# Recognition of decomposable posets by using the poset matrix

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**Abstract.** We introduce the notion of a composition of square matrices. We recall the notion of poset matrix, a square $(0,1)$-matrix, to represent posets. We show that this composition of poset matrices gives generalizations of the ordinal product as well as the direct sum and ordinal sum of poset matrices. We give an interpretation of the composition of poset matrices in posets. We show that the composition of poset matrices is also a poset matrix, and it represents a decomposable poset. This result gives, consequently, a matrix recognition of the decomposable posets.

**Keywords:** decomposable poset, composite poset, matrix recognition, poset matrix, composition, ordinal product.

## 1. Introduction

To maximize efficiency, methods for solving many optimization problems on the structure theory begin with some decomposition techniques. These techniques are used to reduce a bigger structure into smaller ones of the same kind, like posets into autonomous sets [3], graphs into clumps [1], comparability graphs into stable sets [10], schedules into job-modules [4], and networks into simplifiable subnetworks [9]. As a result, due to the computational tractability property of the decomposable posets, various methods for the recognition of this type of posets are considered by numerous authors. Khamis [3] recalled the notion of

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composition of posets and described an algorithmic method for the recognition of prime (indecomposable) posets. In this article, we give a matrix recognition of the decomposable posets by using the poset matrix, an incidence matrix introduced by Mohammad and Talukder [6] to represent posets.

Since the incidence matrices have many computational aspects, these are chosen repeatedly in recognizing different classes of posets [6, 11] and graphs [5, 12]. As a result, special operations on incidence matrices, due to the classical applications in the adjacent fields, are considered in the literature [5, 7, 8]. In this paper, we introduce the notion of a composition of square matrices and give an interpretation of this composition of poset matrices in posets. Tucker [12] recognized the circular-arc graphs and proper circular-arc graphs by using the properties of perfect 0s, circular 1s, and circularly compatible 1s defined on an augmented adjacency matrix. These results give us the idea of defining the property of transitive blocks of 1s on a block poset matrix and giving a matrix recognition of the decomposable posets.

In Section 2, we recall some basic terminologies related to the ordinal product and composition of posets. We also recall the common operations in the poset matrices and their interpretations in posets. In Section 3, we define the aforesaid composition of square matrices. Here, we mainly show that the composition of poset matrices is also a poset matrix, and it represents a decomposable poset. We also show that this composition of poset matrices generalizes the ordinal product of poset matrices, and every composite poset is decomposable. In Section 4, we define the property of transitive blocks of 1s in a block poset matrix and give a matrix recognition of the decomposable posets.

## 2. Preliminaries

A *poset* (*partially ordered set*) is a structure $\mathbf{A} = \langle A, \leqslant \rangle$ consisting of the nonempty set $A$ with the order relation $\leqslant$ on $A$, that is, the relation $\leqslant$ is reflexive, antisymmetric, and transitive on $A$. A poset $\mathbf{A}$ is called *finite* if the underlying set $A$ is finite. Here, we assume that every poset is finite. Let $\mathbf{A} = \langle A, \leqslant_A \rangle$ and $\mathbf{B} = \langle B, \leqslant_B \rangle$ be two posets. A bijective map $\phi : A \to B$ is called an *order isomorphism* if for all $x, y \in A$, we have $x \leqslant_A y$ if and only if $\phi(x) \leqslant_B \phi(y)$. We write $\mathbf{A} \cong \mathbf{B}$ whenever $\mathbf{A}$ and $\mathbf{B}$ are order isomorphic. For further essentials of posets, readers are referred to the classical book by Davey and Priestley [2].

We use the notation $\mathbf{1}$ for the singleton poset, $\mathbf{C}_n (n \geq 1)$ for the $n$-element chain posets, $\mathbf{I}_n (n \geq 1)$ for the $n$-element antichain posets, $\mathbf{D}_n (n \geq 4)$ for the $n$-element diamond posets, $\mathbf{Z}_n (n \geq 4)$ for the $n$-element zigzag posets, and $\mathbf{B}_{m,n} (m \geq 1, n \geq 1)$ for the complete bipartite posets with $m$ minimal elements and $n$ maximal elements.

We also use the notation $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} \oplus \mathbf{B}$ to denote the direct sum and ordinal sum, respectively, of the posets $\mathbf{A}$ and $\mathbf{B}$. For any poset $\mathbf{A}$, we write shortly $n\mathbf{A}$ for $\mathbf{A} + \mathbf{A} + \cdots + \mathbf{A}$ and $\oplus^n \mathbf{A}$ for $\mathbf{A} \oplus \mathbf{A} \oplus \cdots \oplus \mathbf{A}$. In general, for any posets $\mathbf{B}_i, 1 \leq i \leq n$, we write shortly $\sum_{i=1}^{n} \mathbf{B}_i$ for $\mathbf{B}_1 + \mathbf{B}_2 + \cdots + \mathbf{B}_n$ and $\bigoplus_{i=1}^{n} \mathbf{B}_i$ for $\mathbf{B}_1 \oplus \mathbf{B}_2 \oplus \cdots \oplus \mathbf{B}_n$.

A poset $\mathbf{G}$ is called a *$P$-graph* if there exist the singleton or antichain posets $\mathbf{A}_i, 1 \leq i \leq n$ such that $\mathbf{G} \cong \bigoplus_{i=1}^{n} \mathbf{A}_i$. A poset $\mathbf{S}$ is called a *$P$-series* if there exist the $P$-graphs $\mathbf{G}_i, 1 \leq i \leq n$ such that $\mathbf{S} \cong \sum_{i=1}^{n} \mathbf{G}_i$. Every $P$-graph is trivially a $P$-series. A poset is called *series-parallel* if it can be expressed as the sum of the singleton posets using only the direct sum and ordinal sum. Every $P$-series, as well as every $P$-graph, is trivially series-parallel.

The *ordinal product* of the posets $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \otimes \mathbf{B}$, is defined as the poset $\langle A \times B, \leqslant_{\otimes} \rangle$ such that for all $(x, y), (x', y') \in A \times B$, we have $(x, y) \leqslant_{\otimes} (x', y')$ if either (i) $x \leqslant_A x'$ or (ii) $x = x'$ and $y \leqslant_B y'$. Here, the posets $\mathbf{A}$ and $\mathbf{B}$ are called the *ordinal factors* of $\mathbf{A} \otimes \mathbf{B}$. In Figure 1, the ordinal product $\mathbf{B}_{1,2} \otimes \mathbf{B}_{2,1}$ along with the direct sum $\mathbf{B}_{1,2} + \mathbf{B}_{2,1}$ and the ordinal sum $\mathbf{B}_{1,2} \oplus \mathbf{B}_{2,1}$ are shown by using the Hasse diagrams. In general, $\mathbf{A} \otimes \mathbf{B} \ncong \mathbf{B} \otimes \mathbf{A}$.

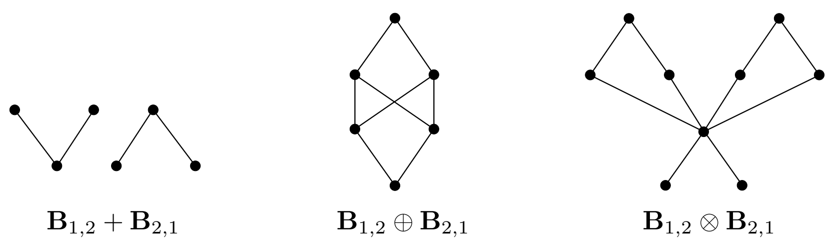


Figure 1: Hasse diagrams of $\mathbf{B}_{1,2} + \mathbf{B}_{2,1}$, $\mathbf{B}_{1,2} \oplus \mathbf{B}_{2,1}$, and $\mathbf{B}_{1,2} \otimes \mathbf{B}_{2,1}$.

A poset $\mathbf{C}$ is said to be *composite* if and only if their exist nonsingleton posets $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{C} \cong \mathbf{A} \otimes \mathbf{B}$. For example, the poset $\mathbf{B}_{1,2} \otimes \mathbf{B}_{2,1}$ (Figure 1) is composite. Also, for any poset $\mathbf{B}$, the poset $n\mathbf{B}$ and $\oplus^n \mathbf{B}$ are composite, because $n\mathbf{B} \cong \mathbf{I}_n \otimes \mathbf{B}$ and $\oplus^n \mathbf{B} \cong \mathbf{C}_n \otimes \mathbf{B}$. A proof by using the poset matrix of the result relating the ordinal sum was given by Mohammad and Talukder [7].

We now recall the definition of the composition of posets. Let $\mathbf{A} = \langle A, \leqslant_A \rangle$ with $A = \{x_1, x_2, ..., x_m\}$ and $\mathbf{B}_r = \langle B_r, \leqslant_{B_r} \rangle$, $1 \leq r \leq m$ with $B_r = \{y_{t+i} : 1 \leq i \leq n_r\}$ where $t = \sum_{k=1}^{r-1} n_k$, be posets on the disjoint sets $A$ and $B_r$, $1 \leq r \leq m$. Then the *composition* of the posets $\mathbf{A}$ and $\mathbf{B}_r$, $1 \leq r \leq m$, denoted by $\mathbf{A}\ [\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_m]$, is defined as the poset $\langle \bigcup_{k=1}^{m} B_k, \leqslant_c \rangle$ such that for all $y_i, y_j \in \bigcup_{r=1}^{m} B_r$, we have $y_i \leqslant_c y_j$ if and only if one of the following conditions is satisfied.

1. $y_{t+i'}, y_{l+j'} \in B_r$ for some $r$ (when $t = l = \sum_{k=1}^{r-1} n_k$, $i' = i - t$ and $j' = j - l$) and $y_{t+i'} \leqslant_{B_r} y_{l+j'}$,

2. $y_{t+i'} \in B_r$ and $y_{l+j'} \in B_s$ for some $r < s$ (when $\sum_{k=1}^{r-1} n_k = t < l = \sum_{k=1}^{s-1} n_k$, $i' = i - t$ and $j' = j - l$) and $x_r \leqslant_A x_s$.

Here, $\mathbf{A}$ is called the *outer* poset or *quotient* poset, and $\mathbf{B}_r, 1 \leq r \leq m$ are called the *inner* posets and their ground sets are called *autonomous* sets. An example of the composition of posets is shown in Figure 2 by using the Hasse diagrams. Obviously, for any posets $\mathbf{B}_i, 1 \leq i \leq n$, we have $\sum_{i=1}^{n} \mathbf{B}_i \cong \mathbf{I}_n[\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_n]$ and $\bigoplus_{i=1}^{n} \mathbf{B}_i \cong \mathbf{C}_n[\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_n]$. In particular, for any poset $\mathbf{A}$ with $|A| = n$, we have $\mathbf{A} \cong \mathbf{A}[\underbrace{\mathbf{1}, \mathbf{1}, ..., \mathbf{1}}_{n \text{ times}}]$.

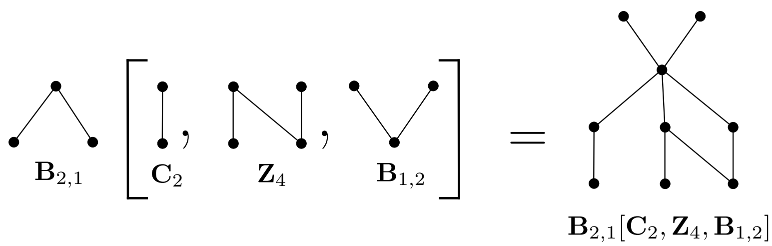


Figure 2: Hasse diagrams giving the composition $\mathbf{B}_{2,1}[\mathbf{C}_2, \mathbf{Z}_4, \mathbf{B}_{1,2}]$.

A poset $\mathbf{D}$ is called *decomposable* if and only if $\mathbf{D}$ is isomorphic to some posets obtained as the composition of two or more inner posets where at least one inner poset is nonsingleton. Thus, a poset $\mathbf{D}$ is decomposable if and only if there exist the poset $\mathbf{A}$ and the posets $\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_n, n \geq 2$, where at least one $\mathbf{B}_i$ is nonsingleton, such that $\mathbf{D} \cong \mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_n]$. For example, the posets $\mathbf{D}_4$ and $\mathbf{Z}_4 \oplus \mathbf{1}$ are decomposable because $\mathbf{D}_4 \cong \mathbf{C}_2[\mathbf{1}, \mathbf{B}_{2,1}] \cong \mathbf{C}_2[\mathbf{B}_{1,2}, \mathbf{1}]$ $\cong \mathbf{C}_3[\mathbf{1}, \mathbf{I}_2, \mathbf{1}]$ and $\mathbf{Z}_4 \oplus \mathbf{1} \cong \mathbf{C}_2[\mathbf{Z}_4, \mathbf{1}]$. Here, we see that the posets $\mathbf{1}$, $\mathbf{I}_2$, and $\mathbf{C}_2$ are not decomposable. We assume that these posets are trivially decomposable. On the other hand, a poset is called *prime* or *indecomposable* if and only if it is not decomposable. For example, the poset $\mathbf{Z}_4$ is a prime poset with the least number of elements.

Note that for any nontrivial $P$-graph $\mathbf{G}$, we have $\mathbf{G} \cong \mathbf{C}_n[\mathbf{I}_{m_1}, \mathbf{I}_{m_2}, ..., \mathbf{I}_{m_n}]$ for some $m_i, 1 \leq i \leq n$. Also, for any nontrivial $P$-series $\mathbf{S}$, we have $\mathbf{S} \cong \mathbf{I}_n[\mathbf{G}_1, \mathbf{G}_2, ..., \mathbf{G}_n]$ for some $P$-graphs $\mathbf{G}_i, 1 \leq i \leq n$. These show that every $P$-series as well as every $P$-graph is decomposable. Similarly, we can show that every series-parallel poset is decomposable. Note also that, since $\mathbf{Z}_4$ is not a $P$-graph, $\mathbf{Z}_4 \oplus \mathbf{1}$ is not series-parallel. Thus, a decomposable poset may not be series-parallel. However, we will show by using the poset matrix that every composite poset is decomposable (Corollary 3.2).

Mohammad and Talukder [6] introduced the notion of poset matrix, where they gave matrix recognitions of some subclasses of series-parallel posets. A square $(0, 1)$-matrix $M = [a_{ij}], 1 \leq i, j \leq m$ is called a *poset matrix* if and only if the following conditions hold.

1. $a_{ii} = 1$ for all $1 \leq i \leq m$ i.e. $M$ is reflexive,

2. $a_{ij} = 1$ and $a_{ji} = 1$ imply $i = j$ i.e. $M$ is antisymmetric,

3. $a_{ij} = 1$ and $a_{jk} = 1$ imply $a_{ik} = 1$ i.e. $M$ is transitive.

Both the matrices $M$ and $M'$ in the following example are poset matrices

**Example 2.1.**

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad M' = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Throughout this paper, we use the notation $M_{m,n}$ for an $m$-by-$n$ matrix and $M_m$ for a square matrix of order $m$. In particular, we use the notation $I_n$, $O_n$, and $Z_n$ for the $n$-th order identity matrix, the matrix with all entries 1s, and the matrix with all entries 0s, respectively. We also use the notation $C_n$ for the matrix $[c_{ij}], 1 \leq i, j \leq n$ defined as $c_{ij} = 1$ for all $i \leq j$ and $c_{ij} = 0$ otherwise. For every $n \geq 1$, both the matrices $I_n$ and $C_n$ are trivially poset matrices.

To each poset matrix $M_m = [a_{ij}]$, $1 \leq i, j \leq m$, a poset $\mathbf{A} = \langle A, \leqslant \rangle$, where $A = \{x_1, x_2, \ldots, x_m\}$ and $x_i$ corresponds the $i$-th row (or column) of $M_m$, is associated by defining the order relation $\leqslant$ on $A$ such that for all $1 \leq i, j \leq m$, we have $x_i \leqslant x_j$ if and only if $a_{ij} = 1$. Then it is said that the poset matrix $M_m$ *represents* the poset $\mathbf{A}$ and vice versa. For example, the poset matrix $I_n$ represents the poset $\mathbf{I}_n$ and the poset matrix $C_n$ represents the poset $\mathbf{C}_n$. Also, the poset matrices $M$ and $M'$, as given in Example 2.1, represent the posets $\mathbf{D}_4$ and $\mathbf{Z}_4$, respectively.

Let $M_m$ be a poset matrix. Then for some $1 \leq i, j \leq m$, interchanges of $i$-th and $j$-th rows along with the interchanges of $i$-th and $j$-th columns in $M_m$ is called the *(i,j)-relabeling* of $M_m$. The following results are obtained by Mohammad and Talukder [6] where the authors gave some interpretations of the relabeling of poset matrices in posets.

**Theorem 2.1.** *Any relabeling of a poset matrix is a poset matrix, and it represents the same poset up to isomorphism.*

**Theorem 2.2.** *Every poset matrix can be relabeled to an upper (or lower) triangular matrix with 1s in the main diagonal by a finite number of relabeling.*

From now on, by a poset matrix we mean a poset matrix in the upper triangular form.

## 3. Composition of poset matrices

In this section, we give the construction of the composition of square matrices. We show that the composition of poset matrices generalizes the ordinal product of poset matrices. We also show that the composition of poset matrices represents a decomposable poset.

**Definition 3.1.** *The* composition *of the square matrices* $M_m = [a_{ij}], 1 \le i, j \le m$ *and* $N_{n_r}, 1 \le r \le m$, *denoted by* $M_m[N_{n_1}, N_{n_2}, ..., N_{n_m}]$, *is a block matrix defined as follows:*

$$M_m[N_{n_1}, ..., N_{n_m}] = \begin{bmatrix} a_{11}N_{n_1} & a_{12}O_{n_1,n_2} & \cdots & a_{1m}O_{n_1,n_m} \\ a_{21}O_{n_2,n_1} & a_{22}N_{n_2} & \cdots & a_{2m}O_{n_2,n_m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}O_{n_m,n_1} & a_{m1}O_{n_m,n_2} & \cdots & a_{mm}N_{n_m} \end{bmatrix}.$$

Let $M_m = [a_{ij}], 1 \le i, j \le m$ and $N_{n_r}, 1 \le r \le m$ be poset matrices. Since $M_m$ is a (0,1)-matrix, the $(i,j)$-th block $Q_{ij}$ of the block matrix $M_m[N_{n_1}, N_{n_2}, ..., N_{n_m}] = [Q_{ij}], 1 \le i, j \le m$ can be expressed as follows:

$$Q_{ij} = \begin{cases} N_{n_i}, & \text{if } i = j, \\ O_{n_i,n_j}, & \text{if } i < j \text{ and } a_{ij} = 1, \\ Z_{n_i,n_j}, & \text{if } i < j \text{ and } a_{ij} = 0, \\ O_{n_j,n_i}, & \text{if } i > j \text{ and } a_{ij} = 1, \\ Z_{n_j,n_i}, & \text{if } i > j \text{ and } a_{ij} = 0. \end{cases} \tag{1}$$

**Example 3.1.**

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \left[ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]$$

$$= \left[ \begin{array}{cc|cccc|ccc} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

In the above example, we give the composition $B[C_2, M', B']$ of the poset matrices $B$, $C_2$, $M'$ (Example 2.1), and $B'$, where the matrices $B$ and $B'$ represent the posets $\mathbf{B}_{2,1}$ and $\mathbf{B}_{1,2}$, respectively.

Mohammad and Talukder [7] introduced the notion of the ordinal product of matrices. The ordinal product $M_m \boxtimes N_n$ of the poset matrices $M_m = [a_{ij}]$, $1 \le i, j \le m$ and $N_n$ is a block matrix where the $(i,j)$-th block $P_{ij}$ of the matrix

$M_m \boxtimes N_n = [P_{ij}]$, $1 \leq i, j \leq m$ is expressed as follows:

$$P_{ij} = \begin{cases} N_n, & \text{if } i = j, \\ O_n, & \text{if } i \neq j \text{ and } a_{ij} = 1, \\ Z_n, & \text{otherwise.} \end{cases} \tag{2}$$

The authors [7] then gave an interpretation of the ordinal product of poset matrices in posets as follows:

**Theorem 3.1.** *Let $M_m$ represent the poset* $\mathbf{A}$ *and $N_n$ represent the poset* $\mathbf{B}$. *Then the matrix $M_m \boxtimes N_n$ is a poset matrix and it represents the poset* $\mathbf{A} \otimes \mathbf{B}$.

**Corollary 3.1.** *Let* $\mathbf{B}$ *be any poset. Then* $\mathbf{C}_n \otimes \mathbf{B} \cong \oplus^n \mathbf{B}$.

The result in Corollary 3.1 was proved by using the fact that the ordinal product of poset matrices gives a generalization of the ordinal sum of poset matrices. Below, we show that the composition of poset matrices generalizes the ordinal product of poset matrices.

**Lemma 3.1.** *Let $M_m$ and $N_n$ be poset matrices. Then*

$$M_m[\underbrace{N_n, N_n, ..., N_n}_{m \text{ times}}] = M_m \boxtimes N_n. \tag{3}$$

**Proof.** Substitute $n_i = n, 1 \leq i \leq m$ in the expression for $Q_{ij}$ in equation (1). Then $(i, j)$-th block $Q_{ij}$ of $M_m[N_{n_1}, N_{n_2}, ..., N_{n_m}] = [Q_{ij}], 1 \leq i, j \leq m$ takes the following form.

$$Q_{ij} = \begin{cases} N_n, & \text{if } i = j, \\ O_{n,n}, & \text{if } i < j \text{ and } a_{ij} = 1, \\ Z_{n,n}, & \text{if } i < j \text{ and } a_{ij} = 0, \\ O_{n,n}, & \text{if } i > j \text{ and } a_{ij} = 1, \\ Z_{n,n}, & \text{if } i > j \text{ and } a_{ij} = 0. \end{cases}$$

This implies

$$Q_{ij} = \begin{cases} N_n, & \text{if } i = j, \\ O_n, & \text{if } i \neq j \text{ and } a_{ij} = 1, \\ Z_n, & \text{otherwise} \end{cases}$$

which equals the expression for $P_{ij}$ in equation (2). Thus, for all $1 \leq i, j \leq m$, the $(i, j)$-th block of the poset matrix $M_m[\underbrace{N_n, N_n, ..., N_n}_{m \text{ times}}]$ equals the $(i, j)$-th block of the poset matrix $M_m \boxtimes N_n$. Hence the equality in equation (3) holds. $\square$

The following result gives an interpretation of the composition of poset matrices in posets.

**Theorem 3.2.** *Let $M_m$ represent the poset* $\mathbf{A}$ *and $N_{n_i}$ represent the poset* $\mathbf{B}_i$*, $1 \leq i \leq m$. Then the matrix $M_m[N_{n_1}, N_{n_2}, ..., N_{n_m}]$ is a poset matrix and it represents the poset* $\mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_m]$.

**Proof.** Let $M_m = [a_{ij}], 1 \leq i, j \leq m$, $N_{n_r} = [b_{ij}], 1 \leq i, j \leq n_r$ and $1 \leq r \leq m$. Also let $M_m[N_{n_1}, N_{n_2}, ..., N_{n_m}] = Q_T = [q_{ij}], 1 \leq i, j \leq T$, where $T = \sum_{r=1}^{m} n_r$, with block representation $[Q_{ij}], 1 \leq i, j \leq m$. Since $M_m$ and $N_{n_r}, 1 \leq r \leq m$ are all upper triangular matrices with 1s in the main diagonal, $Q_{ij} = Z_{n_i,n_j}$ for all $i > j$. Thus $Q_T$ is upper triangular with elements 1s in the main diagonal and hence $Q_T$ is reflexive and antisymmetric. For transitivity of $Q_T$, let $q_{ij} = q_{jk} = 1$ for some $1 \leq i \leq j \leq k \leq T$. Then, we have the following cases:

1. $q_{ij}, q_{jk} \in Q_{rr} = N_{n_r}$ for some $1 \leq r \leq m$. Then there exist $b_{i'j'}$, $b_{j'k'}$, $b_{i'k'} \in N_{n_r}$ such that $b_{i'j'} = q_{ij} = 1$, $b_{j'k'} = q_{jk} = 1$ and $b_{i'k'} = q_{ik}$. Since $N_{n_r}$ is transitive, $q_{ik} = b_{i'k'} = 1$.

2. $q_{ij} \in Q_{rs} = O_{n_r,n_s}$ and $q_{jk} \in Q_{ss} = N_{n_s}$ for some $1 \leq r < s \leq m$. Then $q_{ik} \in Q_{rs} = O_{n_r,n_s}$ and clearly, $q_{ik} = 1$.

3. $q_{ij} \in Q_{rs} = O_{n_r,n_s}$ and $q_{jk} \in Q_{st} = O_{n_s,n_t}$ for some $1 \leq r < s < t \leq m$. Then $q_{ik} \in Q_{rt}$. Then by the definition of composition of poset matrices, $a_{rs}, a_{st} \in M_m$; and $a_{rs} = a_{st} = 1$. Since $M_m$ is transitive, $a_{rt} = 1$. Therefore, $Q_{rt} = O_{n_r,n_t}$ and clearly, $q_{ik} = 1$.

Thus, $Q_T$ is transitive and hence a poset matrix.

Now, we show that $Q_T$ represents the poset $\mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_m]$. Let $A = \{x_1, x_2, ..., x_m\}$ and $B_r = \{y_{t+i} : 1 \leq i \leq n_r\}$ where $t = \sum_{k=1}^{r-1} n_k$. Let $q_{ij} = 1$ in $Q_T$ for some $1 \leq i \leq j \leq T$. Then $q_{ij} \in Q_{rs}$ for some $1 \leq r \leq s \leq m$, and we have the following two cases.

1. $r = s$. Then $Q_{rs} = N_{n_r}$ and $b_{i'j'} = q_{ij} \in Q_{kl} = N_{n_r}$ for $t = \sum_{k=1}^{r-1} n_k$, $i' = i - t$ and $j' = j - t$. Since $b_{i'j'} = 1$ and $N_{n_r}$ represents $\mathbf{B}_r$, we have $y_{t+i'} \leqslant_{B_r} y_{t+j'}$. Then, by the definition of composition of posets, $y_i \leqslant_c y_j$.

2. $r < s$. Then $Q_{rs} = O_{n_r,n_s}$ for $\sum_{k=1}^{r-1} n_k = t < l = \sum_{k=1}^{s-1} n_k$. Then $y_{t+i'} \in B_r$ and $y_{l+j'} \in B_s$. Then by the definition of composition of poset matrices, $1 = a_{rs} \in M_m$. Sine $M_m$ represents $\mathbf{A}$, we have $x_r \leqslant_A x_s$. Then, by the definition of composition of posets, $y_i \leqslant_c y_j$.

For the converse, similarly, we show that $y_i \leqslant_c y_j$ implies $1 = q_{ij} \in Q_T$ for all $1 \leq i, j \leq T$. Hence the matrix $Q_T$ represents the poset $\mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_m]$. □

Below we prove the result that every composite poset is decomposable as an immediate corollary of Theorem 3.2.

**Corollary 3.2.** *Every composite poset is decomposable.*

**Proof.** Let $\mathbf{C}$ be any composite poset. Then there exist the nonsingleton posets $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{C} \cong \mathbf{A} \otimes \mathbf{B}$. Let $|A| = m$. To show that $\mathbf{C}$ is decomposable, we now show that the following isomorphism holds

$$\mathbf{A} \otimes \mathbf{B} \cong \mathbf{A}[\underbrace{\mathbf{B}, \mathbf{B}, \ldots, \mathbf{B}}_{m \text{ times}}]. \tag{4}$$

Let $M_m$ represent the poset $\mathbf{A}$ and $N_n$ represent the poset $\mathbf{B}$. Then, by Theorem 3.1, $M_m \boxtimes N_n$ is a poset matrix and it represents the poset $\mathbf{A} \otimes \mathbf{B}$, and by Theorem 3.2, $M_m[\underbrace{N_n, N_n, ..., N_n}_{m \text{ times}}]$ is a poset matrix and it represents the poset $\mathbf{A}[\underbrace{\mathbf{B}, \mathbf{B}, \ldots, \mathbf{B}}_{m \text{ times}}]$. Therefore, the isomorphism in equation (4) holds by the equality in equation (3), as established in Lemma 3.1. □

## 4. Recognition of decomposable posets

We now define the property of transitive blocks of 1s in a poset matrix.

**Definition 4.1.** *A poset matrix $Q$ is said to have the property of* transitive blocks of 1s *of length $\{m, \{n_1, n_2, \ldots, n_m\}\}$ if and only if there exists a block representation $Q = [M_{ij}], 1 \le i, j \le m$ such that for all $1 \le i, j, k \le m$, the following conditions hold:*

1. $M_{ii} = N_{n_i}$*, a poset matrix,*
2. $M_{ij} = Z_{n_i,n_j}$ *or* $O_{n_i,n_j}$ *for* $i < j$*; and* $M_{ij} = Z_{n_j,n_i}$ *for* $i > j$*,*
3. $M_{ij} = O_{n_i,n_j}$ *and* $M_{jk} = O_{n_j,n_k}$ *implies* $M_{ik} = O_{n_i,n_k}$*.*

Note that if $n_1 = n_2 = \cdots = n_m = n$ (say) then we write shortly $\{m, n\}$ for the length $\{m, \{n_1, n_2, \ldots, n_m\}\}$.

We see that although the poset matrix $N$ in the following example seems not to satisfy the property of the transitive blocks of 1s, the poset matrix $N''$ (Example 4.1), obtained by $(3, 4)$-relabeling of $N$ and then $(2, 3)$-relabeling of $N'$, satisfies the property of transitive blocks of 1s of length $\{3, \{2, 4, 3\}\}$.

**Example 4.1.**

$$N = \left[\begin{array}{cc|cccc|ccc}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
\hline
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]$$

$$\xrightarrow{\textit{(3,4)-relabeling}} \left[\begin{array}{cc|cccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = N'$$

$$\xrightarrow{\textit{(2,3)-relabeling}} \left[\begin{array}{cc|cccc|ccc} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = N''.$$

**Theorem 4.1.** *A matrix satisfies the property of transitive blocks of 1s if and only if it is obtained as the composition of some poset matrices.*

**Proof.** Let the matrix $Q$ be obtained as the composition of the poset matrices $M_m$ and $N_{n_i}, 1 \leq i \leq m$. Then, by the definition of the composition of poset matrices, we have $Q = M_m[N_{n_1}, N_{n_2}, ..., N_{n_m}]$, and by Theorem 3.2, $Q$ is a block poset matrix. This shows that $Q$ is upper triangular having the poset matrices $N_{n_i}, 1 \leq i \leq m$ as diagonal blocks satisfying the first two cases in Definition 4.1. Let $M_m = [a_{ij}]$ and $Q = [Q_{ij}], 1 \leq i, j \leq m$ with $Q_{ij} = O_{n_i,n_j}$ and $Q_{jk} = O_{n_j,n_k}$ for some $1 \leq i < j \leq m$. Then, again by the definition of the composition of poset matrices, we have $a_{ij} = a_{jk} = 1$. Since $M_m$ is transitive, $a_{ik} = 1$. Thus, $Q_{ik} = O_{n_i,n_k}$ which satisfies the last case in Definition 4.1. This shows that $Q$ satisfies the property of transitive blocks of 1s of length $\{m, \{n_1, n_2, \ldots, n_m\}\}$.

Conversely, we suppose that the matrix $Q$ satisfies the property of transitive blocks of 1s of length $\{m, \{n_1, n_2, \ldots, n_m\}\}$ and show similarly that $Q$ can be obtained as the composition of some poset matrices $M_m$ and $N_{n_i}, 1 \leq i \leq m$. $\square$

We observe that the poset matrix $N''$, as given in Example 4.1, represents the decomposable poset $\mathbf{B}_{2,1}[\ \mathbf{C}_2, \mathbf{Z}_4, \mathbf{B}_{1,2}]$ shown in Figure 2. In the following, we establish this result in general where we give a matrix recognition of the decomposable posets.

**Theorem 4.2.** *Let the matrix $Q$ represent the poset $\mathbf{D}$. Then $\mathbf{D}$ is decomposable if and only if $Q$ can be relabeled in such a form that it satisfies the property of transitive blocks of 1s.*

**Proof.** Let $\mathbf{D}$ be a decomposable poset. There exist the posets $\mathbf{A}$ and $\mathbf{B}_i$, $1 \leq i \leq m$, where $m \geq 2$ and at least one $\mathbf{B}_i$ is nonsingleton, such that $\mathbf{D} \cong \mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_m]$. Let $M_m$ represent the poset $\mathbf{A}$ and $N_{n_i}$ represent the poset $\mathbf{B}_i$ for every $1 \leq i \leq m$. Then, by Theorem 3.2, $M_m[N_{n_1}, N_{n_2}, \ldots, N_{n_m}]$ is a poset matrix and it represents the poset $\mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_m] \cong \mathbf{D}$. This shows that $Q$ can be relabeled in such a form that $Q = M_m[N_{n_1}, N_{n_2}, \ldots, N_{n_m}]$. By Theorem 4.1, $Q$ satisfies the property of transitive blocks of 1s of length $\{m, \{n_1, n_2, \ldots, n_m\}\}$.

Conversely, we suppose that the poset matrix $Q$ can be relabeled in such a form that it satisfies the property of transitive blocks of 1s and show similarly that the poset $\mathbf{D}$ is decomposable. $\square$

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