A weighted power distribution mechanism under transferable-utility systems: axiomatic results and dynamic processes

Yu-Hsien Liao
Department of Applied Mathematics
National Pingtung University
Taiwan
twincos@ms25.hinet.net

Abstract. By applying the notion of the efficient Banzhaf value, any additional fixed utility should be distributed equally among the players who are concerned. However, in several applications, this notion seems unrealistic for the situation being modeled. Therefore, we adopt weights to introduce a modification of the efficient Banzhaf value, which we name the weighted Banzhaf value. To present the rationality, we adopt some reasonable properties to characterize this weighted value. Based on different viewpoints, we further define excess functions to propose alternative formulations and related dynamic processes for this weighted value.

Keywords: the weighted Banzhaf value, excess function, dynamic process.

1. Introduction

In the framework of transferable-utility (TU) games, the power indices have been defined to measure the political power of each member of a voting system. A member in a voting system can be a party in a parliament or a country in a confederation. Each member will have a certain number of votes, and so their power indices will differ. The power index results may be found in Algaba et al. [1], Alonso et al. [2], Alonso and Fiestras [3], van den Brink and van der Laan [5], Dubey and Shapley [7], Haller [8], Lehrer [12], Ruiz [18], etc. Banzhaf [4] defined a power index in the framework of voting games that was essentially identical to that given by Coleman [6], and later extended it to arbitrary games by Owen [15, 16], who introduced two formulas. The Banzhaf value defined by Banzhaf [4] does not necessarily distribute the entire utility over all players in a grand coalition. Therefore, the *efficient Banzhaf value* and related results were proposed by Hwang and Liao [11] and Liao et al. [13], respectively.

In real-world situations, players might represent constituencies of different sizes or have different bargaining abilities. In addition, a lack of symmetry may arise when different bargaining abilities for different players are modeled. In various applications of TU games, it seems to be natural to assume that the players are given some *a priori measures of importance*, called *weights*. The study of weighted Banzhaf values was introduced by Radzik et al. [17]. Considering that there are exogenously given some positive weights between players, Radzik et al. [17] proposed an axiomatization of weighted Banzhaf values for a given vector of positive weights of players. Further, the family of all possible weighted Banzhaf values is described axiomatically. However, these weighted Banzhaf values introduced by Radzik et al. [17] are not efficient.

Based on the notion of the efficient Banzhaf value due to Hwang and Liao [11], all players first receive their marginal contributions from all coalitions in which they have participated; the remaining utilities are allocated equally. That is, any additional fixed utility (e.g., the cost of a common facility) is distributed equally among the players who are concerned. However, in several applications, the efficient Banzhaf value seems unrealistic for the situation being modeled. Therefore, we desire that any additional fixed utility could be distributed among players in proportion to their weights.

To modify relative discrimination among players under various situations, we adopt weights to propose different results as follows.

1. In Section 2, we adopt weights to propose the *weighted Banzhaf value*. Further, we present an alternative formulation of the weighted Banzhaf value in terms of *excess functions*. The excess of a coalition could be treated as the *variation* between the productivity and total payoff of the coalition.

2. In Section 3, we adopt the *efficiency-sum-reduced game* to characterize the weighted Banzhaf value. In Section 4, we propose dynamic processes to illustrate that the weighted Banzhaf value can be approached by players who start from an arbitrary efficient payoff vector. In Section 5, more discussions and interpretations are presented in detail.

2. The weighted Banzhaf value

A coalitional game with transferable-utility (TU game) is a pair (N, v) where N is the grand coalition and v is a mapping such that $v : 2^N \longrightarrow I\!R$ and $v(\emptyset) = 0$. Denote the class of all TU games by G. A solution on G is a function ψ which associates with each game $(N, v) \in G$ an element $\psi(N, v)$ of $I\!R^N$.

Definition 2.1. *The efficient Banzhaf value* (*Hwang and Liao* [*10*]), $\overline{\eta}$, *is the solution on* G *which associates with* $(N, v) \in G$ *and each player* $i \in N$ *the value*

$$\overline{\eta_i}(N, v) = \eta_i(N, v) + \frac{1}{|N|} \cdot \big[v(N) - \sum_{k \in N} \eta_k(N, v)\big], \tag{1}$$

where $\eta_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \big[v(S) - v(S \setminus \{i\})\big]$ *is the Banzhaf value* (*Owen* [*15, 16*]) *of* i. *It is known that the Banzhaf value violates EFF, and the efficient Banzhaf value satisfies EFF.*

Let $(N, v) \in G$. A function $w : N \to \mathbb{R}^+$ is called a weight function if w is a non-negative function. In different situations, players in N could be assigned different weights by weight functions. These weights could be interpreted as *a-priori measures of importance*; they are taken to reflect considerations not captured by the characteristic function. For example, we may be dealing with a problem of cost allocation among investment projects. Then the weights could be associated to the profitability of the different projects. In a problem of allocating travel costs among various institutions visited (cf. Shapley [20]), the weights may be the number of days spent at each one.

Given $(N, v) \in G$ and a weight function w, we define $|S|_w = \sum_{i \in S} w(i)$, for all $S \subseteq N$. The weighted Banzhaf value is defined as follows.

Definition 2.2. *Let w be a weight function. The weighted Banzhaf value $\overline{\eta^w}$, is the solution on G which associates with $(N, v) \in G$ and all players $i \in N$ the value*

$$\overline{\eta^w_i}(N, v) = \eta_i(N, v) + \frac{w(i)}{|N|_w} \cdot \big[v(N) - \sum_{k \in N} \eta_k(N, v)\big]. \tag{2}$$

By the definition of $\overline{\eta^w}$, all players firstly receive their marginal contributions from all coalitions, and further allocate the remaining utilities proportionally by applying weights.

Here, we provide a brief application of TU games and the weighted Banzhaf value in the setting of "utility distribution for management systems," such as Microsoft and NBA. In an organization, each department may consider management operation strategies. Besides competing in merchandising, all departments, such as the research department, purchasing department, and logistics department, should develop to increase the utility of the entire organization. Such a utility distribution problem could be formulated as follows. Let $N = \{1, 2, \ldots, n\}$ be a collection of all departments of an organization that could be provided jointly by some coalitions $S \subseteq N$ and let $v(S)$ be the profit of providing the cooperative coalition $S \subseteq N$ jointly. Each coalition $S \subseteq N$ could be formed by considering a specific operational aim. The function v could be treated as a utility function that assigns to each cooperative coalition $S \subseteq N$ the worth that the coalition S can obtain. Modeled in this notion, the utility distribution management system of an organization could be considered a cooperative TU game, with v being its characteristic function. However, as mentioned in the Introduction, it may be inappropriate in many situations if any additional fixed utility should be distributed equally among the players who are concerned. Thus, it is reasonable that weights are assigned to players and any fixed utility should be divided according to these weights. In the following sections, some more results will be proposed to show that the weighted Banzhaf value could be applied in the setting of utility distribution.

A solution ψ satisfies efficiency (EFF) if $\sum_{i\in N}\psi_i(N,v) = v(N)$, for all $(N,v) \in G$. Property EFF asserts that all players distribute all the utility completely.

Lemma 2.1. *The weighted Banzhaf value* $\overline{\eta^w}$ *satisfies EFF.*

Proof of Lemma 2.1. Let $(N,v) \in G$. By Definition 2.2,

$$\begin{aligned}\sum_{i\in N}\overline{\eta^w_i}(N,v) &= \sum_{i\in N}\eta_i(N,v) + \sum_{i\in N}\frac{w(i)}{|N|_w}\cdot\big[v(N) - \sum_{k\in N}\eta_k(N,v)\big]\\ &= \sum_{i\in N}\eta_i(N,v) + \frac{|N|_w}{|N|_w}\cdot\big[v(N) - \sum_{k\in N}\eta_k(N,v)\big]\\ &= v(N).\end{aligned}$$

Hence, the weighted Banzhaf value $\overline{\eta^w}$ satisfies EFF.

Next, we present an alternative formulation for the weighted Banzhaf value in terms of *excess functions*. If $x \in \mathbb{R}^N$ and $S \subseteq N$, write x_S for the restriction of x to S and write $x(S) = \sum_{i\in S} x_i$. Denote that $X(N,v) = \{x \in \mathbb{R}^N : \sum_{i\in N} x_i = v(N)\}$, for all $(N,v) \in G$. The excess of a coalition $S \subseteq N$ at x is the real number $e(S,v,x) = v(S) - x(S)$.

Lemma 2.2. *Let* $(N,v) \in G$*,* $x \in X(N,v)$ *and* w *be a weight function. Then*

$$\begin{aligned}&w(j)\sum_{S\subseteq N\setminus\{i\}}\big[e(S,v,\frac{x}{2^{|N|-1}}) - e(S\cup\{i\},v,\frac{x}{2^{|N|-1}})\big]\\ &= w(i)\sum_{S\subseteq N\setminus\{j\}}\big[e(S,v,\frac{x}{2^{|N|-1}}) - e(S\cup\{j\},v,\frac{x}{2^{|N|-1}})\big]\ \forall i,j\in N\\ &\Longleftrightarrow x = \overline{\eta^w}(N,v).\end{aligned}$$

Proof of Lemma 2.2. Let $(N,v) \in G$, $x \in X(N,v)$ and w be a weight function. For all $i,j \in N$,

$$\begin{aligned}&w(j)\sum_{S\subseteq N\setminus\{i\}}\big[e(S,v,\frac{x}{2^{|N|-1}}) - e(S\cup\{i\},v,\frac{x}{2^{|N|-1}})\big]\\ &= w(i)\sum_{S\subseteq N\setminus\{j\}}\big[e(S,v,\frac{x}{2^{|N|-1}}) - e(S\cup\{j\},v,\frac{x}{2^{|N|-1}})\big]\\ &\Longleftrightarrow w(j)\sum_{S\subseteq N\setminus\{i\}}\big[v(S) - \frac{x}{2^{|N|-1}}(S) + \frac{x}{2^{|N|-1}}(S\cup\{i\}) - v(S\cup\{i\})\big]\\ &= w(i)\sum_{S\subseteq N\setminus\{j\}}\big[v(S) - \frac{x}{2^{|N|-1}}(S) + \frac{x}{2^{|N|-1}}(S\cup\{j\}) - v(S\cup\{j\})\big]\\ (3)\qquad &\Longleftrightarrow w(j)\sum_{S\subseteq N\setminus\{i\}}\big[\frac{x_i}{2^{|N|-1}} - v(S\cup\{i\}) + v(S)\big]\end{aligned}$$

$$= w(i) \sum_{S\subseteq N\setminus\{j\}} \Big[\frac{x_j}{2^{|N|-1}} - v(S\cup\{j\}) + v(S)\Big]$$

$$\Longleftrightarrow w(j)\big[x_i - \sum_{S\subseteq N\setminus\{i\}} [v(S\cup\{i\}) - v(S)]\big]$$

$$= w(i)\big[x_j - \sum_{S\subseteq N\setminus\{j\}} [v(S\cup\{j\}) - v(S)]\big]$$

$$\Longleftrightarrow w(j)\cdot\big[x_i - \eta_i(N,v)\big] = w(i)\cdot\big[x_j - \eta_j(N,v)\big].$$

By Definition 2.2,

$$w(j)\cdot\big[\overline{\eta_i^w}(N,v) - \eta_i(N,v)\big] = w(i)\cdot\big[\overline{\eta_j^w}(N,v) - \eta_j(N,v)\big]. \tag{4}$$

By equations (3) and (4),

$$\big[x_i - \overline{\eta_i^w}(N,v)\big]\sum_{j\in N} w(j) = w(i)\sum_{j\in N}\big[x_j - \overline{\eta_j^w}(N,v)\big].$$

Since $x \in X(N,v)$ and $\overline{\eta^w}$ satisfies EFF,

$$\big[x_i - \overline{\eta_i^w}(N,v)\big]\cdot |N|_w = w(i)\cdot\big[v(N) - v(N)\big] = 0.$$

Therefore, $x_i = \overline{\eta_i^w}(N,v)$, for all $i \in N$.

3. Axiomatic results

In this section, we adopt reductions and excess functions to introduce some axiomatic results and dynamic processes of the weighted Banzhaf value.

Subsequently, we adopt the efficiency-average-reduced game to characterize the weighted Banzhaf value.

Definition 3.1 (Liao et al. [13]). *Let $(N,v)\in G$, $S\subseteq N$ and ψ be a solution. The efficiency-sum-reduced game $(S, v_{S,\psi})$ with respect to ψ and S is defined by*

$$v_{S,\psi}(T) = \begin{cases} 0, & T=\emptyset,\\ v(N) - \sum_{i\in N\setminus S}\psi_i(N,v), & T=S,\\ \sum_{Q\subseteq N\setminus S}\big[v(T\cup Q) - \sum_{i\in Q}\psi_i(N,v)\big], & T\subsetneq S. \end{cases}$$

The efficiency-sum-reduction asserts that given a proposed payoff vector $\psi(N,v)$, the worth of a coalition T in $(S, v_{S,\psi})$ is computed under the assumption that T can secure the cooperation of any subgroup Q of $N\setminus S$, provided each member of Q receives his component of $\psi(N,v)$. After these payments are made, what remains for T is the value $v(T\cup Q) - \sum_{i\in Q}\psi_i(N,v)$. Summing behavior on the part of T involves finding the sum of the values $v(T\cup Q) - \sum_{i\in Q}\psi_i(N,v)$, for all $Q\subseteq N\setminus S$. A solution ψ satisfies bilateral efficiency-sum-consistency (BESCON) if $\psi_i(S, v_{S,\psi}) = \psi_i(N,v)$, for all $(N,v)\in G$ with $|N|\geq 2$, for all $S\subseteq N$ with $|S|=2$ and, for all $i\in S$.

Lemma 3.1. *The weighted Banzhaf value $\overline{\eta^w}$ satisfies BESCON.*

Proof of Lemma 3.1. Let $(N, v) \in G$, $S \subseteq N$ with $|S| = 2$ and w be a weight function. Let $x = \overline{\eta^w}(N, v)$. Suppose $S = \{i, j\}$ then

$$\begin{aligned}
&\sum_{T\subseteq S\setminus\{i\}} \big[e(T, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}}) - e(T\cup\{i\}, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}})\big] \\
&= \big[e(\{j\}, v_{S,\overline{\eta^w}}, \frac{x_S}{2}) - e(S, v_{S,\overline{\eta^w}}, \frac{x_S}{2})\big] + \big[e(\emptyset, v_{S,\overline{\eta^w}}, \frac{x_S}{2}) - e(\{i\}, v_{S,\overline{\eta^w}}, \frac{x_S}{2})\big] \\
&= \big(v_{S,\overline{\eta^w}}(\{j\}) - \frac{x_j}{2}\big) - \big(v_{S,\overline{\eta^w}}(S) - \frac{x_S}{2}(S)\big) + 0 - \big(v_{S,\overline{\eta^w}}(\{i\}) - \frac{x_i}{2}\big) \\
&= \big(v_{S,\overline{\eta^w}}(\{j\}) - \frac{x_j}{2}\big) - 0 + 0 - \big(v_{S,\overline{\eta^w}}(\{i\}) - \frac{x_i}{2}\big) \\
&= \Big(\Big[\sum_{Q\subseteq N\setminus S} \big[v(\{j\}\cup Q) - \sum_{k\in Q}\frac{x_k}{2}\big]\Big] - \frac{x_j}{2}\Big) \\
&- \Big(\Big[\sum_{Q\subseteq N\setminus S} \big[v(\{i\}\cup Q) - \sum_{k\in Q}\frac{x_k}{2}\big]\Big] - \frac{x_i}{2}\Big) \qquad (5) \\
&= \sum_{Q\subseteq N\setminus S} \Big(\Big[v(\{j\}\cup Q) - \frac{x_j}{2^{|N|-1}}\Big] - \Big[v(\{i\}\cup Q) - \frac{x_i}{2^{|N|-1}}\Big]\Big) \\
&= \sum_{Q\subseteq N\setminus S} \Big(\Big[v(\{j\}\cup Q) - \sum_{k\in Q}\frac{x_k}{2^{|N|-1}} - \frac{x_j}{2^{|N|-1}}\Big] \\
&- \Big[v(\{i\}\cup Q) - \sum_{k\in Q}\frac{x_k}{2^{|N|-1}} - \frac{x_i}{2^{|N|-1}}\Big]\Big) \\
&= \sum_{Q\subseteq N\setminus S} \Big(\Big[v(\{j\}\cup Q) - \sum_{k\in\{j\}\cup Q}\frac{x_k}{2^{|N|-1}}\Big] - \Big[v(\{i\}\cup Q) - \sum_{k\in\{j\}\cup Q}\frac{x_k}{2^{|N|-1}}\Big]\Big) \\
&= \sum_{Q\subseteq N\setminus S} \Big[(e(\{j\}\cup Q, v, \frac{x}{2^{|N|-1}}) - (e(\{i\}\cup Q, v, \frac{x}{2^{|N|-1}})\Big] \\
&= \sum_{Q\subseteq N\setminus\{i,j\}} \Big[(e(\{j\}\cup Q, v, \frac{x}{2^{|N|-1}}) - (e(\{i\}\cup Q, v, \frac{x}{2^{|N|-1}})\Big] \\
&= \sum_{Q\subseteq N\setminus\{i\}} \Big[(e(Q, v, \frac{x}{2^{|N|-1}}) - (e(Q\cup\{i\}, v, \frac{x}{2^{|N|-1}})\Big].
\end{aligned}$$

Similar to equation (5),

$$\begin{aligned}
&\sum_{T\subseteq S\setminus\{j\}} \big[e(T, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}}) - e(T\cup\{j\}, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}})\big] \\
&= \sum_{Q\subseteq N\setminus\{j\}} \Big[(e(Q, v, \frac{x}{2^{|N|-1}}) - (e(Q\cup\{j\}, v, \frac{x}{2^{|N|-1}})\Big].
\end{aligned}$$

By EFF of $\overline{\eta^w}$ and the definition of efficiency-sum-reduced game, $x_S \in X(S, v_{S,\overline{\eta^w}})$. Therefore, by Lemma 2.2,

$$\begin{aligned}
&w(j) \cdot \sum_{T \subseteq S\setminus\{i\}} \big[e(T, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}}) - e(T \cup \{i\}, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}})\big] \\
&= w(j) \cdot \sum_{Q \subseteq N\setminus\{i\}} \Big[(e(Q, v, \frac{x}{2^{|N|-1}}) - (e(Q \cup \{i\}, v, \frac{x}{2^{|N|-1}})\Big] \\
&\text{(by equation (5))} \\
&= w(i) \cdot \sum_{Q \subseteq N\setminus\{j\}} \Big[(e(Q, v, \frac{x}{2^{|N|-1}}) - (e(Q \cup \{j\}, v, \frac{x}{2^{|N|-1}})\Big] \\
&\text{(by Lemma 2.2)} \\
&= w(i) \cdot \sum_{T \subseteq S\setminus\{j\}} \big[e(T, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}}) - e(T \cup \{j\}, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}})\big] \\
&\text{(similar to equation (5)).}
\end{aligned}$$

By Lemma 2 and $x_S \in X(S, v_{S,\overline{\eta^w}})$, we have that $x_S = \overline{\eta^w}(S, v_{S,\overline{\eta^w}})$. Hence, $\overline{\eta^w}$ satisfies BESCON.

Inspired by Hart and Mas-Colell [9], we provide an axiomatic result of the weighted Banzhaf value as follows. A solution ψ satisfies weighted Banzhaf standard for games (WBSFG) if $\psi(N, v) = \overline{\eta^w}(N, v)$, for all $(N, v) \in G$ with $|N| \leq 2$. Property WBSFG is a generalization of the two-person standardness axiom of Hart and Mas-Colell [9].

Lemma 3.2. *If a solution ψ satisfies WBSFG and BESCON, then it satisfies EFF.*

Proof of Lemma 3.2. Suppose ψ satisfies WBSFG and BESCON. Let $(N, v) \in G$. If $|N| \leq 2$, then ψ satisfies EFF by BESCON of ψ. Suppose $|N| > 2$, $i, j \in N$ and $S = \{i, j\}$. Since ψ satisfies EFF in two-person games,

$$\psi_i(S, v_{S,\psi}) + \psi_j(S, v_{S,\psi}) = v_{S,\psi}(S) = v(N) - \sum_{k \neq i,j} \psi_k(N, v). \tag{6}$$

By BESCON of ψ,

$$\psi_t(S, v_{S,\psi}) = \psi_t(N, v), \quad \text{for all } t \in S. \tag{7}$$

By equations (6) and (7), $v(N) = \sum_{k \in N} \psi_k(N, v)$, i.e., ψ satisfies EFF.

Theorem 3.1. *A solution ψ satisfies WBSFG and BESCON if and only if $\psi = \overline{\eta^w}$.*

Proof of Theorem 3.1. By Lemma 3.1, $\overline{\eta^w}$ satisfies BESCON. Clearly, $\overline{\eta^w}$ satisfies WBSFG.

To prove uniqueness, suppose ψ satisfies WBSFG and BESCON. By Lemma 3.2, ψ satisfies EFF. Let $(N,v) \in G$. If $|N| \leq 2$, it is trivial that $\psi(N,v) = \overline{\eta^w}(N,v)$ by SFG. Assume that $|N| > 2$. Let $i \in N$ and $S = \{i,j\}$ for some $j \in N \setminus \{i\}$. Then

$$
\begin{aligned}
&\psi_i(N,v) - \overline{\eta_i^w}(N,v) \\
&= \psi_i(S, v_{S,\psi}) - \overline{\eta_i^w}(S, v_{S,\overline{\eta^w}}) \qquad \text{(by BESCON of } \psi, \overline{\eta^w}) \\
&= \overline{\eta_i^w}(S, v_{S,\psi}) - \overline{\eta_i^w}(S, v_{S,\overline{\eta^w}}) \qquad \text{(by WBSFG of } \psi, \overline{\eta^w}) \\
&= \eta_i^w(S, v_{S,\psi}) + \frac{w(i)}{|S|_w} \cdot \Big[v_{S,\psi}(S) - \big[\eta_i^w(S, v_{S,\psi}) + \eta_j^w(S, v_{S,\psi})\big]\Big] \qquad (8)\\
&\quad - \eta_i^w(S, v_{S,\overline{\eta^w}}) - \frac{w(i)}{|S|_w} \cdot \Big[v_{S,\overline{\eta^w}}(S) - \big[\eta_i^w(S, v_{S,\overline{\eta^w}}) + \eta_j^w(S, v_{S,\overline{\eta^w}})\big]\Big] \\
&= \Big[v_{S,\psi}(S) + v_{S,\psi}(\{i\}) - v_{S,\psi}(\{j\})\Big] + \frac{w(i)}{|S|_w} \cdot \Big[- v_{S,\psi}(S)\Big] \\
&\quad - \Big[v_{S,\overline{\eta^w}}(S) + v_{S,\overline{\eta^w}}(\{i\}) - v_{S,\overline{\eta^w}}(\{j\})\Big] - \frac{w(i)}{|S|_w} \cdot \Big[- v_{S,\overline{\eta^w}}(S)\Big].
\end{aligned}
$$

By definitions of $v_{S,\psi}$ and $v_{S,\overline{\eta^w}}$,

$$
\begin{aligned}
v_{S,\psi}(\{i\}) - v_{S,\psi}(\{j\}) &= \sum_{Q \subseteq N \setminus S} \Big[v(\{i\} \cup Q) - v(\{j\} \cup Q)\Big] \\
&= v_{S,\overline{\eta^w}}(\{i\}) - v_{S,\overline{\eta^w}}(\{j\}). \qquad (9)
\end{aligned}
$$

By equations (8) and (9),

$$
\begin{aligned}
\psi_i(N,v) - \overline{\eta_i^w}(N,v) &= \Big[1 - \frac{w(i)}{|S|_w}\Big] \cdot \Big[v_{S,\psi}(S) - v_{S,\overline{\eta^w}}(S)\Big] \\
&= \frac{w(j)}{|S|_w} \cdot \Big[\psi_i(N,v) + \psi_j(N,v) - \overline{\eta_i^w}(N,v) - \overline{\eta_j^w}(N,v)\Big].
\end{aligned}
$$

That is,

$$
w(i) \cdot \Big[\psi_i(N,v) - \overline{\eta_i^w}(N,v)\Big] = w(j) \cdot \Big[\psi_j(N,v) - \overline{\eta_j^w}(N,v)\Big].
$$

By EFF of ψ and $\overline{\eta^w}$,

$$
\begin{aligned}
0 &= v(N) - v(N) \\
&= \sum_{j \in N} \Big[\psi_j(N,v) - \overline{\eta_j^w}(N,v)\Big] \\
&= w(i) \cdot \Big[\psi_i(N,v) - \overline{\eta_i^w}(N,v)\Big] \sum_{j \in N} \frac{1}{w(j)}.
\end{aligned}
$$

Hence, $\psi_i(N,v) = \overline{\eta_i^w}(N,v)$, for all $i \in N$.

The following examples are to show that each of the axioms used in Theorem 3.1 is logically independent of the remaining axioms.

Example 3.1. Define a solution ψ by, for all $(N,v)\in G$ and, for all $i\in N$,

$$\psi_i(N,v)=\frac{v(N)}{|N|}.$$

Clearly, ψ satisfies BESCON, but it violates WBSFG.

Example 3.2. Define a solution ψ by for all $(N,v)\in G$ and, for all $i\in N$,

$$\psi_i(N,v)=\begin{cases}\overline{\eta_i^w}(N,v), & \text{if } |N|\leq 2,\\ 0, & \text{otherwise.}\end{cases}$$

Clearly, ψ satisfies WBSFG, but it violates BESCON.

4. Dynamic results

In this section, we introduce two dynamic processes of the weighted Banzhaf value by applying excess functions and reductions.

In the following, we adopt excess functions to propose a correction function and related dynamic process for the weighted Banzhaf value.

Definition 4.1. *Let $(N,v)\in G$, $i\in N$ and w be a weight function. The e-correction function $f_i^{\overline{\eta^w}}: X(N,v)\to\mathbb{R}$ is defined by*

$$f_i^{\overline{\eta^w}}(x)=x_i+t\sum_{j\in N\setminus\{i\}}\Big(w(i)\sum_{S\subseteq N\setminus\{j\}}\big[e(S,v,\frac{x}{2^{|N|-1}})-e(S\cup\{j\},v,\frac{x}{2^{|N|-1}})\big]$$
$$-w(j)\sum_{S\subseteq N\setminus\{i\}}\big[e(S,v,\frac{x}{2^{|N|-1}})-e(S\cup\{i\},v,\frac{x}{2^{|N|-1}})\big]\Big),$$

where $t\in(0,\infty)$, which reflects the assumption that player i does not ask for full correction (when $t=1$) but only (usually) a fraction of it.

When a player withdraws from the coalitions he/she/it joined, some of the other players may complain. The e-correction function is based on the idea that, each agent shortens the weighted excess relating to his own and others' non-participation in all coalitions, and adopts these regulations to correct the original payoff.

The following lemma shows that the e-correction function is well-defined, i.e., the efficiency is preserved under the e-correction function.

Lemma 4.1. *Let $(N,v)\in G$, w be a weight function and $f^{\overline{\eta^w}}=(f_i^{\overline{\eta^w}})_{i\in N}$. If $x\in X(N,v)$, then $f^{\overline{\eta^w}}(x)\in X(N,v)$.*

Proof of Lemma 4.1. Let $(N,v) \in G$, $i,j \in N$, $x \in X(N,v)$ and w be a weight function. Similar to the equation (3),

$$\begin{aligned}
&w(i)\sum_{S\subseteq N\setminus\{j\}}\big[e(S,v,\frac{x}{2^{|N|-1}})-e(S\cup\{j\},v,\frac{x}{2^{|N|-1}})\big]\\
&-w(j)\sum_{S\subseteq N\setminus\{i\}}\big[e(S,v,\frac{x}{2^{|N|-1}})-e(S\cup\{i\},v,\frac{x}{2^{|N|-1}})\big]\\
&=w(i)\big[x_j-\overline{\eta_j^w}(N,v)\big]-w(j)\big[x_i-\overline{\eta_i^w}(N,v)\big]. \qquad (10)
\end{aligned}$$

By equation (10),

$$\begin{aligned}
&\sum_{j\in N\setminus\{i\}}\Big(w(i)\sum_{S\subseteq N\setminus\{j\}}\big[e(S,v,\frac{x}{2^{|N|-1}})-e(S\cup\{j\},v,\frac{x}{2^{|N|-1}})\big]\\
&-w(j)\sum_{S\subseteq N\setminus\{i\}}\big[e(S,v,\frac{x}{2^{|N|-1}})-e(S\cup\{i\},v,\frac{x}{2^{|N|-1}})\big]\Big)\\
&=w(i)\sum_{j\in N\setminus\{i\}}\big[x_j-\overline{\eta_j^w}(N,v)\big]-\big[x_i-\overline{\eta_i^w}(N,v)\big]\sum_{j\in N\setminus\{i\}}w(j) \qquad (11)\\
&=w(i)\cdot\big[v(N)-v(N)\big]-\big[x_i-\overline{\eta_i^w}(N,v)\big]\cdot|N|_w\\
&\text{(by EFF of }\overline{\eta^w},\ x\in X(N,v))\\
&=|N|_w\cdot\Big(\overline{\eta_i^w}(N,v)-x_i\Big).
\end{aligned}$$

Moreover

$$\begin{aligned}
&\sum_{i\in N}|N|_w\cdot\Big(\overline{\eta_i^w}(N,v)-x_i\Big)\\
&=|N|_w\cdot\big(v(N)-v(N)\big) \qquad \text{(by EFF of }\overline{\eta^w},\ x\in X(N,v)) \qquad (12)\\
&=\ 0.
\end{aligned}$$

So, we have that

$$\begin{aligned}
&\sum_{i\in N}f_i^{\overline{\eta^w}}(x)\\
&=\sum_{i\in N}\Big[x_i+t\sum_{j\in N\setminus\{i\}}\Big(w(i)\sum_{S\subseteq N\setminus\{j\}}\big[e(S,v,\frac{x}{2^{|N|-1}})-e(S\cup\{j\},v,\frac{x}{2^{|N|-1}})\big]\\
&-w(j)\sum_{S\subseteq N\setminus\{i\}}\big[e(S,v,\frac{x}{2^{|N|-1}})-e(S\cup\{i\},v,\frac{x}{2^{|N|-1}})\big]\Big)\Big]\\
&=v(N)+t\cdot 0 \quad \big(\text{by equations (11), (12) and } x\in X(N,v)\big)\\
&=v(N).
\end{aligned}$$

Hence, $f^{\overline{\eta^w}}(x) \in X(N,v)$ if $x \in X(N,v)$.

Based on Lemma 4.1, we can define $x^0 = x$, $x^1 = f^{\overline{\eta^w}}(x^0), \ldots, x^q = f^{\overline{\eta^w}}(x^{q-1})$, for all $(N,v) \in G$, for all $x \in X(N,v)$ and, for all $q \in \mathbb{N}$. Next, we adopt the correction function to propose a dynamic process.

Theorem 4.1. *Let $(N,v) \in G$ and w be a weight function. If $0 < t < \frac{2}{|N|_w}$, then $\{x^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\eta^w}(N,v)$, for all $x \in X(N,v)$.*

Proof of Theorem 4.1. Let $(N,v) \in G$, $i \in N$, $x \in X(N,v)$ and w be a weight function. By equation (11) and definition of $f^{\overline{\eta^w}}$,

$$\begin{aligned} f_i^{\overline{\eta^w}}(x) - x_i = t \sum_{j \in N \setminus \{i\}} \Big(w(i) \sum_{S \subseteq N \setminus \{j\}} \big[e(S, v, \frac{x}{2^{|N|-1}}) - e(S \cup \{j\}, v, \frac{x}{2^{|N|-1}}) \big] \\ - w(j) \sum_{S \subseteq N \setminus \{i\}} \big[e(S, v, \frac{x}{2^{|N|-1}}) - e(S \cup \{i\}, v, \frac{x}{2^{|N|-1}}) \big] \Big) \\ = t \cdot |N|_w \Big(\overline{\eta_i^w}(N,v) - x_i \Big). \end{aligned}$$

Hence,

$$\begin{aligned} \overline{\eta_i^w}(N,v) - f_i^{\overline{\eta^w}}(x) &= \overline{\eta_i^w}(N,v) - x_i + x_i - f_i^{\overline{\eta^w}}(x) \\ &= \overline{\eta_i^w}(N,v) - x_i - t \cdot |N|_w \cdot (\overline{\eta_i^w}(N,v) - x_i) \\ &= \Big(1 - t \cdot |N|_w\Big) \Big[\overline{\eta_i^w}(N,v) - x_i \Big]. \end{aligned}$$

For all $q \in \mathbb{N}$,

$$\overline{\eta^w}(N,v) - x^q = \Big(1 - t \cdot |N|_w\Big)^q \Big[\overline{\eta^w}(N,v) - x \Big].$$

If $0 < t < \frac{2}{|N|_w}$, then $-1 < (1 - t \cdot |N|_w) < 1$ and $\{x^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\eta^w}(N,v)$.

By applying a specific reduction, Maschler and Owen [14] defined a correction function to introduce a dynamic process for the Shapley value [19]. In the following, we propose a dynamic process by applying the notion due to Maschler and Owen [14].

Definition 4.2. *Let ψ be a solution, $(N,v) \in G$, $S \subseteq N$ and $x \in X(N,v)$. The (x,ψ)-reduced game*[1] *$(S, v^r_{\psi,S,x})$ is defined by for all $T \subseteq S$,*

$$v^r_{\psi,S,x}(T) = \begin{cases} v(N) - \sum_{i \in N \setminus S} x_i, & T = S, \\ v_{S,\psi}(T), & \textit{otherwise.} \end{cases}$$

1. For the discussion of x-dependent reduction, please see Maschler and Owen [14].

Inspired by Maschler and Owen [14], we define a correction function as follow. Let $(N,v)\in G$ and w be a weight function. The R-correction function to be $g=(g_i)_{i\in N}$ and $g_i: X(N,v)\to\mathbb{R}$ is define by

$$g_i(x)=x_i+t\sum_{k\in N\setminus\{i\}}\Big(\overline{\eta^w_i}\big(\{i,k\},v^r_{\overline{\eta^w},\{i,k\},x}\big)-x_i\Big),$$

where $t\in(0,\infty)$, which reflects the assumption that player i does not ask for full correction (when $t=1$) but only (usually) a fraction of it. Define $x^0=x,\ x^1=g(x^0),\dots,\ x^q=g(x^{q-1})$, for all $q\in\mathbb{N}$.

Lemma 4.2. *$g(x)\in X(N,v)$, for all $(N,v)\in G$ and, for all $x\in X(N,v)$.*

Proof of Lemma 4.2. Let $(N,v)\in G$, w be a weight function, $i,k\in N$ and $x\in X(N,v)$. Let $S=\{i,k\}$, by EFF of $\overline{\eta^w}$ and Definition 5,

$$\overline{\eta^w_i}(S,v^r_{\overline{\eta^w},S,x})+\overline{\eta^w_k}(S,v^r_{\overline{\eta^w},S,x})=x_i+x_k.$$

By Definition 4.2 and BESCON and WBSFG of $\overline{\beta}$,

$$\begin{aligned}\overline{\eta^w_i}(S,v^r_{\overline{\eta^w},S,x})-\overline{\eta^w_k}(S,v^r_{\overline{\eta^w},S,x})&=\overline{\eta^w_i}(S,v_{S,\overline{\eta^w}})-\overline{\eta^w_k}(S,v_{S,\overline{\eta^w}})\\&=\overline{\eta^w_i}(N,v)-\overline{\eta^w_k}(N,v).\end{aligned}$$

Therefore,

$$2\cdot\Big[\overline{\eta^w_i}(S,v^r_{\overline{\eta^w},S,x})-x_i\Big]=\overline{\eta^w_i}(N,v)-\overline{\eta^w_k}(N,v)-x_i+x_k. \tag{13}$$

By definition of g and equation (13),

$$\begin{aligned}g_i(x)&=x_i+\frac{t}{2}\cdot\Big[\sum_{k\in N\setminus\{i\}}\overline{\eta^w_i}(N,v)-\sum_{k\in N\setminus\{i\}}x_i\\&\quad-\sum_{k\in N\setminus\{i\}}\overline{\eta^w_k}(N,v)+\sum_{k\in N\setminus\{i\}}x_k\Big]\\&=x_i+\frac{w}{2}\cdot\Big[\big(|N|-1\big)\overline{\eta^w_i}(N,v)-\big(|N|-1\big)x_i \qquad (14)\\&\quad-\big(v(N)-\overline{\eta^w_i}(N,v)\big)+\big(v(N)-x_i\big)\Big]\\&=x_i+\frac{|N|\cdot t}{2}\cdot\Big[\overline{\eta^w_i}(N,v)-x_i\Big].\end{aligned}$$

So, we have that

$$\begin{aligned}\sum_{i\in N}g_i(x)&=\sum_{i\in N}\Big[x_i+\frac{|N|\cdot t}{2}\cdot\big[\overline{\eta^w_i}(N,v)-x_i\big]\Big]\\&=\sum_{i\in N}x_i+\frac{|N|\cdot t}{2}\cdot\big[\sum_{i\in N}\overline{\eta^w_i}(N,v)-\sum_{i\in N}x_i\big]\\&=v(N)+\frac{|N|\cdot t}{2}\cdot\big[v(N)-v(N)\big]\\&=v(N).\end{aligned}$$

Thus, $g\big(x\big)\in X(N,v)$, for all $x\in X(N,v)$.

Theorem 4.2. *Let $(N,v) \in G$ and w be a weight function. If $0 < \alpha < \frac{4}{|N|}$, then $\{x^q\}_{q=1}^{\infty}$ converges to $\overline{\eta^w}(N,v)$ for each $x \in X(N,v)$.*

Proof of Theorem 4.2. Let $(N,v) \in G$, w be a weight function and $x \in X(N,v)$. By equation (14), $g_i(x) = x_i + \frac{|N|\cdot t}{2} \cdot [\overline{\eta_i^w}(N,v) - x_i]$, for all $i \in N$. Therefore,

$$\big(1 - \frac{|N| \cdot t}{2}\big) \cdot \big[\overline{\eta_i^w}(N,v) - x_i\big] = \big[\overline{\eta_i^w}(N,v) - g_i(x)\big]$$

So, for all $q \in \mathbb{N}$,

$$\overline{\eta^w}(N,v) - x^q = \Big(1 - \frac{|N| \cdot t}{2}\Big)^q \Big[\overline{\eta^w}(N,v) - x\Big].$$

If $0 < t < \frac{4}{|N|}$, then $-1 < (1 - \frac{|N|\cdot t}{2}) < 1$ and $\{x^q\}_{q=1}^{\infty}$ converges to $\overline{\eta^w}(N,v)$, for all $(N,v) \in G$, for all weight function w and for all $i \in N$.

5. Conclusions

Weights come up naturally in the framework of utility allocation. For example, we may face the problem of utility allocation among investment projects. Then, the weights could be associated with the profitability of the different projects. Weights are also included in contracts signed by the owners of a condominium and used to divide the cost of building or maintaining common facilities. Another example is data or patent pooling among firms where the firms' sizes, measured for instance by their market shares, are natural weights. Therefore, we adopt weight functions to propose the weighted Banzhaf value. To present the rationality of the weighted Banzhaf value, we employ the efficiency-sum-reduction characterization. Based on excess functions, an alternative formulation is proposed to provide an alternative viewpoint for the weighted Banzhaf value. By applying excess functions and reductions, we also define correction functions to propose dynamic processes for the weighted Banzhaf value. Below are the comparisons of our results with related pre-existing results.

- The weighted Banzhaf value and related results are introduced initially in the framework of standard TU games.
- Inspired by Maschler and Owen [14], we propose dynamic processes for the weighted Banzhaf value. The major difference is that our e-correction function (Definition 4.1) is based on "excess functions," and Maschler and Owen's [14] correction function is based on "reductions".

Our results proposed raise two issues.

- Whether there exist weighted modifications and related results for some more solutions.

- Whether there exist different formulations and related results for some more solutions.

These issues are left to the readers.

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