4)$	$(5, \{0.25, 0.3, 0.4\}, 0.2)$

Definition 3.5. *Given a GHFNSS $(\mathcal{H}, \mu)$ over U. The following defines the other special complements of such $(\mathcal{H}, \mu)$.*

a. The Bottom Weak Complement of $(\mathcal{H},\mu)$ *is defined by*

$$(\mathcal{H}^B,\mu)=\{(a,\hbar_f(a)^B,\mu(a))\mid a\in A\},\quad \textit{with}$$

$$\hbar_f(a)^B=\begin{cases}\left\{\frac{(u,0)}{m_{au}}\middle| u\in U\right\}, & \textit{if } r_{au}>0,\\ \left\{\frac{(u,N-1)}{m_{au}}\middle| u\in U\right\}, & \textit{if } r_{au}=0.\end{cases}\tag{9}$$

b. The Bottom Weak Hesitant Fuzzy Complement of $(\mathcal{H},\mu)$ *is defined by*

$$(\mathcal{H}^{B^c},\mu)=\{(a,\hbar_f(a)^{B^c},\mu(a))\mid a\in A\},\quad \textit{with}$$

$$\hbar_f(a)^{B^c}=\begin{cases}\left\{\frac{(u,0)}{\bigcup_{\lambda\in m_{au}}\{1-\lambda\}}\middle| u\in U\right\}, & \textit{if } r_{au}>0,\\ \left\{\frac{(u,N-1)}{\bigcup_{\lambda\in m_{au}}\{1-\lambda\}}\middle| u\in U\right\}, & \textit{if } r_{au}=0.\end{cases}\tag{10}$$

c. The Bottom Weak Preference Complement of $(\mathcal{H},\mu)$ *is defined by*

$$(\mathcal{H}^B,\mu^c)=\{(a,\hbar_f(a)^B,\mu^c(a))\mid a\in A\},$$

where $\hbar_f(a)^B$ *is in equation (9) and* $\mu^c(a)=1-\mu(a)$.

d. The Bottom Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H},\mu)$ *is defined by*

$$(\mathcal{H}^{B^c},\mu^c)=\{(a,\hbar_f(a)^{B^c},\mu^c(a))\mid a\in A\}.$$

where $\hbar_f(a)^{B^c}$ *is in equation (10).*

Note that each complement in Definition 3.4 and Definition 3.5, is unique.

Example 3.4. Based on Example 3.1, the Bottom Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H}_3,\mu)$ is in Table 9.

Table 9: The representation form of the Bottom Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H}_3,\mu)$.

$({\mathcal{H}_3}^{B^c},\mu^c)$	e_1	e_5	e_6
u_1	$(0,\{0.35,0.4\},0.5)$	$(0,\{0.25,0.3,0.35\},0.4)$	$(0,\{0.3,0.4,0.45\},0.2)$
u_2	$(0,\{0.45,0.4,0.5\},0.5)$	$(0,\{0.15,0.2\},0.4)$	$(0,\{0.35,0.45,0.55\},0.2)$
u_3	$(0,\{0.2,0.25,0.3\},0.5)$	$(0,\{0.25,0.35,0.4\},0.4)$	$(0,\{0.25,0.3,0.4\},0.2)$

Now, we propose some further set-theoretic operations in GHFNSSs.

Definition 3.6. *Suppose that U is a set of objects, E is the set of parameters, $A, B \subseteq E$ and $N_1, N_2 \in \{2, 3, ...\}$. Given two GHFNSSs $(\mathcal{H}_1, \mu_1)$ and $(\mathcal{H}_2, \mu_2)$ over U, as follows,*

$$
\begin{aligned}
(\mathcal{H}_1, \mu_1) =& ((\hbar_{f_1}, A, N_1), \mu_1) = \left\{((u, a), \tilde{h}_{f_1}(u, a)) | a \in A, u \in U\right\} \\
(\mathcal{H}_2, \mu_2) =& ((\hbar_{f_2}, B, N_2), \mu_2) = \left\{((u, b), \tilde{h}_{f_2}(u, b)) | b \in B, u \in U\right\}.
\end{aligned}
\tag{11}
$$

Then, the restricted intersection $(\mathcal{H}, \mu)$ of such GHFNSSs is defined by

$$
\begin{aligned}
(\mathcal{H}, \mu) =& (\mathcal{H}_1, \mu_1) \cap_{\Re} (\mathcal{H}_2, \mu_2) = ((\hbar_f, A \cap B, \min(N_1, N_2)), \mu) \\
=& \left\{((u, c), \tilde{h}_f(u, c)) | c \in C, u \in U\right\}
\end{aligned}
$$

where $\forall\, c \in C = A \cap B \neq \emptyset$, and $\forall\, u \in U$, $(r_{cu}, m_{cu}, \mu(c)) = \tilde{h}_f(u, c)$ if and only if

$$
\begin{aligned}
r_{cu} =& \min(r_{cu}^{(1)}, r_{cu}^{(2)}), m_{cu} = \{\lambda \in m_{cu}^{(1)} \cup m_{cu}^{(2)} \mid \lambda \leq \min(m_{cu}^{(1)+}, m_{cu}^{(2)+})\} \\
\mu(c) =& \min(\mu_1(c), \mu_2(c))
\end{aligned}
$$

with $m_{cu}^{(1)+} = \max(m_{cu}^{(1)})$ and $m_{cu}^{(2)+} = \max(m_{cu}^{(2)})$ for $(r_{cu}^{(1)}, m_{cu}^{(1)}, \mu_1(c)) = \tilde{h}_{f_1}(u, c)$ and $(r_{cu}^{(2)}, m_{cu}^{(2)}, \mu_2(c)) = \tilde{h}_{f_2}(u, c)$.

Definition 3.7. *Suppose that U is a set of objects, E is the set of parameters, $A, B \subseteq E$ and $N_1, N_2 \in \{2, 3, ...\}$. Given two GHFNSSs $(\mathcal{H}_1, \mu_1)$ and $(\mathcal{H}_2, \mu_2)$ over U as in equation (11). Then the extended intersection $(\mathcal{H}, \mu)$ of such GHFNSSs is defined by*

$$
\begin{aligned}
(\mathcal{H}, \mu) =& (\mathcal{H}_1, \mu_1) \cap_{\mathcal{E}} (\mathcal{H}_2, \mu_2) = ((\hbar_f, A \cup B, \max(N_1, N_2)), \mu) \\
=& \left\{((u, c), \tilde{h}_f(u, c)) | c \in C, u \in U\right\}
\end{aligned}
$$

where $\forall\, c \in C = A \cup B$ and $\forall\, u \in U$

$$
\tilde{h}_f(u, c) = \begin{cases} \tilde{h}_{f_1}(u, c), & \text{if } c \in A - B, \\ \tilde{h}_{f_2}(u, c), & \text{if } c \in B - A, \\ (r_{cu}, m_{cu}, \mu(c)), & \text{if } c \in A \cap B, \end{cases}
$$

where $r_{cu} = \min(r_{cu}^{(1)}, r_{cu}^{(2)})$, $m_{cu} = \{\lambda \in m_{cu}^{(1)} \cup m_{cu}^{(2)} \mid \lambda \leq \min(m_{cu}^{(1)+}, m_{cu}^{(2)+})\}$ and $\mu(c) = \min(\mu_1(c), \mu_2(c))$ with $(r_{cu}^{(1)}, m_{cu}^{(1)}, \mu_1(c)) = \tilde{h}_{f_1}(u, c)$ and $(r_{cu}^{(2)}, m_{cu}^{(2)}, \mu_2(c)) = \tilde{h}_{f_2}(u, c)$.

Definition 3.8. *Suppose that U is a set of objects, E is the set of parameters, $A, B \subseteq E$ and $N_1, N_2 \in \{2, 3, ...\}$. Given two GHFNSSs $(\mathcal{H}_1, \mu_1)$ and $(\mathcal{H}_2, \mu_2)$*

over U as in equation (11). Then the restricted union $(\mathcal{H}, \mu)$ of such GHFNSSs is defined by

$$\begin{aligned}(\mathcal{H}, \mu) =&(\mathcal{H}_1, \mu_1) \cup_{\Re} (\mathcal{H}_2, \mu_2) = ((\hbar_f, A \cap B, \max(N_1, N_2)), \mu)\\ =&\left\{((u, c), \tilde{h}_f(u, c)) | c \in C, u \in U\right\}\end{aligned}$$

where $\forall\, c \in C = A \cap B \neq \emptyset$, $\forall\, u \in U$, $(r_{cu}, m_{cu}, \mu(c)) = \tilde{h}_f(u, c)$ *if and only if*

$$\begin{aligned} r_{cu} =& \max(r_{cu}^{(1)}, r_{cu}^{(2)}) m_{cu} = \{\lambda \in m_{cu}^{(1)} \cup m_{cu}^{(2)} \mid \lambda \geq \max(m_{cu}^{(1)^-}, m_{cu}^{(2)^-})\}\\ \mu(c) =& \max(\mu_1(c), \mu_2(c))\end{aligned}$$

with $m_{cu}^{(1)^-} = \min(m_{cu}^{(1)})$ *and* $m_{cu}^{(2)^-} = \min(m_{cu}^{(2)})$ *for* $(r_{cu}^{(1)}, m_{cu}^{(1)}, \mu_1(c)) = \tilde{h}_{f_1}(u, c)$ *and* $(r_{cu}^{(2)}, m_{cu}^{(2)}, \mu_2(c)) = \tilde{h}_{f_2}(u, c)$.

Definition 3.9. *Suppose that U is a set of objects, E is the set of parameters, $A, B \subseteq E$ and $N_1, N_2 \in \{2, 3, ...\}$. Given two GHFNSSs $(\mathcal{H}_1, \mu_1)$ and $(\mathcal{H}_2, \mu_2)$ over U as in equation (11). Then the extended union $(\mathcal{H}, \mu)$ of such GHFNSSs is defined by*

$$\begin{aligned}(\mathcal{H}, \mu) =&(\mathcal{H}_1, \mu_1) \cup_{\mathcal{E}} (\mathcal{H}_2, \mu_2) = ((\hbar_f, A \cup B, \max(N_1, N_2)), \mu)\\ =&\left\{((u, c), \tilde{h}_f(u, c)) | c \in C, u \in U\right\},\end{aligned}$$

where $\forall\, c \in C = A \cup B$, $\forall\, u \in U$, $(r_{cu}, m_{cu}, \mu(c)) \in \tilde{h}_f(u, c)$ *if and only if*

$$\tilde{h}_f(u, c) = \begin{cases} \tilde{h}_{f_1}(u, c), & \text{if } c \in A - B,\\ \tilde{h}_{f_2}(u, c), & \text{if } c \in B - A,\\ (r_{cu}, m_{cu}, \mu(c)), & \text{if } c \in A \cap B,\end{cases}$$

where $r_{cu} = \max(r_{cu}^{(1)}, r_{cu}^{(2)})$, $m_{cu} = \{\lambda \in m_{cu}^{(1)} \cup m_{cu}^{(2)} \mid \lambda \geq \max(m_{cu}^{(1)^-}, m_{cu}^{(2)^-})\}$ *and* $\mu(c) = \max(\mu_1(c), \mu_2(c))$ *with* $(r_{cu}^{(1)}, m_{cu}^{(1)}, \mu_1(c)) = \tilde{h}_{f_1}(u, c)$ *and* $(r_{cu}^{(2)}, m_{cu}^{(2)}, \mu_2(c)) = \tilde{h}_{f_2}(u, c)$.

4. Some properties of GHFNSSs

Referring to the operations in the previous section, we derive the following properties, such as associative and distributive. However, the commutative property of GHFNSSs is trivial.

Theorem 4.1 (Associative)**.** *Given three GHFNSSs $(\mathcal{H}_1, \mu_1)$, $(\mathcal{H}_2, \mu_2)$ and $(\mathcal{H}_3, \mu_3)$ over U, with $\mathcal{H}_1 = (\hbar_{f_1}, A, N_1)$, $\mathcal{H}_2 = (\hbar_{f_2}, B, N_2)$ and $\mathcal{H}_3 = (\hbar_{f_3}, C, N_3)$ are HFNSSs over U. Then*

1. $(\mathcal{H}_1, \mu_1) \cap_{\Re} ((\mathcal{H}_2, \mu_2) \cap_{\Re} (\mathcal{H}_3, \mu_3)) = ((\mathcal{H}_1, \mu_1) \cap_{\Re} (\mathcal{H}_2, \mu_2)) \cap_{\Re} (\mathcal{H}_3, \mu_3)$.

2. $(\mathcal{H}_1, \mu_1) \cap_{\mathcal{E}} ((\mathcal{H}_2, \mu_2) \cap_{\mathcal{E}} (\mathcal{H}_3, \mu_3)) = ((\mathcal{H}_1, \mu_1) \cap_{\mathcal{E}} (\mathcal{H}_2, \mu_2)) \cap_{\mathcal{E}} (\mathcal{H}_3, \mu_3)$.

3. $(\mathcal{H}_1, \mu_1) \cup_{\Re} ((\mathcal{H}_2, \mu_2) \cup_{\Re} (\mathcal{H}_3, \mu_3)) = ((\mathcal{H}_1, \mu_1) \cup_{\Re} (\mathcal{H}_2, \mu_2)) \cup_{\Re} (\mathcal{H}_3, \mu_3)$.

4. $(\mathcal{H}_1, \mu_1) \cup_{\mathcal{E}} ((\mathcal{H}_2, \mu_2) \cup_{\mathcal{E}} (\mathcal{H}_3, \mu_3)) = ((\mathcal{H}_1, \mu_1) \cup_{\mathcal{E}} (\mathcal{H}_2, \mu_2)) \cup_{\mathcal{E}} (\mathcal{H}_3, \mu_3)$.

Proof. We only give the proof of *2*. The others are similar. Suppose that $(\mathcal{H}_4, \mu_4) = (\mathcal{H}_2, \mu_2) \cap_{\mathcal{E}} (\mathcal{H}_3, \mu_3)$ and $D = B \cup C$. By using Definition 3.7

$$(\mathcal{H}_4, \mu_4) = ((\hbar_{f_4}, B \cup C, \max(N_2, N_3)), \mu_4). = \left\{((u, d), \tilde{h}_{f_4}(u, d)) | d \in D, u \in U\right\},$$

where any $d \in D = B \cup C$, $\forall\, u \in U$,

$$\tilde{h}_{f_4}(u, d) = \begin{cases} \tilde{h}_{f_2}(u, d), & \text{if } d \in B - C, \\ \tilde{h}_{f_3}(u, d), & \text{if } d \in C - B, \\ (r_{cu}, m_{cu}, \mu(c)), & \text{if } d \in B \cap C, \end{cases}$$

where $r_{du} = \min(r_{du}^{(2)}, r_{du}^{(3)})$, $m_{du} = \{\lambda_4 \in m_{du}^{(2)} \cup m_{du}^{(3)} \mid \lambda_4 \leq \min({m_{du}^{(2)}}^+, {m_{du}^{(3)}}^+)\}$ and $\mu_4(d) = \min(\mu_2(d), \mu_3(d))$ with $(r_{du}^{(2)}, m_{du}^{(2)}, \mu_2(d)) = \tilde{h}_{f_2}(u, d)$ and $(r_{du}^{(3)}, m_{du}^{(3)}, \mu_3(d)) = \tilde{h}_{f_3}(u, d)$.

Suppose that $(\mathcal{H}, \mu) = (\mathcal{H}_1, \mu_1) \cap_{\mathcal{E}} (\mathcal{H}_4, \mu_4)$ and $G = A \cup D$.
Based on Definition 3.7,

$$\begin{aligned} (\mathcal{H}, \mu) =& ((\hbar_f, A \cup D, \min(N_1, N_4)), \mu) \\ =& ((\hbar_f, A \cup (B \cup C), \min(N_1, \min(N_2, N_3))), \mu) \\ =& ((\hbar_f, (A \cup B) \cup C, \min(\min(N_1, N_2), N_3))), \mu) \\ =& \left\{((u, d), \tilde{h}_f(u, d)) | d \in G, u \in U\right\}, \end{aligned}$$

where any $d \in A \cup D$, $\forall\, u \in U$,

$$\tilde{h}_f(u, d) = \begin{cases} \tilde{h}_{f_1}(u, d), & \text{if } d \in A - D, \\ \tilde{h}_{f_4}(u, d), & \text{if } d \in D - A, \\ (r_{du}, m_{du}, \mu(d)), & \text{if } d \in A \cap D, \end{cases}$$

where $r_{du} = \min(r_{du}^{(1)}, r_{du}^{(4)})$, $m_{du} = \{\lambda \in m_{du}^{(1)} \cup m_{du}^{(4)} \mid, \lambda \leq \min({m_{du}^{(1)}}^+, {m_{du}^{(4)}}^+)\}$ and $\mu(d) = \min(\mu_1(d), \mu_4(d))$ with $(r_{du}^{(1)}, m_{du}^{(1)}, \mu_1(d)) = \tilde{h}_{f_1}(u, d)$ and $(r_{du}^{(4)}, m_{du}^{(4)}, \mu_4(d)) = \tilde{h}_{f_4}(u, d)$.

Since

$$\begin{aligned} r_{du} =& \min(r_{du}^{(1)}, \min(r_{du}^{(2)}, r_{du}^{(3)})) = \min(\min(r_{du}^{(1)}, r_{du}^{(2)}), r_{du}^{(3)}) \\ m_{du} =& \{\lambda \in m_{du}^{(1)} \cup (m_{du}^{(2)} \cup m_{du}^{(3)}) \mid \lambda \leq \min({m_{du}^{(1)}}^+, \min({m_{du}^{(2)}}^+, {m_{du}^{(3)}}^+))\} \\ =& \{\lambda \in (m_{du}^{(1)} \cup (m_{du}^{(2)}) \cup m_{du}^{(3)} \mid \lambda \leq \min(\min({m_{du}^{(1)}}^+, {m_{du}^{(2)}}^+), {m_{du}^{(3)}}^+)\} \text{ and} \\ \mu(d) =& \min(\mu_1(d), \min(\mu_2(d), \mu_3(d)) = \min(\min(\mu_1(d), \mu_2(d)), \mu_3(d)), \end{aligned}$$

then it is proved that $(\mathcal{H}_1, \mu_1) \cap_{\mathcal{E}} ((\mathcal{H}_2, \mu_2) \cap_{\mathcal{E}} (\mathcal{H}_3, \mu_3)) = ((\mathcal{H}_1, \mu_1) \cap_{\mathcal{E}} (\mathcal{H}_2, \mu_2)) \cap_{\mathcal{E}} (\mathcal{H}_3, \mu_3)$. $\square$

Theorem 4.2. *(Distributive) Given three GHFNSSs* $(\mathcal{H}_1, \mu_1)$, $(\mathcal{H}_2, \mu_2)$ *and* $(\mathcal{H}_3, \mu_3)$ *over* U, *with* $\mathcal{H}_1 = (\hbar_{f_1}, A, N_1)$, $\mathcal{H}_2 = (\hbar_{f_2}, B, N_2)$ *and* $\mathcal{H}_3 = (\hbar_{f_3}, C, N_3)$ *are HFNSSs over* U. *Then*

1. $(\mathcal{H}_1, \mu_1) \cap_{\Re} ((\mathcal{H}_2, \mu_2) \cup_{\Re} (\mathcal{H}_3, \mu_3)) =$
   $((\mathcal{H}_1, \mu_1) \cap_{\Re} (\mathcal{H}_2, \mu_2)) \cup_{\Re} ((\mathcal{H}_1, \mu_1) \cap_{\Re} (\mathcal{H}_3, \mu_3))$.
2. $(\mathcal{H}_1, \mu_1) \cup_{\Re} ((\mathcal{H}_2, \mu_2) \cap_{\Re} (\mathcal{H}_3, \mu_3)) =$
   $((\mathcal{H}_1, \mu_1) \cup_{\Re} (\mathcal{H}_2, \mu_2)) \cap_{\Re} ((\mathcal{H}_1, \mu_1) \cup_{\Re} (\mathcal{H}_3, \mu_3))$.
3. $(\mathcal{H}_1, \mu_1) \cap_{\mathcal{E}} ((\mathcal{H}_2, \mu_2) \cup_{\mathcal{E}} (\mathcal{H}_3, \mu_3)) =$
   $((\mathcal{H}_1, \mu_1) \cap_{\mathcal{E}} (\mathcal{H}_2, \mu_2)) \cup_{\mathcal{E}} ((\mathcal{H}_1, \mu_1) \cap_{\mathcal{E}} (\mathcal{H}_3, \mu_3))$.
4. $(\mathcal{H}_1, \mu_1) \cup_{\mathcal{E}} ((\mathcal{H}_2, \mu_2) \cap_{\mathcal{E}} (\mathcal{H}_3, \mu_3)) =$
   $((\mathcal{H}_1, \mu_1) \cup_{\mathcal{E}} (\mathcal{H}_2, \mu_2)) \cap_{\mathcal{E}} ((\mathcal{H}_1, \mu_1) \cup_{\mathcal{E}} (\mathcal{H}_3, \mu_3))$.

Proof. Here, we give the proof of *1*. The others are similar. Suppose that $(\mathcal{H}_4, \mu_4) = (\mathcal{H}_2, \mu_2) \cup_{\Re} (\mathcal{H}_3, \mu_3)$, $D = B \cap C$ and $N_4 = \max(N_2, N_3)$. Based on Definition 3.8,

$$(\mathcal{H}_4, \mu_4) = ((\hbar_{f_4}, B \cap C, \max(N_2, N_3)), \mu_4) = \left\{((u, d), \tilde{h}_{f_4}(u, d)) | d \in D, u \in U\right\},$$

where, for any $d \in D = B \cap C$, $\forall\, u \in U$, $(r_{du}, m_{du}, \mu(d)) = \tilde{h}_{f_4}(u, d)$ if and only if

$$\begin{aligned} r_{du} &= \max(r_{du}^{(2)}, r_{du}^{(3)}), m_{du} = \{\lambda_4 \in m_{du}^{(2)} \cup m_{du}^{(3)} \mid \lambda_4 \geq \max(m_{du}^{(2)^-}, m_{du}^{(3)^-})\} \\ \mu_4(d) &= \max(\mu_2(d), \mu_3(d)) \end{aligned}$$

for $(r_{du}^{(2)}, m_{du}^{(2)}, \mu_2(d)) = \tilde{h}_{f_2}(u, d)$ and $(r_{du}^{(3)}, m_{du}^{(3)}, \mu_3(d)) = \tilde{h}_{f_3}(u, d)$.

Suppose that $(\mathcal{H}, \mu) = (\mathcal{H}_1, \mu_1) \cap_{\Re} (\mathcal{H}_4, \mu_4)$ and $G = A \cap D$. By using Definition 3.6,

$$\begin{aligned} (\mathcal{H}, \mu) &= ((\hbar_f, A \cap D, \min(N_1, N_4)), \mu) \\ &= ((\hbar_f, A \cap (B \cap C), \min(N_1, \max(N_2, N_3))), \mu). \\ &= \left\{((u, g), \tilde{h}_f(u, g)) | g \in G, u \in U\right\}. \end{aligned}$$

where, for any $g \in G = A \cap D$, $\forall\, u \in U$, $(r_{gu}, m_{gu}, \mu(g)) = \tilde{h}_f(u, g)$ if and only if

$$\begin{aligned} r_{gu} &= \min(r_{gu}^{(1)}, r_{gu}^{(4)}) = \min(r_{gu}^{(1)}, \max(r_{gu}^{(2)}, r_{gu}^{(3)})) \\ m_{gu} &= \{\lambda \in m_{gu}^{(1)} \cup m_{gu}^{(4)} \mid \lambda \leq \min(m_{gu}^{(1)^+}, m_{gu}^{(4)^+})\} \\ \mu(g) &= \min(\mu_1(g), \mu_4(g)) = \min(\mu_1(g), \max(\mu_2(g), \mu_3(g))) \end{aligned}$$

for $(r_{gu}^{(1)}, m_{gu}^{(1)}, \mu_1(g)) = \tilde{h}_{f_1}(u, g)$ and $(r_{gu}^{(4)}, m_{gu}^{(4)}, \mu_4(g)) = \tilde{h}_{f_4}(u, g)$.

Suppose that $(\mathcal{H}_5, \mu_5) = (\mathcal{H}_1, \mu_1) \cap_{\Re} (\mathcal{H}_2, \mu_2)$, $P = A \cap B$ and $N_5 = \min(N_1, N_2)$. Based on Definition 3.6

$$\begin{aligned}(\mathcal{H}_5, \mu_5) =& ((\hbar_{f_5}, A \cap B, \min(N_1, N_2)), \mu_5)\\ =& \Big\{((u,p), \tilde{h}_{f_5}(u,p)) | p \in P, u \in U\Big\}\end{aligned}$$

where, for any $p \in A \cap B$, $\forall\, u \in U$, $(r_{pu}, m_{pu}, \mu(p)) = \tilde{h}_{f_5}(u,p)$ if and only if

$$\begin{aligned}r_{pu} =& \min(r_{pu}^{(1)}, r_{pu}^{(2)})\\ m_{pu} =& \{\lambda_5 \in m_{pu}^{(1)} \cup m_{pu}^{(2)} \mid \lambda_5 \le \min({m_{pu}^{(1)}}^+, {m_{pu}^{(2)}}^+)\}\\ \mu_5(p) =& \min(\mu_1(p), \mu_2(p)).\end{aligned}$$

for $(r_{pu}^{(1)}, m_{pu}^{(1)}, \mu_1(p)) = \tilde{h}_{f_1}(u,p)$ and $(r_{pu}^{(2)}, m_{pu}^{(2)}, \mu_2(p)) = \tilde{h}_{f_2}(u,p)$.

Suppose that $(\mathcal{H}_6, \mu_6) = (\mathcal{H}_1, \mu_1) \cap_{\Re} (\mathcal{H}_3, \mu_3)$, $Q = A \cap C$ and $N_6 = \min(N_1, N_3)$. Based on Definition 3.6

$$\begin{aligned}(\mathcal{H}_6, \mu_6) =& ((\hbar_{f_6}, A \cap C, \min(N_1, N_3)), \mu_6)\\ =& \Big\{((u,q), \tilde{h}_{f_6}(u,q)) | q \in Q, u \in U\Big\}\end{aligned}$$

where, for any $q \in Q = A \cap C$, $\forall\, u \in U$, $(r_{qu}, m_{qu}, \mu(q)) = \tilde{h}_{f_6}(u,q)$ if and only if

$$\begin{aligned}r_{qu} =& \min(r_{qu}^{(1)}, r_{qu}^{(3)}),\\ m_{qu} =& \{\lambda_6 \in m_{qu}^{(1)} \cup m_{qu}^{(3)} \mid \lambda_6 \le \min({m_{qu}^{(1)}}^+, {m_{cu}^{(3)}}^+)\},\\ \mu_6(q) =& \min(\mu_1(q), \mu_3(q))\end{aligned}$$

for $(r_{qu}^{(1)}, m_{qu}^{(1)}, \mu_1(q)) = \tilde{h}_{f_1}(u,q)$ and $(r_{qu}^{(3)}, m_{qu}^{(3)}, \mu_3(q)) = \tilde{h}_{f_3}(u,q)$.

Suppose that $(\mathcal{H}_7, \mu_7) = (\mathcal{H}_5, \mu_5) \cup_{\Re} (\mathcal{H}_6, \mu_6)$, $S = P \cap Q$ and $N_7 = \max(N_5, N_6)$. Based on Definition 3.8

$$\begin{aligned}(\mathcal{H}_7, \mu_7) =& ((\hbar_{f_7}, P \cap Q, \max(N_5, N_6)), \mu_7)\\ =& \Big\{((u,s), \tilde{h}_{f_7}(u,s)) | s \in S, u \in U\Big\},\end{aligned}$$

where, for any $s \in P \cap Q$, $\forall\, u \in U$, $(r_{su}, m_{su}, \mu(s)) = \tilde{h}_{f_7}(u,s)$ if and only if

$$\begin{aligned}r_{su} =& \max(r_{su}^{(5)}, r_{su}^{(6)}),\\ m_{su} =& \{\lambda_7 \in m_{su}^{(5)} \cup m_{su}^{(6)} \mid \lambda_7 \ge \max({m_{su}^{(5)}}^-, {m_{su}^{(6)}}^-)\},\\ \mu_7(s) =& \max(\mu_5(s), \mu_6(s)),\end{aligned}$$

for $(r_{su}^{(5)}, m_{su}^{(5)}, \mu_5(s)) = \tilde{h}_{f_5}(u,s)$ and $(r_{su}^{(6)}, m_{su}^{(6)}, \mu_6(s)) = \tilde{h}_{f_6}(u,s)$.

Now, we will prove $(\mathcal{H}, \mu) = (\mathcal{H}_7, \mu_7)$. Consider that

$$\begin{aligned}(\mathcal{H}_7, \mu_7) =&((\hbar_{f_7}, P \cap Q, \max(N_5, N_6)), \mu_7)\\ =&((\hbar_{f_7}, (A \cap B) \cap (A \cap C), \max(\min(N_1, N_2), \min(N_1, N_3)), \mu_7)\\ =&((\hbar_{f_7}, A \cap (B \cap C), \min(N_1, \max(N_2, N_3)), \mu_7) = (\mathcal{H}, \mu),\end{aligned}$$

where, for any $s \in A \cap (B \cap C)$, $\forall\, u \in U$, $(r^{(7)}_{su}, m^{(7)}_{su}, \mu_7(s)) = \tilde{h}_{f_7}(u, s)$ if and only if

$$\begin{aligned}r^{(7)}_{su} =&\max(r^{(5)}_{su}, r^{(6)}_{su}) = \max(\min(r^{(1)}_{su}, r^{(2)}_{su}), \min(r^{(1)}_{su}, r^{(3)}_{su}))\\ =&\min(r^{(1)}_{su}, \max(r^{(2)}_{su}, r^{(3)}_{su})) = r_{su},\\ m^{(7)}_{su} =&\{\lambda_7 \in m^{(5)}_{su} \cup m^{(6)}_{su} \mid \lambda_7 \geq \max(m^{(5)^-}_{su}, m^{(6)^-}_{su})\}\\ =&\{\lambda_7 \in m^{(1)}_{su} \cup (m^{(2)}_{su} \cup m^{(3)}_{su}) \mid \lambda_7 \leq \min(m^{(1)^+}_{su}, m^{(4)^+}_{su})\} = m_{su}\\ \mu_7(s) =&\max(\mu_5(s), \mu_6(s)) = \max(\min(\mu_1(s), \mu_2(s)), \min(\mu_1(s), \mu_3(s)))\\ =&\min(\mu_1(s), \max(\mu_2(s), \mu_3(s))) = \mu(s).\end{aligned}$$

Therefore $(\mathcal{H}_1, \mu_1) \cap_{\Re} ((\mathcal{H}_2, \mu_2) \cup_{\Re} (\mathcal{H}_3, \mu_3)) = ((\mathcal{H}_1, \mu_1) \cap_{\Re} (\mathcal{H}_2, \mu_2)) \cup_{\Re} ((\mathcal{H}_1, \mu_1) \cap_{\Re} (\mathcal{H}_3, \mu_3))$. □

5. Application of GHFNSSs

Hwang and Yoon, in 1981 [9] introduced an algorithm for decision-making problems concerning parameters or attributes. This algorithm is called TOPSIS *(Technique for Order Preference by Similarity to Ideal Solution)*. Under HFNSS information, Akram et al. [2] have extended this method. When a decision-maker wants to rank objects to obtain the best performance, the chosen alternative has the shortest distance from the positive ideal solution (PIS) and the longest distance from the negative ideal solution (NIS).

We propose the two following algorithms by extending the TOPSIS method to apply under GHFNSS information. Algorithm 1 could apply for a condition that the number of elements of m_{ij} is not necessary the same for all i and j, while in Algorithm 2, that is the same. Algorithm 2 is a new extended method based on GHFNSSs as a generalization of the method introduced by Akram et al. [2]. In our method, we use the information on the preference degree of parameters. The sum of all the preference degrees does not need equal to one as in the definition of the weight of the parameters. On the other hand, in determining the ranking order of objects in choosing the best one, Akram et al. [2] refer to pairs of values called relative adjacency to ideal solution. It is impossible to determine the ranking order of a collection of pairs of values (a_i, a_j) for $i, j \in \mathbb{N}$, except in the condition that $a_i > a_j$ and $b_i > b_j$ for $i \neq j$. Because of this, in Algorithm 2, we give a modification of the Akram's method.

Algorithm 1

1. Input a subset A of a parameter set E. Given a set of objects $U = \{u_1, u_2, \dots, u_p\}$ and the set of parameters or attributes $A = \{a_1, a_2, \dots, a_q\}$.

2. Represent a GHFNSS in the representation form.

3. The matrix of the representation form of the corresponding GHFNSS over U is

$$D = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2q} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{ij} & \cdots & b_{iq} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pq} \end{pmatrix} = [b_{ij}],$$

   where $b_{ij} = (\frac{r_{ij}}{m_{ij}}, \mu(a_j))$, with r_{ij} is the grade, $m_{ij} = \{\lambda_{ij}^1, \lambda_{ij}^2, \dots, \lambda_{ij}^{k_{ij}}\}$ is the set of membership values of u_i with respect to the parameter a_j, and $\mu(a_j)$ is the degree of preference of the parameter a_j.

4. Transform the matrix $D = [b_{ij}]$ to be the matrix $D' = [b'_{ij}]$ where $b'_{ij} = (\frac{r_{ij}}{m'_{ij}}, \mu(a_j))$, with $m'_{ij} = \frac{1}{k_{ij}} \sum_{l=1}^{k_{ij}} \lambda_{ij}^l$, $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$.

5. Transform matrix D' to be normalized decision matrix $V = [(\frac{V_{ij}}{v_{ij}}, \sigma_j)]$ by using

$$V_{ij} = \frac{r_{ij}}{\sqrt{\sum_{i=1}^{p} {r_{ij}}^2}}, \; v_{ij} = \frac{m'_{ij}}{\sqrt{\sum_{i=1}^{p} {m'_{ij}}^2}} \text{ and } \sigma_j = \frac{\mu(e_j)}{\sum_{j=1}^{q} \mu(e_j)}.$$

6. Define matrix $W = [\frac{W_{ij}}{w_{ij}}]$ by $W_{ij} = V_{ij}\sigma_j$ and $w_{ij} = v_{ij}\sigma_j$.

7. Find the positive ideal solution D^+ and the negative ideal solution D^- defined by

$$\begin{aligned} D^+ =& \{(\frac{\max_i(W_{ij})}{\max_i(w_{ij})} \mid j \in J), (\frac{\min_i(W_{ij})}{\min_i(w_{ij})} \mid j \in J')\} \\ =& \{\frac{{W_j}^+}{{w_j}^+} \mid j = 1, 2, \dots, q\} \\ D^- =& \{(\frac{\min_i(W_{ij})}{\min_i(w_{ij})} \mid j \in J), (\frac{\max_i(W_{ij})}{\max_i(w_{ij})} \mid j \in J')\} \\ =& \{\frac{{W_j}^-}{{w_j}^-} \mid j = 1, 2, \dots, q\}, \end{aligned}$$

   where $J = \{j \mid j \text{ is a supporting parameter}\}$, $J' = \{j \mid j \text{ is not a supporting parameter}\}$, and $\mid J \mid + \mid J' \mid = q$.

8. Calculate separation measures $({S_i}^+, {s_i}^+)$ and $({S_i}^-, {s_i}^-)$

$$(12)\quad \begin{aligned} ({S_i}^+, {s_i}^+) &= \left(\sqrt{\sum_{j=1}^{q}(W_{ij} - {W_j}^+)^2}, \sqrt{\sum_{j=1}^{q}(w_{ij} - {w_j}^+)^2}\right), \\ & i = 1, 2, \ldots, p \\ ({S_i}^-, {s_i}^-) &= \left(\sqrt{\sum_{j=1}^{q}(W_{ij} - {W_j}^-)^2}, \sqrt{\sum_{j=1}^{q}(w_{ij} - {w_j}^-)^2}\right), \\ & i = 1, 2, \ldots, p. \end{aligned}$$

9. Calculate relative adjacency to ideal solution

$$(13)\quad \begin{aligned} &(C_i, c_i) = \left(\frac{{S_i}^-}{{S_i}^+ + {S_i}^-}, \frac{{s_i}^-}{{s_i}^+ + {s_i}^-}\right) \\ &0 < C_i < 1, 0 < c_i < 1, i = 1, 2, \ldots, p. \end{aligned}$$

10. Form matrix $E = [E_i]$ with $E_i = \frac{C_i + c_i}{2}$, $i = 1, 2, \ldots, p$.

11. The best choice is an object u_t such that $E_t \geq E_j$ for all $j \neq t$.

Algorithm 2.

1. Repeat steps 1-3 of Algorithm 1.

2. Using matrix D in Algorithm 1, determine positive ideal solution B^+ and negative ideal solution B^-

$$\begin{aligned} B^+ =& \{({r_j}^+, \{({\lambda_j}^1)^+, ({\lambda_j}^2)^+, \ldots, ({\lambda_j}^k)^+\})\ j = 1, 2, \ldots, q\}, \\ B^- =& \{({r_j}^-, \{({\lambda_j}^1)^-, ({\lambda_j}^2)^-, \ldots, ({\lambda_j}^k)^-\})\ j = 1, 2, \ldots, q\} \end{aligned}$$

where,

$${r_j}^+ = \max_i(r_{ij}),\ \ {r_j}^- = \min_i(r_{ij}),\ \ \lambda_{ij}^1 \leq \lambda_{ij}^2 \leq \cdots \leq \lambda_{ij}^k,$$

and for each i, j

$$\begin{aligned} &({\lambda_j}^1)^+ = \max_i({\lambda_{ij}}^1),\ ({\lambda_j}^1)^- = \min_i({\lambda_{ij}}^1), \\ &({\lambda_j}^2)^+ = \max_i({\lambda_{ij}}^2),\ ({\lambda_j}^2)^- = \min_i({\lambda_{ij}}^2), \\ &\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ &({\lambda_j}^k)^+ = \max_i({\lambda_{ij}}^k),\ ({\lambda_j}^k)^- = \min_i({\lambda_{ij}}^k). \end{aligned}$$

3. Calculate separation measures ${S_i}^+$ and ${S_i}^-$,

$${S_i}^+ = ({R_i}^+, {M_i}^+), \quad i = 1, 2, \ldots, p$$

where

$${R_i}^+ = \sum_{j=1}^{q} \sigma_j \mid r_{ij} - r_j^+ \mid, \;\; {M_i}^+ = \sum_{j=1}^{q} \sigma_j \sqrt{\frac{1}{k} \sum_{l=1}^{k} \mid {\lambda_{ij}}^l - ({\lambda_j}^l)^+ \mid^2}, \text{ and}$$

$${S_i}^- = ({R_i}^-, {M_i}^-), \quad j = 1, 2, \ldots, p$$

where

$${R_i}^- = \sum_{j=1}^{q} \sigma_j \mid r_{ij} - r_j^- \mid, \;\; {M_i}^- = \sum_{j=1}^{q} \sigma_j \sqrt{\frac{1}{k} \sum_{l=1}^{k} \mid {\lambda_{ij}}^l - ({\lambda_j}^l)^- \mid^2}, \text{ and}$$

$$\sigma_j = \frac{\mu(e_j)}{\sum_{j=1}^{q} \mu(e_j)}.$$

4. Calculate relative adjacency to ideal solution

$$(C_i, c_i) = \left(\frac{{R_i}^-}{{R_i}^+ + {R_i}^-}, \frac{{M_i}^-}{{M_i}^+ + {M_i}^-} \right),$$
$$0 < C_i < 1,\ 0 < c_i < 1, i = 1, 2, \ldots, p.$$

5. Form matrix $E = [E_i]$ with $E_i = \frac{C_i + c_i}{2}$, $i = 1, 2, \ldots, p$.

6. The best choice is an object u_t such that $E_t \geq E_j$, for all $j \neq t$..

Example 5.1. An Educational institution assesses several universities in order to choose the best university. Let $U = \{u_1, u_2, u_3, u_4\}$ be a set of universities and $A = \{e_1, e_2, e_3, e_4, e_5\}$ is the set of assessment criteria, namely, e_1=Teacher Credibility, e_2=facility, e_3=accreditation, e_4=research and $e_5 = alumni$. The assessment was carried out by two trusted teams and provided an assessment of the university in terms of 5 parameters. The assessment is expressed in the form of membership values. On the other hand, the assessment is also carried out by members of the university who concern about conditions in the university and the assessment is expressed in the form of grades. On the other hand, the Educational institution assumes that degree of important of parameters are 0.8, 0.6, 0.7, 0.7, and 0.6 for e_1, e_2, e_3, e_4 and e_5 respectively. The evaluation results by evaluators is given in Table 10.

We use Algorithm 1 to determine the best university by the following steps.
1. Input the evaluation result in matrix D below (or see Table 10).

$$D = \begin{pmatrix} \left(\frac{3}{\{0.7,0.8\}}, 0.8\right) & \left(\frac{3}{\{0.7,0.75\}}, 0.6\right) & \left(\frac{2}{\{0.6,0.7\}}, 0.7\right) & \left(\frac{2}{\{0.55,0.65\}}, 0.7\right) & \left(\frac{3}{\{0.7,0.75\}}, 0.6\right) \\ \left(\frac{2}{\{0.65,0.75\}}, 0.8\right) & \left(\frac{2}{\{0.6,0.75\}}, 0.6\right) & \left(\frac{2}{\{0.6,0.7\}}, 0.7\right) & \left(\frac{3}{\{0.75,0.8\}}, 0.7\right) & \left(\frac{2}{\{0.6,0.75\}}, 0.6\right) \\ \left(\frac{2}{\{0.65,0.75\}}, 0.8\right) & \left(\frac{1}{\{0.55,0.65\}}, 0.6\right) & \left(\frac{2}{\{0.65,0.8\}}, 0.7\right) & \left(\frac{3}{\{0.7,0.75\}}, 0.7\right) & \left(\frac{3}{\{0.7,0.85\}}, 0.6\right) \\ \left(\frac{3}{\{0.65,0.75\}}, 0.8\right) & \left(\frac{2}{\{0.6,0.7\}}, 0.6\right) & \left(\frac{1}{\{0.55,0.7\}}, 0.7\right) & \left(\frac{2}{\{0.6,0.75\}}, 0.7\right) & \left(\frac{3}{\{0.7,0.85\}}, 0.6\right) \end{pmatrix}.$$

Table 10: Assessment data from several universities

$U_i \backslash e_j$	e_1	e_2	e_3	e_4	e_5
u_1	$\left(\frac{3}{\{0.7,0.8\}}, 0.8\right)$	$\left(\frac{3}{\{0.7,0.75\}}, 0.6\right)$	$\left(\frac{2}{\{0.6,0.7\}}, 0.7\right)$	$\left(\frac{2}{\{0.55,0.65\}}, 0.7\right)$	$\left(\frac{3}{\{0.7,0.75\}}, 0.6\right)$
u_2	$\left(\frac{2}{\{0.65,0.75\}}, 0.8\right)$	$\left(\frac{2}{\{0.6,0.75\}}, 0.6\right)$	$\left(\frac{2}{\{0.6,0.7\}}, 0.7\right)$	$\left(\frac{3}{\{0.75,0.8\}}, 0.7\right)$	$\left(\frac{2}{\{0.6,0.75\}}, 0.6\right)$
u_3	$\left(\frac{2}{\{0.65,0.75\}}, 0.8\right)$	$\left(\frac{1}{\{0.55,0.65\}}, 0.6\right)$	$\left(\frac{2}{\{0.65,0.8\}}, 0.7\right)$	$\left(\frac{3}{\{0.7,0.75\}}, 0.7\right)$	$\left(\frac{3}{\{0.7,0.85\}}, 0.6\right)$
u_4	$\left(\frac{3}{\{0.65,0.75\}}, 0.8\right)$	$\left(\frac{2}{\{0.6,0.7\}}, 0.6\right)$	$\left(\frac{1}{\{0.55,0.7\}}, 0.7\right)$	$\left(\frac{2}{\{0.6,0.75\}}, 0.7\right)$	$\left(\frac{3}{\{0.7,0.85\}}, 0.6\right)$

2. Transform the matrix D to be the matrix D'

$$D^{'} = \begin{pmatrix} (\frac{3}{0.75}, 0.8) & (\frac{3}{0.725}, 0.6) & (\frac{2}{0.65}, 0.7) & (\frac{2}{0.6}, 0.7) & (\frac{3}{0.725}, 0.6) \\ (\frac{2}{0.7}, 0.8) & (\frac{2}{0.675}, 0.6) & (\frac{2}{0.65}, 0.7) & (\frac{3}{0.775}, 0.7) & (\frac{2}{0.675}, 0.6) \\ (\frac{2}{0.7}, 0.8) & (\frac{1}{0.6}, 0.6) & (\frac{2}{0.725}, 0.7) & (\frac{3}{0.725}, 0.7) & (\frac{3}{0.775}, 0.6) \\ (\frac{3}{0.7}, 0.8) & (\frac{2}{0.65}, 0.6) & (\frac{1}{0.625}, 0.7) & (\frac{2}{0.625}, 0.7) & (\frac{3}{0.775}, 0.6) \end{pmatrix}.$$

3. Transform matrix D' to be normalized decision matrix V.

$$V = \begin{pmatrix} (\frac{0.5882}{0.5245}, 0.24) & (\frac{0.7143}{0.5451}, 0.18) & (\frac{0.5556}{0.4887}, 0.2) & (\frac{0.3922}{0.4380}, 0.2) & (\frac{0.5357}{0.4899}, 0.18) \\ (\frac{0.3922}{0.4895}, 0.24) & (\frac{0.4762}{0.5075}, 0.18) & (\frac{0.5556}{0.4887}, 0.2) & (\frac{0.5882}{0.5657}, 0.2) & (\frac{0.3571}{0.4561}, 0.18) \\ (\frac{0.3922}{0.4895}, 0.24) & (\frac{0.2381}{0.4511}, 0.18) & (\frac{0.5556}{0.5451}, 0.2) & (\frac{0.5882}{0.5292}, 0.2) & (\frac{0.5357}{0.5237}, 0.18) \\ (\frac{0.5882}{0.4895}, 0.24) & (\frac{0.4762}{0.4887}, 0.18) & (\frac{0.2778}{0.4699}, 0.2) & (\frac{0.3922}{0.4562}, 0.2) & (\frac{0.5357}{0.5237}, 0.18) \end{pmatrix}.$$

4. Calculate matrix W

$$W = \begin{pmatrix} (\frac{0.1412}{0.1259}) & (\frac{0.1286}{0.0981}) & (\frac{0.1111}{0.0977}) & (\frac{0.0784}{0.0876}) & (\frac{0.0964}{0.0882}) \\ (\frac{0.0941}{0.1175}) & (\frac{0.0857}{0.0914}) & (\frac{0.1111}{0.0977}) & (\frac{0.1176}{0.1131}) & (\frac{0.0643}{0.0821}) \\ (\frac{0.0941}{0.1175}) & (\frac{0.0429}{0.0812}) & (\frac{0.1111}{0.1090}) & (\frac{0.1176}{0.1058}) & (\frac{0.0964}{0.0943}) \\ (\frac{0.1412}{0.1175}) & (\frac{0.0857}{0.0880}) & (\frac{0.0556}{0.0940}) & (\frac{0.0784}{0.0912}) & (\frac{0.0964}{0.0943}) \end{pmatrix}.$$

5. Find the positive ideal solution D^+ and the negative ideal solution D^-

$$D^+ = ((\tfrac{0.1412}{0.1259}) \quad (\tfrac{0.1286}{0.0981}) \quad (\tfrac{0.1111}{0.1090}) \quad (\tfrac{0.1176}{0.1131}) \quad (\tfrac{0.0964}{0.0943}))$$
$$D^- = ((\tfrac{0.0941}{0.1175}) \quad (\tfrac{0.0429}{0.0812}) \quad (\tfrac{0.0556}{0.0940}) \quad (\tfrac{0.0784}{0.0876}) \quad (\tfrac{0.0643}{0.0821})).$$

Here, we assume that all parameters are supporting ones.

6. Calculate separation measures

$$\begin{aligned} ({S_1}^+, {s_1}^+) =& (0.0387, 0.0030), ({S_2}^+, {s_2}^+) = (0.0707, 0.0224), \\ ({S_3}^+, {s_3}^+) =& (0.0975, 0.0224), ({S_4}^+, {s_4}^+) = (0.0800, 0.0300), \\ ({S_1}^-, {s_1}^-) =& (0.1131, 0.0220), ({S_2}^-, {s_2}^-) = (0.0800, 0.0283), \\ ({S_3}^-, {s_3}^-) =& (0.0748, 0.0265), ({S_4}^-, {s_4}^-) = (0.0640, 0.0173). \end{aligned}$$

7. Calculate relative adjacency to ideal solution

$$\begin{aligned} (C_1, c_1) =& (0.7451, 0.4231), (C_2, c_2) = (0.5309, 0.5582), \\ (C_3, c_3) =& (0.4341, 0.5419), (C_4, c_4) = (0.4444, 0.3658). \end{aligned}$$

8. Find E_i, $E_1 = 0.5841, E_2 = 0.5445, E_3 = 0.4880, E_4 = 0.4051$. We obtain $E_4 < E_3 < E_2 < E_1$.

9. The order of universities from the best is u_1, u_2, u_3 and u_4.

If we apply Algorithm 2, for Example 5.1, we will get separation measures as follows (see Table 11).

Table 11: Separation measures

U_i	R_i^+	M_i^+	R_i^-	M_i^-	E_i
u_1	0.2	0.012	0.98	0.011	0.64
u_2	0.6	0.011	0.58	0.011	0.49
u_3	0.6	0.009	0.58	0.012	0.53
u_4	0.4	0.014	0.78	0.008	0.51

Based on Table 11, we obtain that the ranking order of E_i is $E_1 > E_3 > E_4 > E_2$. Hence the best university is u_1.

We see that when the problem in Example 5.1 was solved by the two algorithms above, we obtained a different conclusion. This clearly can happen because the two algorithms use different approaches, especially in using membership values in the calculation and formulation of the Separation Measures.

6. Conclusion

In this article, we proposed the concept of Generalized Hesitant Fuzzy N-Soft sets (GHFNSSs) and defined some of their complements and operations, such as restricted and extended intersections and restricted and extended unions of two GHFNSSs. Based on the operations, we prove some properties, such as associative and distributive laws. Lastly, we propose two algorithms for decision-making problems by extending the TOPSIS method to apply under GHFNSS information. Since the GHFNSS is a generalization of Generalized Hesitant Fuzzy Soft sets, there are many further studies for scholars on the issue of studying NSSs, such as a generalization of Hesitant Intuitionistic Fuzzy Soft Sets and Interval-valued Hesitant Intuitionistic Fuzzy Soft Sets.

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