On the sub-η-n-polynomial convexity and its applications

Lei Xu
Three Gorges Mathematical Research Center
China Three Gorges University
Yichang 443002
P. R. China
leixu9903@163.com

Tingsong Du*
Three Gorges Mathematical Research Center
China Three Gorges University
Yichang 443002
P. R. China
and
Department of Mathematics
College of Science
China Three Gorges University
Yichang 443002
P. R. China
tingsongdu@ctgu.edu.cn

Abstract. This study addresses a new family of functions, to be named as the sub-η-n-polynomial convex functions, which is defined as a general form of the n-polynomial convex functions and the sub-η-convex functions, and some of their significant properties are presented as well. In addition, by means of the sub-η-n-polynomial convexity, certain Hermite–Hadamard-type inequalities are established here. The sufficient conditions regarding optimality for sub-η-n-polynomial convex programming are discussed as applications.
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1. Introduction

Convexity, as well as generalized convexity, provide forceful principles and approaches in both mathematics and certain areas of engineering, in particular, in optimization theory, see [13, 29, 15, 31, 33] and the references therein cited in them. With regard to generalizations and extensions of classical convexity, a variety of interesting articles have been published by plenty of mathematicians. For example, Bector and Singh [5] considered a type of B-vex functions. Long and Peng [24] discussed a family of functions, which is a general form of the B-

*. Corresponding author

vex mappings, called semi-B-preinvex mappings. Chao et al. [8] investigated a group of extented sub-b-convex mappings, as well as demonstrated the sufficient optimality criteria regarding sub-b-convex programming within unconstrained and inequality constrained conditions. Ahmad et al. [2] proposed the concept of geodesic sub-b-s-convex mappings, as well as gave certain properties on Riemannian manifolds. Liao and Du considered two groups of mappings in [21] and [22], named as the sub-b-s-convex mappings and sub-(b, m)-convex mappings, respectively, from which certain significant properties were studied, and optimality conditions for the introduced families of generalized convex programming were reported.

On the other hand, convexity acts on a crucial role in the area of inequalities by its significance of mathematics definition. Recently, a large number of researchers, including mathematicians, engineers and scientists, have tried to conduct an in-depth research regarding properties and inequalities in association with convexity from distinct directions. For instance, Toplu et al. [32] found a class of non-negative mappings, called n-polynomial convex mappings, as well as several related Hermite–Hadamard-type inequalities have been discussed. Deng et al. [10] constructed an integral identity, as well as received certain error bounds involving integral inequalities with regard to a family of strongly convex mappings, which is named as strongly n-polynomial preinvex mappings. By virtue of n-polynomial s-type preinvexity, Butt et al. [7] studied certain refinements of Hermite–Hadamard-type integral inequalities. For more significant findings in connection with n-polynomial convex mappings, we recommend the minded readers to consult [6, 27] and the bibliographies quoted in them.

Trying to get the further discussion, let us consider to the subsequent extraordinary Hermite–Hadamard's inequality in association with convexity.

Suppose that $\psi : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on the interval Ω, for each $\zeta_1, \zeta_2 \in \Omega$ together with $\zeta_1 \neq \zeta_2$. The subsequent inequalities, to be named as Hermite–Hadamard's inequalities, are frequently put into use in engineering mathematical and applied analysis

$$\psi\left(\frac{\zeta_1+\zeta_2}{2}\right) \leq \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma \leq \frac{\psi(\zeta_1)+\psi(\zeta_2)}{2}. \tag{1}$$

The distinguished integral inequalities, which have given rise to considerable attention from plenty of authors, provide error bounds for the mean value regarding a continuous convex mapping $\psi : [\zeta_1, \zeta_2] \to \mathbb{R}$. There have been a large amount of studies, with regard to the Hermite–Hadamard-type inequalities involving other diverse types of convex mappings, such as N-quasiconvex mappings [1], s-convex mappings [20], (α, m)-convex mappings [30], strongly exponentially generalized preinvex mappings [17], h-convex mappings [9], γ-preinvex mappings [4] and so on. For more vital outcomes pertaining to the Hermite–Hadamard-type inequalities, the reader may refer to [3, 11, 16, 23, 28, 25, 34] and the bibliographies quoted in them.

Enlightened by the above-mentioned research works, in particular, those created in [18, 8, 32], we study a new group of generalized convex sets, as well as generalized convex functions, to be called as sub-η-n-polynomial convex sets and sub-η-n-polynomial convex functions, respectively. And we explore certain fascinating properties of such group of sets and functions. Moreover, we investigate quite a few Hermite–Hadamard's type inequalities in relation to the sub-η-n-polynomial convex functions. As applications, we pursue the sufficient optimality conditions for unconstrained, as well as inequality constrained programming, which are under the sub-η-n-polynomial convexity.

Through out the paper, let us suppose that Λ is a nonempty convex set in $\mathbb{R}^n$. To this end, this section retrospects certain conceptions regarding generalized convexity, and related momentous results.

Definition 1.1 ([8])**.** *The real function $\psi\colon \Lambda \to \mathbb{R}$ is named as a sub-η-convex mapping defined on the interval Λ with regard to the mapping $\eta : \Lambda \times \Lambda \times [0,1] \to \mathbb{R}$, if the successive inequality*

$$\psi\big(\nu\gamma + (1-\nu)\varrho\big) \le \nu\psi(\gamma) + (1-\nu)\psi(\varrho) + \eta(\gamma, \varrho, \nu)$$

holds true for all $\gamma, \varrho \in \Lambda$ and $\nu \in [0,1]$.

Definition 1.2 ([32])**.** *Assume that $n \in \mathbb{N}$, the nonnegative mapping $\psi\colon \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is named as an n-polynomial convex mapping if the subsequent inequality*

$$\psi\big(\nu\gamma + (1-\nu)\varrho\big) \le \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\psi(\gamma) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\psi(\varrho)$$

holds true for all $\gamma, \varrho \in \Omega$ and $\nu \in [0,1]$.

In the published article [14], the author proposed a refinement version with regard to the extraordinary Hölder's integral inequality, called as Hölder–İşcan's integral inequality as below.

Theorem 1.1 ([14])**.** *Suppose that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If ψ and ρ are two real mappings defined on the interval $[\zeta_1, \zeta_2]$, as well as if $|\psi|^p$, $|\rho|^q$ are both integrable mappings on the interval $[\zeta_1, \zeta_2]$, then we have the coming inequality*

$$\begin{aligned}\int_{\zeta_1}^{\zeta_2} |\psi(x)\rho(x)|\mathrm{d}\gamma \le \frac{1}{\zeta_2-\zeta_1}\Bigg[&\left(\int_{\zeta_1}^{\zeta_2}(\zeta_2-\gamma)|\psi(\gamma)|^p\mathrm{d}\gamma\right)^{\frac{1}{p}}\left(\int_{\zeta_1}^{\zeta_2}(\zeta_2-\gamma)|\rho(\gamma)|^q\mathrm{d}\gamma\right)^{\frac{1}{q}}\\ &+\left(\int_{\zeta_1}^{\zeta_2}(\gamma-\zeta_1)|\psi(\gamma)|^p\mathrm{d}\gamma\right)^{\frac{1}{p}}\left(\int_{\zeta_1}^{\zeta_2}(\gamma-\zeta_1)|\rho(\gamma)|^q\mathrm{d}\gamma\right)^{\frac{1}{q}}\Bigg].\end{aligned}$$

2. Sub-η-n-polynomial convex functions and their properties

The fact that the convexity, n-polynomial convexity, and sub-η-convexity have almost the analogous structures impels us to generalize these distinct families of convex functions. Now, let us consider to introduce the conception of the sub-η-n-polynomial convex functions and sub-η-n-polynomial convex sets as below. Then certain basic characterization theorems are proposed, as well as preservation of the sub-η-n-polynomial convexity with regard to some functional operations such as composition, sum and maximum are studied. In particular, two property theorems with regard to differentiable sub-η-n-polynomial convex functions are investigated in this section.

Definition 2.1. *Assume that $n \in \mathbb{N}$, the non-negative function $\psi\colon \Lambda \to \mathbb{R}$ is named as sub-η-n-polynomial convex defined on the interval Λ with regard to the mapping $\eta : \Lambda \times \Lambda \times [0,1] \to \mathbb{R}$, if the subsequent inequality*

$$(2)\quad \psi\big(\nu\gamma+(1-\nu)\varrho\big) \le \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\psi(\gamma)+\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\psi(\varrho)+\eta(\gamma,\varrho,\nu)$$

holds true for each $\gamma, \varrho \in \Lambda$ and $\nu \in [0,1]$. On the other hand, if the successive inequality

$$(3)\quad \psi\big(\nu\gamma+(1-\nu)\varrho\big) \ge \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\psi(\gamma)+\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\psi(\varrho)+\eta(\gamma,\varrho,\nu)$$

holds true for each $\gamma, \varrho \in \Lambda$ and $\nu \in [0,1]$, then the function ψ is named as sub-η-n-polynomial concave. If the inequality notations in the above-mentioned inequalities are strict, then the function ψ is named as strictly sub-η-n-polynomial convex, as well as strictly sub-η-n-polynomial concave, respectively.

Remark 2.1. If we consider to take $n = 1$, then the sub-η-n-polynomial convex function reduces to the sub-η convex functions. Moreover, when we attempt to put $n = 1$ and claim $\eta(\gamma, \varrho, \nu) \le 0$, the sub-$\eta$-$n$-polynomial convex function transforms to convex functions.

Remark 2.2. In accordance with Remark 3 in Ref. [32], we know that each nonnegative convex function is an n-polynomial convex function. When the mapping $\eta(\gamma, \varrho, \nu) \ge 0$, each nonnegative convex function is also a sub-η-n-polynomial convex function. In the same way, when we claim $\eta(\gamma, \varrho, \nu) \ge 0$, it is obvious that each n-polynomial convex function is also a sub-η-n-polynomial convex function.

Now, we try to study certain operations that preserve the sub-η-n-polynomial convexity with regard to positive linear combination and securing pointwise maximum. Because the proofs of these properties are simplified, they are omitted.

Proposition 2.1. *If the functions ψ, ρ: $\Lambda \to \mathbb{R}$ are both sub-η-n-polynomial convex with regard to the same mapping η, then $\psi + \rho$ is sub-η-n-polynomial convex with regard to the mapping 2η, and $\alpha\psi(\alpha > 0)$ is sub-η-n-polynomial convex with regard to the mapping $\alpha\eta$.*

Corollary 2.1. *If ψ_κ: $\Lambda \to \mathbb{R}$ $(\kappa = 1, 2, \dots, \delta)$ are a series of sub-η-n-polynomial convex functions regarding the mappings $\eta_\kappa : \Lambda \times \Lambda \times [0, 1] \to \mathbb{R}$ $(\kappa = 1, 2, \dots, \delta)$, correspondingly, then the function*

$$\psi = \sum_{\kappa=1}^{\delta} a_\kappa \psi_\kappa, a_\kappa \geq 0, (\kappa = 1, 2, \dots, \delta) \tag{4}$$

is sub-η-n-polynomial convex with regard to $\eta = \sum_{\kappa=1}^{\delta} a_\kappa \eta_\kappa$.

Proposition 2.2. *If ψ_κ: $\Lambda \to \mathbb{R}$ $(\kappa = 1, 2, \dots, \delta)$ are a series of sub-η-n-polynomial convex functions with respect to the mappings $\eta_\kappa : \Lambda \times \Lambda \times [0, 1] \to \mathbb{R}$ $(\kappa{=}1, 2, \dots, \delta)$, correspondingly, then the function $\psi{=}\max\{\psi_\kappa, i{=}1, 2, \dots, \delta\}$ is a sub-η-n-polynomial convex function with regard to the mapping $\eta = \max\{\eta_\kappa, \kappa = 1, 2, \dots, \delta\}$.*

Theorem 2.1. *Assume that ψ: $\Lambda \to \mathbb{R}$ is a sub-η-n-polynomial convex function with regard to the mapping $\eta : \Lambda \times \Lambda \times [0, 1] \to \mathbb{R}$, as well as ρ: $\mathbb{R} \to \mathbb{R}$ is an increasing function. If ρ meets the coming conditions:*

$$\text{(i) } \rho(\alpha\gamma) = \alpha\rho(\gamma), \forall \gamma \in \mathbb{R}, \alpha > 0, \tag{5}$$

$$\text{(ii) } \rho(\gamma + \varrho) = \rho(\gamma) + \rho(\varrho), \forall \gamma, \varrho \in \mathbb{R}, \tag{6}$$

then the function $\psi^\triangle = \rho \circ \psi$ is sub-η-n-polynomial convex with regard to $\eta^\triangle = \rho \circ \eta$.

Proof. Since the function ψ is sub-η-n-polynomial convex regarding the mapping η and the function ρ is increasing, it follows that

$$\begin{aligned}
&(\rho \circ \psi)\big(\nu\gamma + (1 - \nu)\varrho\big) \\
&= \rho\big(\psi\big(\nu\gamma + (1 - \nu)\varrho\big)\big) \\
&\leq \rho\left(\frac{1}{n}\sum_{\kappa=1}^{n}\big[1 - (1 - \nu)^\kappa\big]\psi(\gamma) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1 - \nu^\kappa\big]\psi(\varrho) + \eta(\gamma, \varrho, \nu)\right).
\end{aligned}$$

By virtue of the provided conditions in (5) and (6), it readily yields that

$$\begin{aligned}
&(\rho \circ \psi)\big(\nu\gamma + (1 - \nu)\varrho\big) \\
&\leq \frac{1}{n}\sum_{\kappa=1}^{n}\big[1 - (1 - \nu)^\kappa\big]\rho\big(\psi(\gamma)\big) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1 - \nu^\kappa\big]\rho\big(\psi(\varrho)\big) + \rho\big(\eta(\gamma, \varrho, \nu)\big) \\
&= \frac{1}{n}\sum_{\kappa=1}^{n}\big[1 - (1 - \nu)^\kappa\big](\rho \circ \psi)(\gamma) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1 - \nu^\kappa\big](\rho \circ \psi)(\varrho) + (\rho \circ \eta)(\gamma, \varrho, \nu).
\end{aligned}$$

That is, the function $\psi^{\triangle} = \rho \circ \psi$ is sub-η-n-polynomial convex with regard to $\eta^{\triangle} = \rho \circ \eta$. This ends the proof. $\square$

Theorem 2.2. *Assume that $\eta_1 : \Lambda \times \Lambda \times [0,1] \to \mathbb{R}$ and $\eta_2 : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}$ are two mappings along with $\eta_1(\gamma, \varrho, \nu) \leq \eta_2(\zeta_1, \zeta_2, \nu)$. If $\psi : \Lambda \subset \mathbb{R}^n \to \mathbb{R}$ is a sub-η_1-n-polynomial convex function on Λ with regard to η_1, then for all $\gamma, \varrho \in \Lambda$, the function $\Phi : [0,1] \to \mathbb{R}$, $\Phi(t) = \psi(\nu\gamma + (1-\nu)\varrho)$ is sub-η_2-n-polynomial convex on $[0,1]$ with regard to the mapping η_2.*

Proof. Assume that ψ is a sub-η_1-n-polynomial convex function on Λ regarding the mapping η_1. Let $\gamma, \varrho \in \Lambda$, $\nu \in [0,1]$ and $\zeta_1, \zeta_2 \in [0,1]$. Then, we know that

$$0 \leq \nu\zeta_1 + (1-\nu)\zeta_2 \leq 1,$$

and

$$\begin{aligned}
&\Phi(\nu\zeta_1 + (1-\nu)\zeta_2) \\
&= \psi\big[\big(\nu\zeta_1 + (1-\nu)\zeta_2\big)\gamma + \big(1 - \nu\zeta_1 - (1-\nu)\zeta_2\big)\varrho\big] \\
&= \psi\big[\nu(\zeta_1\gamma + (1-\zeta_1)\varrho) + (1-\nu)(\zeta_2\gamma + (1-\zeta_2)\varrho)\big] \\
&\leq \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\psi(\zeta_1\gamma + (1-\zeta_1)\varrho) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\psi(\zeta_2\gamma + (1-\zeta_2)\varrho) \\
&\quad + \eta_1(\zeta_1\gamma + (1-\zeta_1)\varrho, \zeta_2\gamma + (1-\zeta_2)\varrho, \nu) \\
&= \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\Phi(\zeta_1) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\Phi(\zeta_2) \\
&\quad + \eta_1(\zeta_1\gamma + (1-\zeta_1)\varrho, \zeta_2\gamma + (1-\zeta_2)\varrho, \nu) \\
&\leq \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\Phi(\zeta_1) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\Phi(\zeta_2) + \eta_2(\zeta_1, \zeta_2, \nu).
\end{aligned}$$

Hence, the function Φ is sub-η_2-n-polynomial convex on $[0,1]$ with regard to η_2. The proof of Theorem 2.2 is completed. $\square$

In what following, let us consider a novel concept regarding sub-η-n-polynomial convex set.

Definition 2.2. *Assume that the set $X \subseteq \mathbb{R}^{n+1}$ is a nonempty set. A set X is named as a sub-η-n-polynomial convex set with regard to the mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}$, if the subsequent inclusion relation*

$$\left(\nu\gamma + (1-\nu)\varrho,\ \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\alpha + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\beta + \eta(\gamma, \varrho, \nu)\right) \in X \tag{7}$$

holds true for $\forall$ $(\gamma, \alpha), (\varrho, \beta) \in X, \gamma, \varrho \in \mathbb{R}^n$ and $\nu \in [0,1]$.

Here, let us take into account a characterization of sub-η-n-polynomial convex function ψ: $\Lambda \to \mathbb{R}$, by means of its epigraph $E(\psi)$, which is described by

$$E(\psi) = \{(\gamma, \alpha) | \gamma \in \Lambda, \alpha \in \mathbb{R}; \psi(\gamma) \leq \alpha\}. \tag{8}$$

Theorem 2.3. *A function $\psi\colon \Lambda \to \mathbb{R}$ is a sub-η-n-polynomial convex function regarding the mapping $\eta : \Lambda \times \Lambda \times [0,1] \to \mathbb{R}$ when, and only when $E(\psi)$ is a sub-η-n-polynomial convex set regarding the same mapping η.*

Proof. Suppose that the function ψ is sub-η-n-polynomial convex regarding the mapping η. Let $(\gamma_1,\alpha_1),(\gamma_2,\alpha_2) \in E(\psi)$. Then $\psi(\gamma_1) \le \alpha_1, \psi(\gamma_2) \le \alpha_2$, we know that

$$\begin{aligned}
&\psi\big(\nu\gamma_1 + (1-\nu)\gamma_2\big)\\
&\le \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\psi(\gamma_1) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\psi(\gamma_2) + \eta(\gamma_1,\gamma_2,\nu)\\
&\le \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\alpha_1 + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\alpha_2 + \eta(\gamma_1,\gamma_2,\nu)
\end{aligned}$$

holds true for $\forall \gamma_1,\gamma_2 \in \Lambda, \nu \in [0,1]$.

Hence, it is not difficult to check that

$$\Big(\nu\gamma_1+(1-\nu)\gamma_2,\ \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\alpha_1+\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\alpha_2+\eta(\gamma_1,\gamma_2,\nu)\Big) \in E(\psi).$$

Therefore, the set $E(\psi)$ is a sub-η-n-polynomial convex set regarding the mapping η.

In turn, let us assume that $E(\psi)$ is a sub-η-n-polynomial convex set regarding the mapping η. Let $\gamma_1,\gamma_2 \in \Lambda$, we have $(\gamma_1,\alpha_1),(\gamma_2,\alpha_2) \in E(\psi)$. Thus, for $\nu \in [0,1]$, we find that

$$\left(\nu\gamma_1+(1-\nu)\gamma_2,\ \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\alpha_1+\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\alpha_2+\eta(\gamma_1,\gamma_2,\nu\right) \in E(\psi).$$

It suffices to show that

$$\psi\big(\nu\gamma_1+(1-\nu)\gamma_2\big) \le \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\psi(\gamma_1)+\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\psi(\gamma_2)+\eta(\gamma_1,\gamma_2,\nu).$$

That is, the function ψ is a sub-η-n-polynomial convex regarding the mapping η. This finishes the proof. □

We have the succedent propositions without proof.

Proposition 2.3. *If $X_\kappa (\kappa \in \Omega)$ is a series of sub-η-n-polynomial convex sets regarding the same mapping $\eta(\gamma,\varrho,\nu)$, then $\bigcap_{\kappa\in\Omega} X_\kappa$ is a sub-η-n-polynomial convex set with regard to the same mapping $\eta(\gamma,\varrho,\nu)$.*

Proposition 2.4. *If $\{\psi_\kappa | \kappa \in \Omega\}$ is a group of numerical functions, as well as any ψ_κ is a sub-η-n-polynomial convex function regarding the same mapping $\eta(\gamma,\varrho,\nu)$, then the numerical function $\psi = \sup_{\kappa\in\Omega} \psi_\kappa(\gamma)$ is a sub-η-n-polynomial convex function regarding the same mapping $\eta(\gamma,\varrho,\nu)$.*

To explore the optimal conditions regarding sub-η-n-polynomial convex programming, we next discuss certain properties in relation to a family of the differentiable sub-η-n-polynomial convex functions. Further, we assume that the limit $\lim_{\nu\to 0^+}\frac{\eta(\gamma,\varrho,\nu)}{\nu}$ exists for certain fixed $\gamma,\varrho\in\Lambda$.

Theorem 2.4. *Suppose that the function $\psi\colon \Lambda\to\mathbb{R}$ is differentiable and sub-η-n-polynomial convex regarding the mapping η. Then we have*

$$\nabla\psi(\gamma^*)^T(\gamma-\gamma^*)\le\frac{n+1}{2}\psi(\gamma)-\frac{1}{n}\psi(\gamma^*)+\lim_{\nu\to 0^+}\frac{\eta(\gamma,\gamma^*,\nu)}{\nu}. \tag{9}$$

Proof. By virtue of Taylor expansion and the sub-η-n-polynomial convexity of ψ defined on Λ, we find that

$$\begin{aligned}&\psi\big(\nu\gamma+(1-\nu)\gamma^*\big)\\&=\psi(\gamma^*)+\nu\nabla\psi(\gamma^*)^T(\gamma-\gamma^*)+o(\nu)\\&\le\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^\kappa\big]\psi(\gamma)+\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^\kappa\big]\psi(\gamma^*)+\eta(\gamma,\gamma^*,\nu).\end{aligned}$$

This implies that

$$\begin{aligned}&\nu\nabla\psi(\gamma^*)^T(\gamma-\gamma^*)+o(\nu)\\&\le\frac{1}{n}\Big[\sum_{\kappa=1}^{n}[1-(1-\nu)^\kappa]\psi(\gamma)-\sum_{\kappa=1}^{n}\nu^\kappa\psi(\gamma^*)\Big]+\eta(\gamma,\gamma^*,\nu).\end{aligned}\tag{10}$$

Dividing the above inequality (10) by ν and taking $\nu\to 0^+$, it yields that

$$\begin{aligned}\nabla\psi(\gamma^*)^T(\gamma-\gamma^*)\le{}&\lim_{\nu\to 0^+}\frac{\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^\kappa\big]}{\nu}\psi(\gamma)\\&-\lim_{\nu\to 0^+}\frac{\frac{1}{n}\sum_{\kappa=1}^{n}\nu^\kappa}{\nu}\psi(\gamma^*)+\lim_{\nu\to 0^+}\frac{\eta(\gamma,\gamma^*,\nu)}{\nu}.\end{aligned}\tag{11}$$

Employing the L'Hospital's rule, we can figure out that

$$\lim_{\nu\to 0^+}\frac{\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^\kappa\big]}{\nu}=\frac{n+1}{2},$$

and

$$\lim_{\nu\to 0^+}\frac{\frac{1}{n}\sum_{\kappa=1}^{n}\nu^\kappa}{\nu}=\frac{1}{n}.$$

Making use of the inequality (11), we deduce that

$$\nabla\psi(\gamma^*)^T(\gamma-\gamma^*)\le\frac{n+1}{2}\psi(\gamma)-\frac{1}{n}\psi(\gamma^*)+\lim_{\nu\to 0^+}\frac{\eta(\gamma,\gamma^*,\nu)}{\nu},$$

which proves the required inequality in (9). This concludes the proof. $\square$

Remark 2.3. If one attempts to pick up $n = 1$, in Theorem 2.4, then one receives Theorem 1.3 proven by Chao et al. in [8].

Theorem 2.5. *With the same hypotheses considered in Theorem 2.4, we have*

$$\begin{aligned}&\big(\nabla\psi(\varrho)-\nabla\psi(\gamma)\big)^T(\gamma-\varrho)\\ &\le \frac{(n-1)(n+2)}{2n}\big[\psi(\varrho)+\psi(\gamma)\big]+\lim_{\nu\to0^+}\frac{\eta(\gamma,\varrho,\nu)}{\nu}+\lim_{\nu\to0^+}\frac{\eta(\varrho,\gamma,\nu)}{\nu}.\end{aligned}\tag{12}$$

Proof. In accordance with Theorem 2.4, it follows that

$$\nabla\psi(\varrho)^T(\gamma-\varrho)\le\frac{n+1}{2}\psi(\gamma)-\frac{1}{n}\psi(\varrho)+\lim_{\nu\to0^+}\frac{\eta(\gamma,\varrho,\nu)}{\nu},\tag{13}$$

and

$$\nabla\psi(\gamma)^T(\varrho-\gamma)\le\frac{n+1}{2}\psi(\varrho)-\frac{1}{n}\psi(\gamma)+\lim_{\nu\to0^+}\frac{\eta(\varrho,\gamma,\nu)}{\nu}.\tag{14}$$

Adding the above two inequalities, we obtain that

$$\begin{aligned}&\big(\nabla\psi(\varrho)-\nabla\psi(\gamma)\big)^T(\gamma-\varrho)\\ &\le \frac{(n-1)(n+2)}{2n}\big[\psi(\varrho)+\psi(\gamma)\big]+\lim_{\nu\to0^+}\frac{\eta(\gamma,\varrho,\nu)}{\nu}+\lim_{\nu\to0^+}\frac{\eta(\varrho,\gamma,\nu)}{\nu}.\end{aligned}$$

This ends the proof. □

Remark 2.4. If one attempts to pick up $n = 1$, in Theorem 2.5, then one captures Theorem 1.4 presented by Chao et al. in [8].

3. Inequalities in connection with sub-η-n-polynomial convexity

In this part, we construct the successive Hermite–Hadamard-type inequalities under sub-η-n-polynomial convexity.

Theorem 3.1. *Assume that the function $\psi\colon [\zeta_1,\zeta_2]\to\mathbb{R}$ is sub-η-n-polynomial convex with $\zeta_1<\zeta_2$, and the mapping $\eta\colon [\zeta_1,\zeta_2]\times[\zeta_1,\zeta_2]\times[0,1]\to\mathbb{R}$ is continuous. If the function $\psi\in L([\zeta_1,\zeta_2])$, then the subsequent Hermite–Hadamard-type inequalities*

$$\begin{aligned}&\frac12\left(\frac{n}{n+2^{-n}-1}\right)\left[\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)-\eta\left(\xi_0\zeta_1+(1-\xi_0)\zeta_2,(1-\xi_0)\zeta_1+\xi_0\zeta_2,\frac12\right)\right]\\ &\le\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\le\left(\frac{\psi(\zeta_1)+\psi(\zeta_2)}{n}\right)\sum_{\kappa=1}^{n}\frac{\kappa}{\kappa+1}+\eta(\zeta_1,\zeta_2,\xi_0)\end{aligned}\tag{15}$$

hold true for certain fixed $\xi_0\in(0,1)$.

Proof. On account of the sub-η-n-polynomial convexity of ψ defined over the interval $[\zeta_1,\zeta_2]$, we can figure out that

$$\begin{aligned}
&\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\\
&=\psi\left(\frac{[\nu\zeta_1+(1-\nu)\zeta_2]+[(1-\nu)\zeta_1+\nu\zeta_2]}{2}\right)\\
&\leq \frac{1}{n}\sum_{\kappa=1}^{n}\left[1-\left(1-\frac{1}{2}\right)^{\kappa}\right]\psi\big(\nu\zeta_1+(1-\nu)\zeta_2\big)\\
&\quad+\frac{1}{n}\sum_{\kappa=1}^{n}\left[1-\left(\frac{1}{2}\right)^{\kappa}\right]\psi\big((1-\nu)\zeta_1+\nu\zeta_2\big)\\
&\quad+\eta\left(\nu\zeta_1+(1-\nu)\zeta_2,(1-\nu)\zeta_1+\nu\zeta_2,\frac{1}{2}\right)\\
&=\frac{1}{n}\sum_{\kappa=1}^{n}\left[1-\left(\frac{1}{2}\right)^{\kappa}\right]\Big[\psi\big(\nu\zeta_1+(1-\nu)\zeta_2\big)+\psi\big((1-\nu)\zeta_1+\nu\zeta_2\big)\Big]\\
&\quad+\eta\left(\nu\zeta_1+(1-\nu)\zeta_2,(1-\nu)\zeta_1+\nu\zeta_2,\frac{1}{2}\right).
\end{aligned}$$

Integrating the resulting inequality above regarding the variate ν over $[0,1]$, it follows that

$$\begin{aligned}
&\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\\
&\leq \frac{1}{n}\sum_{\kappa=1}^{n}\left[1-\left(\frac{1}{2}\right)^{\kappa}\right]\left[\int_0^1 f\big(\nu\zeta_1+(1-\nu)\zeta_2\big)\mathrm{d}\nu+\int_0^1\psi\big((1-\nu)\zeta_1+\nu\zeta_2\big)\mathrm{d}\nu\right]\\
&\quad+\int_0^1\eta\left(\nu\zeta_1+(1-\nu)\zeta_2,(1-\nu)\zeta_1+\nu\zeta_2,\frac{1}{2}\right)\mathrm{d}\nu\\
&=\frac{2}{\zeta_2-\zeta_1}\left(\frac{n+2^{-n}-1}{n}\right)\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\\
&\quad+\int_0^1\eta\left(\nu\zeta_1+(1-\nu)\zeta_2,(1-\nu)\zeta_1+\nu\zeta_2,\frac{1}{2}\right)\mathrm{d}\nu.
\end{aligned}$$

According to the mean value theorem of integrals, it yields that

$$\begin{aligned}
&\int_0^1\eta\left(\nu\zeta_1+(1-\nu)\zeta_2,(1-\nu)\zeta_1+\nu\zeta_2,\frac{1}{2}\right)\mathrm{d}\nu\\
&=\eta\left(\xi_0\zeta_1+(1-\xi_0)\zeta_2,(1-\xi_0)\zeta_1+\xi_0\zeta_2,\frac{1}{2}\right),\ \xi_0\in(0,1).
\end{aligned}$$

This finishes the proof of the first inequality in (15).

In the same way, by taking advantage of the sub-η-n-polynomial convexity of ψ on the interval $[\zeta_1,\zeta_2]$, as well as the mean value theorem of integrals, if the variable is changed as $\gamma = \nu\zeta_1 + (1-\nu)\zeta_2$, then we know that

$$\begin{aligned}
&\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\\
&=\int_0^1 \psi\big(\nu\zeta_1+(1-\nu)\zeta_2\big)\mathrm{d}\nu\\
&\le \int_0^1 \left[\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\psi(\zeta_1)+\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\psi(\zeta_2)+\eta(\zeta_1,\zeta_2,\nu)\right]\mathrm{d}\nu\\
&=\frac{\psi(\zeta_1)}{n}\int_0^1\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu+\frac{\psi(\zeta_2)}{n}\int_0^1\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\mathrm{d}\nu+\int_0^1\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu\\
&=\frac{\psi(\zeta_1)}{n}\sum_{\kappa=1}^{n}\int_0^1\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu+\frac{\psi(\zeta_2)}{n}\sum_{\kappa=1}^{n}\int_0^1\big[1-\nu^{\kappa}\big]\mathrm{d}\nu\\
&\quad+\eta(\zeta_1,\zeta_2,\xi_0),\ \ \xi_0\in(0,1).
\end{aligned}$$

Also, we observe that

$$\int_0^1\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu=\int_0^1\big[1-\nu^{\kappa}\big]\mathrm{d}\nu=\frac{\kappa}{\kappa+1}.$$

This finishes the proof. □

Remark 3.1. If one attempts to pick up the mapping $\eta = 0$ in Theorem 3.1, then one receives Theorem 4 deduced by Toplu et al. in [32]. In particular, if one considers to pick up $\eta = 0$ and $n = 1$, then the inequalities (15) coincides with the extraordinary Hermite–Hadamard's inequalities (1).

Theorem 3.2. *Suppose that $\psi\colon \Omega\subseteq\mathbb{R}\to\mathbb{R}$ is a differentiable mapping on the interval Ω° with $\zeta_1<\zeta_2$. If ψ is a sub-η-n-polynomial convex function regarding continuous mapping $\eta\colon \Omega\times\Omega\times[0,1]\to\mathbb{R}$, then the successive inequalities*

$$\begin{aligned}
&\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\left[\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)-\eta\left(\xi_0\zeta_1+(1-\xi_0)\zeta_2,(1-\xi_0)\zeta_1+\xi_0\zeta_2,\frac{1}{2}\right)\right]\\
&\le\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)d\gamma\\
&\le\left(\frac{n+2^{-n}-1}{n}\right)\left[\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)+\left(\frac{\psi\left(\frac{3\zeta_1-\zeta_2}{2}\right)+\psi\left(\frac{3\zeta_2-\zeta_1}{2}\right)}{n}\right)\sum_{\kappa=1}^{n}\frac{\kappa}{\kappa+1}\right.\\
&\left.+\eta\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\xi_0\right)\right]+\eta\left(\xi_1,\frac{\zeta_1+\zeta_2}{2},\frac{1}{2}\right),
\end{aligned}\tag{16}$$

and

$$\begin{aligned}
&\left|\left[\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)d\gamma-\eta\left(\xi_1,\frac{\zeta_1+\zeta_2}{2},\frac{1}{2}\right)\right]-\left(\frac{n+2^{-n}-1}{n}\right)\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right|\\
&\leq\left|\left(\frac{n+2^{-n}-1}{n}\right)\left[\left(\frac{\psi\left(\frac{3\zeta_1-\zeta_2}{2}\right)+\psi\left(\frac{3\zeta_2-\zeta_1}{2}\right)}{n}\right)\sum_{\kappa=1}^{n}\frac{\kappa}{\kappa+1}\right.\right.\\
&\qquad\left.\left.+\eta\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\xi_0\right)\right]\right|
\end{aligned}\tag{17}$$

hold true for certain fixed $\xi_0\in(0,1)$ *and* $\xi_1\in\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2}\right)$.

Proof. Applying the mean value theorem of integrals, as well as by substituting the variables $\gamma=\frac{3}{4}\nu+\frac{\zeta_1+\zeta_2}{4}$, $\nu\in[\frac{3\zeta_1-\zeta_2}{3},\frac{3\zeta_2-\zeta_1}{3}]$, we deduce that

$$\begin{aligned}
&\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\\
&=\frac{3}{4(\zeta_2-\zeta_1)}\int_{\frac{3\zeta_1-\zeta_2}{3}}^{\frac{3\zeta_2-\zeta_1}{3}}\psi\left(\frac{3}{4}\nu+\frac{\zeta_1+\zeta_2}{4}\right)\mathrm{d}\nu\\
&=\frac{3}{4(\zeta_2-\zeta_1)}\int_{\frac{3\zeta_1-\zeta_2}{3}}^{\frac{3\zeta_2-\zeta_1}{3}}\psi\left(\frac{1}{2}\left(\frac{3}{2}\nu\right)+\frac{1}{2}\left(\frac{\zeta_1+\zeta_2}{2}\right)\right)\mathrm{d}\nu\\
&\leq\frac{3}{4(\zeta_2-\zeta_1)}\int_{\frac{3\zeta_1-\zeta_2}{3}}^{\frac{3\zeta_2-\zeta_1}{3}}\left(\frac{1}{n}\sum_{\kappa=1}^{n}\left[1-\left(\frac{1}{2}\right)^{\kappa}\right]\left[\psi\left(\frac{3}{2}\nu\right)+\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right]\right.\\
&\qquad\left.+\eta\left(\frac{3}{2}\nu,\frac{\zeta_1+\zeta_2}{2},\frac{1}{2}\right)\right)\mathrm{d}\nu\\
&=\left(\frac{n+2^{-n}-1}{n}\right)\left[\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)+\frac{1}{2(\zeta_2-\zeta_1)}\int_{\frac{3\zeta_1-\zeta_2}{2}}^{\frac{3\zeta_2-\zeta_1}{2}}\psi(\nu)\mathrm{d}\nu\right]\\
&+\eta\left(\xi_1,\frac{\zeta_1+\zeta_2}{2},\frac{1}{2}\right).
\end{aligned}\tag{18}$$

According to the right hand side of outcome (15), we find that

$$\begin{aligned}
&\frac{1}{2(\zeta_2-\zeta_1)}\int_{\frac{3\zeta_1-\zeta_2}{2}}^{\frac{3\zeta_2-\zeta_1}{2}}f(\nu)\mathrm{d}\nu\\
&\leq\left(\frac{\psi\left(\frac{3\zeta_1-\zeta_2}{2}\right)+\psi\left(\frac{3\zeta_2-\zeta_1}{2}\right)}{n}\right)\sum_{\kappa=1}^{n}\frac{\kappa}{\kappa+1}+\eta\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\xi_0\right).
\end{aligned}\tag{19}$$

Combining the above-mentioned inequalities (18) and (19), one achieves the findings (16) and (17). This ends the proof. □

Remark 3.2. Under the assumptions mentioned in Theorem 3.2 with $\eta = 0$ and $n = 1$, we receive Lemma 3 presented by Mehrez in [25].

For mappings whose derivatives in absolute value are sub-η-n-polynomial convex, we will try to develop a series of Hermite–Hadamard-type integral inequalities. To achieve this object, we need the successive lemmas.

Lemma 3.1 ([12])**.** *Assume that the mapping $\psi\colon \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable defined over the interval Ω°, $\zeta_1, \zeta_2 \in \Omega^\circ$ with $\zeta_1 < \zeta_2$. If the mapping $\psi' \in L([\zeta_1, \zeta_2])$, then we have the subsequent identity*

$$\frac{\psi(\zeta_1)+\psi(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma = \frac{\zeta_2-\zeta_1}{2}\int_0^1 (1-2\nu)\psi'(\nu\zeta_1+(1-\nu)\zeta_2)\mathrm{d}\nu.$$

Lemma 3.2 ([19])**.** *Assume that $f\colon \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping defined over the interval Ω°, $\zeta_1, \zeta_2 \in \Omega^\circ$ with $\zeta_1 < \zeta_2$. If the mapping $\psi' \in L([\zeta_1, \zeta_2])$, then we have the coming identity*

$$\begin{aligned}
&\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma - \psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\\
&= (\zeta_2-\zeta_1)\left[\int_0^{\frac{1}{2}}\nu\psi'\big(\zeta_2+(\zeta_1-\zeta_2)\nu\big)\mathrm{d}\nu + \int_{\frac{1}{2}}^1(\nu-1)\psi'\big(\zeta_2+(\zeta_1-\zeta_2)\nu\big)\mathrm{d}\nu\right].
\end{aligned}$$

Theorem 3.3. *Assume that $\psi\colon \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable function defined on the interval Ω°, $\zeta_1, \zeta_2 \in \Omega^\circ$ with $\zeta_1 < \zeta_2$, and let the function $\psi' \in L([\zeta_1, \zeta_2])$. If the function $|\psi'|$ is sub-η-n-polynomial convex defined over the interval $[\zeta_1, \zeta_2]$ and the mapping $\eta\colon \Omega\times\Omega\times[0,1] \to \mathbb{R}^+$ is continuous, then the coming inequality*

$$\begin{aligned}
&\left|\frac{\psi(\zeta_1)+\psi(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right|\\
&\le \frac{\zeta_2-\zeta_1}{2n}\left(\sum_{\kappa=1}^{n}\left[\frac{(\kappa^2+\kappa+2)2^\kappa-2}{(\kappa+1)(\kappa+2)2^{\kappa+1}}\right]\big(|\psi'(\zeta_1)|+|\psi'(\zeta_2)|\big)+\frac{n}{2}\eta(\zeta_1,\zeta_2,\xi_0)\right)
\end{aligned} \tag{20}$$

holds true for some fixed $\xi_0 \in (0,1)$.

Proof. Taking advantage of Lemma 3.1, as well as the sub-η-n-polynomial convexity of $|\psi'|$ defined on the interval $[\zeta_1, \zeta_2]$, it yields that

$$\begin{aligned}
&\left|\frac{\psi(\zeta_1)+\psi(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right|\\
&\le \frac{\zeta_2-\zeta_1}{2}\int_0^1|1-2\nu||\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|\mathrm{d}\nu\\
&\le \frac{\zeta_2-\zeta_1}{2}\int_0^1|1-2\nu|\left(\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^\kappa\big]|\psi'(\zeta_1)|\right.\\
&\qquad\left.+\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^\kappa\big]|\psi'(\zeta_2)|+\eta(\zeta_1,\zeta_2,\nu)\right)\mathrm{d}\nu
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\zeta_2-\zeta_1}{2n}\Bigg(|\psi'(\zeta_1)|\int_0^1 |1-2\nu|\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu \\
&\quad +|\psi'(\zeta_2)|\int_0^1 |1-2\nu|\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\mathrm{d}\nu\Bigg) + \frac{\zeta_2-\zeta_1}{2}\int_0^1 |1-2\nu|\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu \\
&= \frac{\zeta_2-\zeta_1}{2n}\Bigg(|\psi'(\zeta_1)|\sum_{\kappa=1}^{n}\int_0^1 |1-2\nu|\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu \\
&\quad +|\psi'(\zeta_2)|\sum_{\kappa=1}^{n}\int_0^1 |1-2\nu|\big[1-\nu^{\kappa}\big]\mathrm{d}\nu\Bigg) + \frac{\zeta_2-\zeta_1}{2}\int_0^1 |1-2\nu|\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu.
\end{aligned}
$$

According to the mean value theorem of generalized integrals, we derive that

$$\int_0^1 |1-2\nu|\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu = \frac{1}{2}\eta(\zeta_1,\zeta_2,\xi_0),\ \ \xi_0\in(0,1).$$

Also, we observe that

$$\int_0^1 |1-2\nu|\mathrm{d}\nu = \frac{1}{2},$$

and

$$\int_0^1 |1-2\nu|\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu = \int_0^1 |1-2\nu|\big[1-\nu^{\kappa}\big]\mathrm{d}\nu = \frac{(\kappa^2+\kappa+2)2^{\kappa}-2}{(\kappa+1)(\kappa+2)2^{\kappa+1}}.$$

Therefore, the proof of Theorem 3.3 is completed. □

Remark 3.3. If one considers to pick up $\eta = 0$ in Theorem 3.3, then one receives Theorem 5 established by Toplu et al. in [32]. In particular, if we attempt to take $\eta = 0$ and $n = 1$, then we gain Theorem 2.2 provided by Dragomir et al. in [12].

Theorem 3.4. *Assume that $\psi\colon \Omega\subseteq\mathbb{R}\to\mathbb{R}$ is a differentiable function on Ω°, $\zeta_1,\zeta_2\in\Omega^\circ$ with $\zeta_1<\zeta_2$, and $\psi'\in L([\zeta_1,\zeta_2])$. If the function $|\psi'|^q$ is sub-η-n-polynomial convex on the interval $[\zeta_1,\zeta_2]$ for $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, and the mapping $\eta\colon \Omega\times\Omega\times[0,1]\to\mathbb{R}^+$ is continuous, then the succedent inequality*

$$
\begin{aligned}
&\left|\frac{\psi(\zeta_1)+\psi(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right| \\
&\le \frac{\zeta_2-\zeta_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{1}{n}\sum_{\kappa=1}^{n}\frac{\kappa}{\kappa+1}\big(|\psi'(\zeta_1)|^q+|\psi'(\zeta_2)|^q\big)+\eta(\zeta_1,\zeta_2,\xi_0)\right]^{\frac{1}{q}}
\end{aligned}
\tag{21}
$$

holds true for some fixed $\xi_0\in(0,1)$.

Proof. By means of Lemma 3.1, Hölder's integral inequality, and the sub-η-n-polynomial convexity of $|\psi'|^q$ defined on the interval $[\zeta_1,\zeta_2]$, it follows that

$$\begin{aligned}
&\left|\frac{\psi(\zeta_1)+\psi(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right| \\
&\le \frac{\zeta_2-\zeta_1}{2}\int_0^1 |1-2\nu||\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|\mathrm{d}\nu \\
&\le \frac{\zeta_2-\zeta_1}{2}\left(\int_0^1|1-2\nu|^p\mathrm{d}\nu\right)^{\frac{1}{p}}\left(\int_0^1|\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|^q\mathrm{d}\nu\right)^{\frac{1}{q}} \\
&\le \frac{\zeta_2-\zeta_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\psi'(\zeta_1)|^q}{n}\sum_{\kappa=1}^{n}\int_0^1[1-(1-\nu)^\kappa]\mathrm{d}\nu\right. \\
&\quad\left.+\frac{|\psi'(\zeta_2)|^q}{n}\sum_{\kappa=1}^{n}\int_0^1[1-\nu^\kappa]\mathrm{d}\nu+\int_0^1\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu\right)^{\frac{1}{q}}.
\end{aligned}$$

According to the mean value theorem of integrals, we obtain that

$$\int_0^1\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu=\eta(\zeta_1,\zeta_2,\xi_0),\ \ \xi_0\in(0,1).$$

Direct computation yields that,

$$\int_0^1|1-2\nu|^p\mathrm{d}\nu=\frac{1}{p+1},$$

and

$$\int_0^1[1-(1-\nu)^\kappa]\mathrm{d}\nu=\int_0^1[1-\nu^\kappa]\mathrm{d}\nu=\frac{\kappa}{\kappa+1}.$$

This finishes the proof. □

Remark 3.4. If one attempts to pick up the mapping $\eta=0$ in Theorem 3.4, then one acquires Theorem 6 derived by Toplu et al. in [32]. In particular, if we consider to take $\eta=0$ and $n=1$, we capture Theorem 2.3 provided by Dragomir et al. in [12].

Theorem 3.5. *Assume that ψ: $\Omega\subseteq\mathbb{R}\to\mathbb{R}$ is a differentiable function on the interval Ω°, $\zeta_1,\zeta_2\in\Omega^\circ$ with $\zeta_1<\zeta_2$, and $\psi'\in L([\zeta_1,\zeta_2])$. If $|\psi'|^q$ is a sub-η-n-polynomial convex function on $[\zeta_1,\zeta_2]$ for $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, and the mapping η: $\Omega\times\Omega\times[0,1]\to\mathbb{R}^+$ is continuous, then the succeding inequality*

$$\begin{aligned}
&\left|\frac{\psi(\zeta_1)+\psi(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right| \\
&\le \frac{\zeta_2-\zeta_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} \qquad (22)\\
&\times\left[\left(\frac{|\psi'(\zeta_1)|^q}{n}\sum_{\kappa=1}^{n}\frac{\kappa}{2(\kappa+2)}+\frac{|\psi'(\zeta_2)|^q}{n}\sum_{\kappa=1}^{n}\frac{\kappa(\kappa+3)}{2(\kappa+1)(\kappa+2)}+\frac{1}{2}\eta(\zeta_1,\zeta_2,\xi_0)\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\frac{|\psi'(\zeta_1)|^q}{n}\sum_{\kappa=1}^{n}\frac{\kappa(\kappa+3)}{2(\kappa+1)(\kappa+2)}+\frac{|\psi'(\zeta_2)|^q}{n}\sum_{\kappa=1}^{n}\frac{\kappa}{2(\kappa+2)}+\frac{1}{2}\eta(\zeta_1,\zeta_2,\xi_0)\right)^{\frac{1}{q}}\Bigg]$$

holds true for certain fixed $\xi_0\in(0,1)$.

Proof. By taking advantage of Lemma 3.1, as well as the Hölder–İşcan's integral inequality, it yields that

$$\begin{aligned}
&\left|\frac{\psi(\zeta_1)+\psi(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right|\\
&\le\frac{\zeta_2-\zeta_1}{2}\int_0^1|1-2\nu||\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|\mathrm{d}\nu\\
&\le\frac{\zeta_2-\zeta_1}{2}\left[\left(\int_0^1(1-\nu)|1-2\nu|^p\mathrm{d}\nu\right)^{\frac{1}{p}}\left(\int_0^1(1-\nu)|\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|^q\mathrm{d}\nu\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\nu|1-2\nu|^p\mathrm{d}\nu\right)^{\frac{1}{p}}\left(\int_0^1\nu|\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|^q\mathrm{d}\nu\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Making use of the sub-η-n-polynomial convexity of $|\psi'|^q$, it follows that

$$\begin{aligned}
&\int_0^1(1-\nu)|\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|^q\mathrm{d}\nu\\
&\le\frac{|\psi'(\zeta_1)|^q}{n}\sum_{\kappa=1}^{n}\int_0^1(1-\nu)[1-(1-\nu)^\kappa]\mathrm{d}\nu+\frac{|\psi'(\zeta_2)|^q}{n}\sum_{\kappa=1}^{n}\int_0^1(1-\nu)[1-\nu^\kappa]\mathrm{d}\nu\\
&+\int_0^1(1-\nu)\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\nu|\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|^q\mathrm{d}\nu\\
&\le\frac{|\psi'(\zeta_1)|^q}{n}\sum_{\kappa=1}^{n}\int_0^1\nu[1-(1-\nu)^\kappa]\mathrm{d}\nu+\frac{|\psi'(\zeta_2)|^q}{n}\sum_{\kappa=1}^{n}\int_0^1\nu[1-\nu^\kappa]\mathrm{d}\nu\\
&+\int_0^1\nu\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu.
\end{aligned}$$

According to the mean value theorem of generalized integrals, we know that

$$\int_0^1(1-\nu)\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu=\int_0^1\nu\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu=\frac{1}{2}\eta(\zeta_1,\zeta_2,\xi_0),\ \ \xi_0\in(0,1).$$

Direct computation yields that

$$\int_0^1(1-\nu)|1-2\nu|^p\mathrm{d}\nu=\int_0^1\nu|1-2\nu|^p\mathrm{d}\nu=\frac{1}{2(p+1)},$$

$$\int_0^1 (1-\nu)[1-(1-\nu)^\kappa]\mathrm{d}\nu = \int_0^1 \nu[1-\nu^\kappa]\mathrm{d}\nu = \frac{\kappa}{2(\kappa+2)},$$

and

$$\int_0^1 \nu[1-(1-\nu)^\kappa]\mathrm{d}\nu = \int_0^1 (1-\nu)[1-\nu^\kappa]\mathrm{d}\nu = \frac{\kappa(\kappa+3)}{2(\kappa+1)(\kappa+2)}.$$

Thus, this concludes the proof. □

Remark 3.5. If one attempts to pick up the mapping $\eta = 0$, in Theorem 3.5, then one receives Theorem 8 constructed by Toplu et al. in [32]. In particular, if we consider to take $\eta = 0$ and $n = 1$, we capture Theorem 8 presented by İşcan in [14].

Theorem 3.6. *Suppose that the mapping η_1: $\Lambda \times \Lambda \times [0,1] \to \mathbb{R}^+$ and the mapping $\eta_2 : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}^+$ are two continuous mappings together with $\eta_1(\gamma,\varrho,\nu) \leq \eta_2(\zeta_1,\zeta_2,\nu)$, and the function ψ: $\Lambda \subseteq \mathbb{R}^n \to \mathbb{R}^+$ is sub-η_1-n-polynomial convex on Λ with regard to η_1. Then for any $\gamma,\varrho \in \Lambda$ and $\zeta_1,\zeta_2 \in [0,1]$ with $\zeta_1 < \zeta_2$, the subsequent inequality*

$$\begin{aligned}
&\left|\frac{1}{2}\int_0^{\zeta_1} \psi(s\gamma+(1-s)\varrho)\mathrm{d}s + \frac{1}{2}\int_0^{\zeta_2} \psi(s\gamma+(1-s)\varrho)\mathrm{d}s\right.\\
&\left.-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\left(\int_0^\theta \psi(s\gamma+(1-s)\varrho)\mathrm{d}s\right)\mathrm{d}\theta\right|\\
&\leq \frac{\zeta_2-\zeta_1}{2n}\left[\sum_{\kappa=1}^{n}\left(\frac{(\kappa^2+\kappa+2)2^\kappa-2}{(\kappa+1)(\kappa+2)2^{\kappa+1}}\right)\big(\psi(\zeta_1\gamma+(1-\zeta_1)\varrho)\right.\\
&\left.+\psi(\zeta_2\gamma+(1-\zeta_2)\varrho)\big)+\frac{n}{2}\eta_2(\zeta_1,\zeta_2,\xi_0)\right]
\end{aligned} \tag{23}$$

holds true for certain fixed $\xi_0 \in (0,1)$.

Proof. Assume that $\gamma,\varrho \in \Lambda$ and $\zeta_1,\zeta_2 \in [0,1]$ with $\zeta_1 < \zeta_2$. Since ψ is a sub-η_1-n-polynomial convex function, by Theorem 2.2, it yields that the function

$$\Phi : [0,1] \to \mathbb{R},\ \Phi(\nu) = \psi(\nu\gamma+(1-\nu)\varrho)$$

is a sub-η_2-n-polynomial convex function on $[0,1]$ with regard to η_2.

Define Ψ: $[0,1] \to \mathbb{R}$

$$\Psi(\nu) = \int_0^\nu \Phi(s)\mathrm{d}s = \int_0^\nu \psi(s\gamma+(1-s)\varrho)\mathrm{d}s.$$

Evidently, $\Psi'(\nu) = \Phi(\nu)$ for $\forall\ \nu \in (0,1)$.

Owing to $\psi(\Lambda) \subseteq \mathbb{R}^+$, it shows that $\Phi \geq 0$ on $[0,1]$. Thus, $\Psi' \geq 0$ on $[0,1]$. If one employs Theorem 3.3 to the function Ψ, then one knows that

$$\begin{aligned}
&\left|\frac{\Psi(\zeta_1)+\Psi(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\Psi(\theta)\mathrm{d}\theta\right|\\
&\leq \frac{\zeta_2-\zeta_1}{2n}\left(\sum_{\kappa=1}^{n}\left[\frac{(\kappa^2+\kappa+2)2^\kappa-2}{(\kappa+1)(\kappa+2)2^{\kappa+1}}\right]\big(|\Psi'(\zeta_1)|+|\Psi'(\zeta_2)|\big)+\frac{n}{2}\eta_2(\zeta_1,\zeta_2,\xi_0)\right),
\end{aligned}$$

and we conclude that the desired outcome (23) holds true. $\square$

Theorem 3.7. *Suppose that the function $\psi\colon \Omega \subseteq \mathbb{R} \to \mathbb{R}$ has differentiable sub-η_1-n-polynomial convexity on Ω° regarding continuous mapping $\eta_1\colon \Omega\times\Omega\times[0,1] \to \mathbb{R}^+$, $\zeta_1, \zeta_2 \in \Omega^\circ$ with $\zeta_1 < \zeta_2$, and its derivative $\psi'\colon [\frac{3\zeta_1-\zeta_2}{2}, \frac{3\zeta_2-\zeta_1}{2}] \to \mathbb{R}$ is a continuous function on $[\frac{3\zeta_1-\zeta_2}{2}, \frac{3\zeta_2-\zeta_1}{2}]$. For $q \geq 1$, if the function $|\psi'|^q$ is sub-η_2-n-polynomial convex on $[\frac{3\zeta_1-\zeta_2}{2}, \frac{3\zeta_2-\zeta_1}{2}]$ regarding continuous mapping $\eta_2\colon [\frac{3\zeta_1-\zeta_2}{2}, \frac{3\zeta_2-\zeta_1}{2}] \times [\frac{3\zeta_1-\zeta_2}{2}, \frac{3\zeta_2-\zeta_1}{2}] \times [0,1] \to \mathbb{R}^+$, then the successive inequality*

$$\begin{aligned}
&\left|\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\left[\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right.\right.\\
&\left.\left.-\eta_1\left(\xi_1, \frac{\zeta_1+\zeta_2}{2}, \frac{1}{2}\right)\right] - \psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right| \leq (\zeta_2-\zeta_1)\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\frac{|\psi'(\frac{3\zeta_1-\zeta_2}{2})|^q}{n}K_1+\frac{|\psi'(\frac{3\zeta_2-\zeta_1}{2})|^q}{n}K_2+\frac{1}{8}\eta_2\left(\frac{3\zeta_1-\zeta_2}{2}, \frac{3\zeta_2-\zeta_1}{2}, \xi_2\right)\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{|\psi'(\frac{3\zeta_1-\zeta_2}{2})|^q}{n}K_2+\frac{|\psi'(\frac{3\zeta_2-\zeta_1}{2})|^q}{n}K_1+\frac{1}{8}\eta_2\left(\frac{3\zeta_1-\zeta_2}{2}, \frac{3\zeta_2-\zeta_1}{2}, \xi_3\right)\right)^{\frac{1}{q}}\right]
\end{aligned} \tag{24}$$

holds for certain fixed $\xi_1 \in (\frac{3\zeta_1-\zeta_2}{2}, \frac{3\zeta_2-\zeta_1}{2})$, $\xi_2 \in (0, \frac{1}{2})$, and $\xi_3 \in (\frac{1}{2}, 1)$, where

$$K_1 = \sum_{\kappa=1}^{n}\left[\frac{1}{8}+\frac{\kappa+3-2^{\kappa+2}}{(\kappa+1)(\kappa+2)2^{\kappa+2}}\right],$$

and

$$K_2 = \sum_{\kappa=1}^{n}\left[\frac{1}{8}-\frac{1}{(\kappa+2)2^{\kappa+2}}\right].$$

Proof. Making use of inequality (18), we know that

$$\begin{aligned}
&\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\\
&\leq\left(\frac{n+2^{-n}-1}{n}\right)\left[\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)+\frac{1}{2(\zeta_2-\zeta_1)}\int_{\frac{3\zeta_1-\zeta_2}{2}}^{\frac{3\zeta_2-\zeta_1}{2}}\psi(\nu)\mathrm{d}\nu\right]\\
&+\eta_1\left(\xi_1, \frac{\zeta_1+\zeta_2}{2}, \frac{1}{2}\right).
\end{aligned} \tag{25}$$

Taking advantage of Lemma 3.2, we derive that

$$
\begin{aligned}
&\frac{1}{2(\zeta_2-\zeta_1)}\int_{\frac{3\zeta_1-\zeta_2}{2}}^{\frac{3\zeta_2-\zeta_1}{2}}\psi(\nu)\mathrm{d}\nu\\
&=\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)+2(\zeta_2-\zeta_1)\left[\int_0^{\frac{1}{2}}\nu\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\mathrm{d}\nu\right.\\
&\left.+\int_{\frac{1}{2}}^{1}(\nu-1)\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\mathrm{d}\nu\right].
\end{aligned}\tag{26}
$$

By putting (26) into (25), and by virtue of the properties of modulus, it yields that

$$
\begin{aligned}
&\left|\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\left[\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right.\right.\\
&\left.\left.-\eta_1\left(\xi_1,\frac{\zeta_1+\zeta_2}{2},\frac{1}{2}\right)\right]-\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right|\\
&\le(\zeta_2-\zeta_1)\left[\int_0^{\frac{1}{2}}\nu\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\right|\mathrm{d}\nu\right.\\
&\left.+\int_{\frac{1}{2}}^{1}(1-\nu)\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\right|\mathrm{d}\nu\right].
\end{aligned}\tag{27}
$$

Let us take into account the coming two cases. Suppose that $q=1$. We observe that

$$
\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)=\psi'\left(\nu\left(\frac{3\zeta_1-\zeta_2}{2}\right)+(1-\nu)\left(\frac{3\zeta_2-\zeta_1}{2}\right)\right).
$$

Since the function $|\psi'|$ is a sub-η_2-n-polynomial convex on $[\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2}]$, we know that for any $\nu\in[0,1]$

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\nu\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\right|\mathrm{d}\nu\\
&\le\frac{\left|\psi'\left(\frac{3\zeta_1-\zeta_2}{2}\right)\right|}{n}\sum_{\kappa=1}^{n}\int_0^{\frac{1}{2}}\nu\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu\\
&+\frac{\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}\right)\right|}{n}\sum_{\kappa=1}^{n}\int_0^{\frac{1}{2}}\nu\big[1-\nu^{\kappa}\big]\mathrm{d}\nu\\
&+\int_0^{\frac{1}{2}}\nu\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\nu\right)\mathrm{d}\nu.
\end{aligned}\tag{28}
$$

Similarly, it follows that

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}(1-\nu)\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\right|\mathrm{d}\nu\\
&\leq \frac{\left|\psi'\left(\frac{3\zeta_1-\zeta_2}{2}\right)\right|}{n}\sum_{\kappa=1}^{n}\int_{\frac{1}{2}}^{1}(1-\nu)\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu\\
&+\frac{\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}\right)\right|}{n}\sum_{\kappa=1}^{n}\int_{\frac{1}{2}}^{1}(1-\nu)\big[1-\nu^{\kappa}\big]\mathrm{d}\nu\\
&+\int_{\frac{1}{2}}^{1}(1-\nu)\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\nu\right)\mathrm{d}\nu.
\end{aligned}\tag{29}$$

According to the mean value theorem of generalized integrals, we derive that

$$\int_{0}^{\frac{1}{2}}\nu\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\nu\right)\mathrm{d}\nu=\frac{1}{8}\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\xi_2\right),\xi_2\in\left(0,\frac{1}{2}\right),$$

and

$$\int_{\frac{1}{2}}^{1}(1-\nu)\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\nu\right)\mathrm{d}\nu=\frac{1}{8}\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\xi_3\right),\xi_3\in\left(\frac{1}{2},1\right).$$

Direct computation yields that

$$\sum_{\kappa=1}^{n}\int_{0}^{\frac{1}{2}}\nu[1-(1-\nu)^{\kappa}]\mathrm{d}\nu=\sum_{\kappa=1}^{n}\int_{\frac{1}{2}}^{1}(1-\nu)[1-\nu^{\kappa}]\mathrm{d}\nu=\sum_{\kappa=1}^{n}\left[\frac{1}{8}+\frac{\kappa+3-2^{\kappa+2}}{(\kappa+1)(\kappa+2)2^{\kappa+2}}\right],$$

and

$$\sum_{\kappa=1}^{n}\int_{0}^{\frac{1}{2}}\nu[1-\nu^{\kappa}]\mathrm{d}\nu=\sum_{\kappa=1}^{n}\int_{\frac{1}{2}}^{1}(1-\nu)[1-(1-\nu)^{\kappa}]\mathrm{d}\nu=\sum_{\kappa=1}^{n}\left[\frac{1}{8}-\frac{1}{(\kappa+2)2^{\kappa+2}}\right].$$

Consequently, this concludes the proof for this case.

Assume that $q>1$. On account of the power–mean inequality, as well as the sub-η_2-n-polynomial convexity of $|\psi'|^q$, we deduce that

$$\begin{aligned}
&\int_{0}^{\frac{1}{2}}\nu\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\right|\mathrm{d}\nu\\
&=\int_{0}^{\frac{1}{2}}\nu\left|\psi'\left(\nu\left(\frac{3\zeta_1-\zeta_2}{2}\right)+(1-\nu)\left(\frac{3\zeta_2-\zeta_1}{2}\right)\right)\right|\mathrm{d}\nu\\
&\leq\left(\int_{0}^{\frac{1}{2}}\nu\mathrm{d}\nu\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{1}{2}}\nu\left|\psi'\left(\nu\left(\frac{3\zeta_1-\zeta_2}{2}\right)+(1-\nu)\left(\frac{3\zeta_2-\zeta_1}{2}\right)\right)\right|^{q}\mathrm{d}\nu\right)^{\frac{1}{q}}\\
&\leq\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left(\frac{|\psi'(\frac{3\zeta_1-\zeta_2}{2})|^q}{n}K_1+\frac{|\psi'(\frac{3\zeta_2-\zeta_1}{2})|^q}{n}K_2\right.\\
&\left.+\frac{1}{8}\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\xi_2\right)\right)^{\frac{1}{q}}.
\end{aligned}\tag{30}$$

In the same way, it yields that

$$
\begin{aligned}
&\int_{\frac{1}{2}}^{1} \nu \left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\right| \mathrm{d}\nu \\
&\leq \left(\frac{1}{8}\right)^{1-\frac{1}{q}} \left(\frac{\left|\psi'\left(\frac{3\zeta_1-\zeta_2}{2}\right)\right|^q}{n} K_2 + \frac{\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}\right)\right|^q}{n} K_1 \right. \\
&\quad \left. + \frac{1}{8}\eta_2\left(\frac{3\zeta_1-\zeta_2}{2}, \frac{3\zeta_2-\zeta_1}{2}, \xi_3\right)\right)^{\frac{1}{q}}.
\end{aligned} \tag{31}
$$

Employing (30) and (31) in (27), one achieves the desired outcome (24), which concludes the proof. □

Remark 3.6. Under the same assumptions considered in Theorem 3.7 with $\eta_1 = \eta_2 = 0$ and $n = 1$, we successfully gain Theorem 1 presented by Mehrez in [25].

4. Applications

In order to identify the applications of the outcomes derived in the study, the unconstraint nonlinear programming is considered as below:

$$
(P) \qquad \min\left\{\psi(\gamma) | \gamma \in \Lambda \subset \mathbb{R}^n\right\}, \tag{32}
$$

where $\psi : \Lambda \to \mathbb{R}$ is a differentiable sub-η-n-polynomial convex function on Λ.

Theorem 4.1. *Assume that the function ψ: $\Lambda \to \mathbb{R}$ has differentiable sub-η-n-polynomial convexity with regard to the mapping $\eta : \Lambda \times \Lambda \times [0,1] \to \mathbb{R}$. If $\gamma^* \in \Lambda$ and the successive condition*

$$
\nabla\psi(\gamma^*)^T(\gamma-\gamma^*) - \lim_{\nu\to 0^+} \frac{\eta(\gamma,\gamma^*,\nu)}{\nu} \geq \frac{n-1}{n}\psi(\gamma^*) + \frac{n-1}{2}\psi(\gamma), \tag{33}
$$

holds true for each $\gamma \in \Lambda$, $\nu \in [0,1]$, then γ^* is the optimal solution of ψ on Λ.

Proof. For any $\gamma \in \Lambda$, by Theorem 2.4, we find that

$$
\nabla\psi(\gamma^*)^T(\gamma-\gamma^*) - \lim_{\nu\to 0^+} \frac{\eta(\gamma,\gamma^*,\nu)}{\nu} \leq \frac{n+1}{2}\psi(\gamma) - \frac{1}{n}\psi(\gamma^*).
$$

In combination with the condition (33), it readily yields that

$$
\frac{n+1}{2}\psi(\gamma) - \frac{1}{n}\psi(\gamma^*) \geq \frac{n-1}{n}\psi(\gamma^*) + \frac{n-1}{2}\psi(\gamma),
$$

i.e., $\psi(\gamma) - \psi(\gamma^*) \geq 0$. Therefore, γ^* is an optimal solution of ψ on Λ. This concludes the proof. □

Remark 4.1. If one considers to pick up $n = 1$, in Theorem 4.1, then one successfully receives Theorem 2.1 deduced by Chao et al. in [8].

Now, let us apply the outcomes investigated in this study to the nonlinear programming along with the subsequent inequality constraints:

$$(P_g) \quad \begin{array}{ll} \min & \psi(\gamma) \\ s.t. & \omega_i(\gamma) \leq 0, \ i \in U = \{1, 2, \ldots, m\}, \\ & \gamma \in \mathbb{R}^n, \end{array} \tag{34}$$

where ψ and ω_i are all differentiable defined on the set $D = \{\gamma \in \mathbb{R}^n | \omega_i(\gamma) \leq 0, \ i \in U\}$, which is assumed to be a nonempty feasible set of (P_g). In addition, for $\gamma^* \in D$, we define $U^* = \{\gamma \in \mathbb{R}^n | \omega_i(\gamma^*) = 0, \ i \in U\}$, $\lambda_i = (\lambda_1, \ldots, \lambda_m)^T$.

The successive theorem displays the Karush–Kuhn–Tucker (KKT) sufficient conditions.

Theorem 4.2. (KKT sufficient conditions) *Assume that $\psi(\gamma) : \mathbb{R}^n \to \mathbb{R}$ is a differentiable and sub-η-n-polynomial convex function with regard to the mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}$, and the functions $\omega_i : \mathbb{R}^n \to \mathbb{R}$ $(i \in U)$ are a series of differentiable sub-η-n-polynomial convex with regard to the mappings $\eta_i : \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}$ $(i \in U)$. Assume that $\gamma^* \in D$ is a KKT point regarding (P_g), that is, there exist multipliers $\lambda_i \geq 0$ $(i \in U)$ satisfying that*

$$\begin{aligned} &\nabla\psi(\gamma^*) + \sum_{i \in U} \lambda_i \nabla\omega_i(\gamma^*) = 0, \\ &\lambda_i \omega_i(\gamma^*) = 0. \end{aligned} \tag{35}$$

If the subsequent condition

$$\begin{aligned} &\frac{n-1}{n}\psi(\gamma^*) + \frac{n-1}{2}\psi(\gamma) + \lim_{\nu \to 0^+} \frac{\eta(\gamma, \gamma^*, \nu)}{\nu} \\ &\quad \leq -\sum_{i \in U} \lambda_i \lim_{\nu \to 0^+} \frac{\eta_i(\gamma, \gamma^*, \nu)}{\nu}, \forall \ \gamma \in D, \end{aligned} \tag{36}$$

also holds true, then γ^* is an optimal solution regarding the problem (P_g).

Proof. For each $\gamma \in D$, one observes that

$$\omega_i(\gamma) \leq 0 = \omega_i(\gamma^*), \ i \in U^* = \{i \in U | \omega_i(\gamma^*) = 0\}.$$

Making use of the sub-η-n-polynomial convexity of ω_i and Theorem 2.4, for $i \in U^*$, we find that

$$\nabla\omega_i(\gamma^*)^T(\gamma - \gamma^*) - \lim_{\nu \to 0^+} \frac{\eta_i(\gamma, \gamma^*, \nu)}{\nu} \leq \frac{n+1}{2}\omega_i(\gamma) - \frac{1}{n}\omega_i(\gamma^*) \leq 0. \tag{37}$$

According to the conditions (35), we know that

$$\nabla\psi(\gamma^*)^T(\gamma-\gamma^*) = -\sum_{i\in U}\lambda_i\nabla\omega_i(\gamma^*)^T(\gamma-\gamma^*) = -\sum_{i\in U^*}\lambda_i\nabla\omega_i(\gamma^*)^T(\gamma-\gamma^*). \tag{38}$$

By virtue of the inequality (36), we can figure out that

$$\begin{aligned}
&\nabla\psi(\gamma^*)^T(\gamma-\gamma^*) - \frac{n-1}{n}\psi(\gamma^*) - \frac{n-1}{2}\psi(\gamma) - \lim_{\nu\to 0^+}\frac{\eta(\gamma,\gamma^*,\nu)}{\nu} \\
&\geq -\sum_{i\in U^*}\lambda_i\nabla\omega_i(\gamma^*)^T(\gamma-\gamma^*) + \sum_{i\in U^*}\lambda_i\lim_{\nu\to 0^+}\frac{\eta_i(\gamma,\gamma^*,\nu)}{\nu} \\
&\geq -\sum_{i\in U^*}\lambda_i\left[\nabla\omega_i(\gamma^*)^T(\gamma-\gamma^*) - \lim_{\nu\to 0^+}\frac{\eta_i(\gamma,\gamma^*,\nu)}{\nu}\right].
\end{aligned} \tag{39}$$

Here, we use (37) and (39) to derive the coming inequality

$$\nabla\psi(\gamma^*)^T(\gamma-\gamma^*) - \frac{n-1}{n}\psi(\gamma^*) - \frac{n-1}{2}\psi(\gamma) - \lim_{\nu\to 0^+}\frac{\eta(\gamma,\gamma^*,\nu)}{\nu} \geq 0,$$

that is,

$$\nabla\psi(\gamma^*)^T(\gamma-\gamma^*) - \lim_{\nu\to 0^+}\frac{\eta(\gamma,\gamma^*,\nu)}{\nu} \geq \frac{n-1}{n}\psi(\gamma^*) + \frac{n-1}{2}\psi(\gamma),$$

and in accordance with Theorem 4.1 , it yields that

$$\psi(\gamma) \geq \psi(\gamma^*), \forall\ \gamma \in D.$$

Therefore, γ^* is an optimal solution regarding the problem (P_g). This concludes the proof. □

5. Conclusions

Sub-η-n-polynomial convexity, as well as sub-η-n-polynomial convex sets, are introduced in the present paper. Because of their significance, a series of interesting properties for newly defined functions and sets are discussed, respectively. Certain Hermite–Hadamard-type integral inequalities, in connection with sub-η-n-polynomial convex functions, are also presented. We conclude the article by showing that the derived inequalities also hold for convex functions and n-polynomial convex functions. As applications, under the sub-η-n-polynomial convexity, the KKT sufficient optimality conditions, under the sub-η-n-polynomial convex programming with unconstrained and constrained inequalities, are deduced in the present paper, respectively. We have reason to confirm that it is an interesting and innovative problem, for forthcoming researchers who will enable them to establish analogous integral inequalities for other diverse types of sub-η-convexity, and corresponding KKT optimality conditions for the generalized sub-η-convex programming in their future work.

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References

[1] S. Abramovich, L. E. Persson, *Fejér and Hermite-Hadamard type inequalities for N-quasiconvex functions,* Math. Notes, 102 (2017), 599-609.

[2] I. Ahmad, A. Jayswal, B. Kumari, *Characterizations of geodesic sub-b-s-convex functions on Riemannian manifolds,* J. Nonlinear Sci. Appl., 11 (2018), 189-197.

[3] B. Ahmad, A. Alsaedi, M. Kirane, B. T. Torebek, *Hermite-Hadamard, Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals,* J. Comput. Appl. Math., 353 (2019), 120-129.

[4] M. U. Awan, S. Talib, M. A. Noor, K. I. Noor, *On γ-preinvex functions,* Filomat, 34 (2020), 4137-4159.

[5] C. R. Bector, C. Singh, *B-vex functions,* J. Optim. Theory Appl., 71 (1991), 237-253.

[6] S. I. Butt, S. Rashid, M. Tariq, M. K. Wang, *Novel refinements via n-polynomial harmonically s-type convex functions and application in special functions,* J. Funct. Spaces, 2021 (2021) Article ID 6615948, 17 pages.

[7] S. I. Butt, H. Budak, M. Tariq, M. Nadeem, *Integral inequalities for n-polynomial s-type preinvex functions with applications,* Math. Methods Appl. Sci., 44 (2021), 11006-11021.

[8] M. T. Chao, J. B. Jian, D. Y. Liang, *Sub-b-convex functions and sub-b-convex programming,* J. Operations Research Transactions (China), 16 (2012), 1-8.

[9] M. R. Delavar, M. De La Sen, *A mapping associated to h-convex version of the Hermite-Hadamard inequality with applications,* J. Math. Inequal., 14 (2020), 329-335.

[10] Y. P. Deng, M. U. Awan, S. Talib, M. A. Noor, K. I. Noor, P. O. Mohammed, S. H. Wu, *Inequalities for estimations of integrals related to higher-order strongly n-polynomial preinvex functions,* Journal of Mathematics, 2020 (2020), Article ID 8841356, 12 pages.

[11] T. S. Du, H. Wang, M. A. Khan, Y. Zhang, *Certain integral inequalities considering generalized m-convexity on fractal sets and their applications,* Fractals, 27 (2019), Article ID 1950117, 17 pages.

[12] S. S. Dragomir, R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula,* Appl. Math. Lett., 11 (1998), 91-95.

[13] S. Dragomir, M. Jleli, B. Samet, *Generalized convexity and integral inequalities,* Math. Methods Appl. Sci., 2020, 1-15, Online: https://doi.org/10.1002/mma.6293.

[14] İ. İşcan, *New refinements for integral and sum forms of Hölder inequality,* J. Inequal. Appl., 2019 (2019) Article ID 304, 11 pages.

[15] B. C. Joshi, *Generalized convexity and mathematical programs,* Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 82 (2020), 151-160.

[16] M. Kadakal, İ. İşcan, P. Agarwal, M. Jleli, *Exponential trigonometric convex functions and Hermite-Hadamard type inequalities,* Math. Slovaca, 71 (2021), 43-56.

[17] A. Kashuri, T. M. Rassias, *Fractional trapezium-type inequalities for strongly exponentially generalized preinvex functions with applications,* Appl. Anal. Discrete Math., 14 (2020), 560-578.

[18] A. Kiliçman, W. Saleh, *Generalized preinvex functions and their applications,* Symmetry, 2018 (2018), Article ID 493, 13 pages.

[19] U. S. Kirmaci, M. E. Özdemir, *Some inequalities for mappings whose derivatives are bounded and applications to special means of real numbers,* Appl. Math. Lett., 17 (2004), 641-645.

[20] M. A. Latif, *On some new inequalities of Hermite-Hadamard type for functions whose derivatives are s-convex in the second sense in the absolute value,* Ukrainian Math. J., 67 (2016), 1552-1571.

[21] J. G. Liao, T. S. Du, *On some characterizations of sub-b-s-convex functions,* Filomat, 30 (2016), 3885-3895.

[22] J. G. Liao, T. S. Du, *Optimality conditions in sub-(b, m)-convex programming,* Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 79 (2017), 95-106.

[23] J. G. Liao, S. H. Wu, T. S. Du, *The Sugeno integral with respect to α-preinvex functions,* Fuzzy Sets and Systems, 379 (2020), 102-114.

[24] X. J. Long, J. W. Peng, *Semi-B-preinvex functions,* J. Optim. Theory Appl., 131 (2006), 301-305.

[25] K. Mehrez, P. Agarwal, *New Hermite-Hadamard type integral inequalities for convex functions and their applications,* J. Comput. Appl. Math., 350 (2019), 274-285.

[26] P. O. Mohammed, *Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals of a convex function with respect to a monotone function,* Math. Meth. Appl. Sci., 44 (2021), 2314-2324.

[27] E. R. Nwaeze, M. A. Khan, A. Ahmadian, M. N. Ahmad, A. K. Mahmood, *Fractional inequalities of the Hermite-Hadamard type for m-polynomial convex and harmonically convex functions,* AIMS Mathematics, 6 (2020), 1889-1904.

[28] S. Sezer, *The Hermite-Hadamard inequality for s-convex functions in the third sense,* AIMS Mathematics, 6 (2021), 7719-7732.

[29] S. Sezer, Z. Eken, G. Tınaztepe, G. Adilov, *p-convex functions and some of their properties*, Numer. Funct. Anal. Optim., 42 (2021), 443-459.

[30] W. B. Sun, Q. Lin, *New Hermite-Hadamard type inequalities for (α, m)-convex functions and applications to special means,* J. Math. Inequal., 11 (2017), 383-397.

[31] S. Suzuki, *Optimality conditions and constraint qualifications for quasiconvex programming,* J. Optim. Theory Appl., 183 (2019), 963-976.

[32] T. Toplu, M. Kadakal, İ. İşcan, *On n-polynomial convexity and some related inequlities,* AIMS Mathematics, 5 (2020), 1304-1318.

[33] B. Yu, J. G. Liao, T. S. Du, *Optimality and duality with respect to b-(ε, m)-convex programming,* Symmetry, 2018 (2018), Article ID 774, 16 pages.

[34] S. H. Wu, M. U. Awan, *Estimates of upper bound for a function associated with Riemann-Liouville fractional integral via h-convex functions,* J. Funct. Spaces, 2019 (2019), Article ID 9861790, 7 pages.

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