Novel concepts in fuzzy graphs

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Abstract. Today, fuzzy graphs have a variety of applications in other fields of study, including medicine, engineering, and psychology, and for this reason many researchers around the world are trying to identify their properties and use them in computer science as well as finding the shortest problem in a network. So, in this paper, some new fuzzy graphs are introduced and some properties of them are investigated. As a consequence of our results, some well-known assertions in the graph theory are obtained.

Keywords: fuzzy set, fuzzy graph, fuzzy line graph, fuzzy common neighborhood graph.

1. Introduction

The concept of graph theory was first introduced by Euler. In 1965, L. A. Zadeh discussed the fuzzy set [37]. Graphs are basically the bonding of objects. To emphasis on a real life problem, the objects are being bonded by some relations, such as friendship is the bonding of people. But when the ambiguousness or uncertainty in bonding exists, then the corresponding graph can be modeled as fuzzy graph model.

The first definition of a fuzzy graph was given by Kaufmann, which was based on Zadeh's fuzzy relations in 1973. A fuzzy graph has good capabilities in dealing with problems that cannot be explained by weight graphs. They have been able to have wide applications even in fields such as psychology and identifying people based on cancerous behaviors. One of the advantages of fuzzy graph is its flexibility in reducing time and costs in economic issues, which has been welcomed by all managers of institutions and companies. Fuzzy graph models are advantageous mathematical tools for dealing with combinatorial problems

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of various domains including operations research, optimization, social science, algebra, computer science, and topology. They are obviously better than graphical models due to natural existence of vagueness and ambiguity. Mordeson studied fuzzy line graphs and developed its basic properties, in 1993 [14]. The theory of fuzzy graph is growing rapidly, with numerous applications in many domains, including networking, communication, data mining, clustering, image capturing, image segmentation, planning, and scheduling. Rashmanlou et al. [21, 22, 25, 26, 27, 28] defined bipolar fuzzy graphs with categorical properties, product vague graphs, and shortest path problem in vague graphs. Akram et al. [1, 2] introduced certain types of vague graphs and strong intuitionistic fuzzy graphs. Borzooei et al. [4, 5, 6, 7, 8, 9, 10] investigated new concepts on vague graphs. Parvathi et al. [16, 17] introduced intuitionistic fuzzy graphs and domination in intuitionistic fuzzy graphs. Kou et al. [11] given novel description on vague graph with application in transportation systems. Samanta et al. [18, 19, 20, 23, 24]presented new definitions on fuzzy graphs. Kosari et al. [12] introduced vague graph structure with application in medical diagnosis. Talebi et al. [34, 35, 36] studied interval-valued intuitionistic fuzzy competition graph, and new concept of an intuitionistic fuzzy graph with applications. Rao et al. [29, 30, 31, 32] defined domination and equitable domination in vague graphs. Zeng et al. [38] investigated certain properties of single-valued neutrosophic graphs. In this paper, we introduce many basic notions concerning a fuzzy graph and investigate a few related properties.

First we go through some basic definitions from [14, 15]

Definition 1.1. *A fuzzy subset of a non-empty set S is a map $\sigma : S \to [0,1]$ which assigns to each element x in S a degree of membership $\sigma(x)$ in $[0,1]$ such that $0 \leq \sigma(x) \leq 1$.*

If S represents a set, a fuzzy relation μ on S is a fuzzy subset of $S \times S$. In symbols, $\mu : S \times S \to [0,1]$ such that $0 \leq \mu(x,y) \leq 1$ for all $(x,y) \in S \times S$.

Definition 1.2. *Let σ be a fuzzy subset of a set S and μ a fuzzy relation on S. Then μ is called a fuzzy relation on σ if $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in S$ where $\wedge$ denote minimum.*

Let V be a nonempty set. Define the relation $\sim$ on $V \times V$ by for all $(x,y),(u,v) \in V \times V$, $(x,y) \sim (u,v)$ if and only if $x = u$ and $y = v$ or $x = v$ and $y = u$. Then it is easily shown that $\sim$ is an equivalence relation on $V \times V$. For all $x, y \in V$, let $[(x,y)]$ denote the equivalence class of (x,y) with respect to $\sim$. Then $[(x,y)] = \{(x,y),(y,x)\}$. Let $\mathcal{E}_V = \{[(x,y)] | x, y \in V, x \neq y\}$. For simplicity, we often write $\mathcal{E}$ for $\mathcal{E}_V$ when V is understood. Let $E \subseteq \mathcal{E}$. A graph is a pair (V,E). The elements of V are thought of as vertices of the graph and the elements of E as the edges. For $x, y \in V$, we let xy denote $[(x,y)]$. Then clearly $xy = yx$. We note that graph (V,E) has no loops or parallel edges.

Definition 1.3. *A fuzzy graph $G = (V, \sigma_G, \mu_G)$ is a triple consisting of a nonempty set V together with a pair of functions $\sigma := \sigma_G : V \to [0,1]$ and $\mu := \mu_G : \mathcal{E} \to [0,1]$ such that for all $x, y \in V$, $\mu(xy) \leq \sigma(x) \wedge \sigma(y)$.*

The fuzzy set σ is called the fuzzy vertex set of G and μ the fuzzy edge set of G. Clearly μ is a fuzzy relation on σ.

Definition 1.4. *A path P in a fuzzy graph $G = (V, \sigma, \mu)$ is a sequence of distinct vertices $x_0, x_1, \cdots, x_n$ (except possibly x_0 and x_n) such that $\mu(x_{i-1}x_i) > 0$ for $i = 1, \cdots, n$. Here n is called the length of the path. We call P a cycle if $x_0 = x_n$ and $n \geq 3$. Two vertices that are joined by a path are called connected.*

Definition 1.5. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The degree $x \in V$ is denoted by $d_G(x)$ and defined as $d_G(x) = \sum\limits_{y \in V} \mu(xy)$.*

2. Introducing some new fuzzy graphs

In this section after introducing some new fuzzy graphs, we study some properties of them. These new fuzzy graphs and their properties are important not only as fuzzy graphs, but also for the crisp graph in the special case.

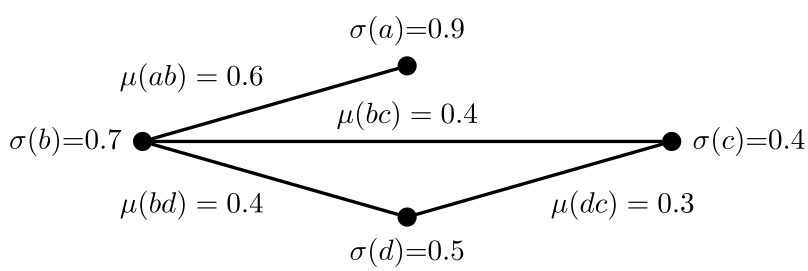
Definition 2.1. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. We define the complement of G by $\overline{G} = (\overline{V}, \overline{\sigma}, \overline{\mu})$ such that*

a) $\overline{V} = V$ and $\overline{\sigma}(v) = \sigma(v)$ for all $v \in V$;

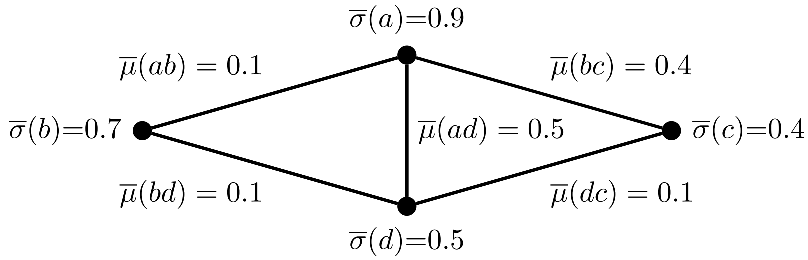
b) $\overline{\mu}(uv) = \sigma(u) \wedge \sigma(v) - \mu(uv)$, for all $u, v \in V$.

It is easy to show that $\overline{G}$ is a fuzzy graph on V.

Example 2.1. Let $V = \{a, b, c, d\}$ and $\sigma : V \to [0,1]$ be a map such that $\sigma(a) = 0.9$, $\sigma(b) = 0.7$, $\sigma(c) = 0.4$ and $\sigma(d) = 0.5$. Also, let $\mu : V \times V \to [0,1]$ be a map such that $\mu(ab) = 0.6$, $\mu(bc) = 0.4$, $\mu(bd) = 0.4$ and $\mu(dc) = 0.3$. We have the following diagram for the fuzzy graph $G = (V, \sigma, \mu)$.



Also, the diagram of the fuzzy graph $\overline{G} = (\overline{V}, \overline{\sigma}, \overline{\mu})$ is as follows:



Lemma 2.1. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then $\overline{\overline{G}} = G$.*

Proof. Suppose that $\overline{G} = (\overline{V}, \overline{\sigma}, \overline{\mu})$ and $\overline{\overline{G}} = (\overline{\overline{V}}, \overline{\overline{\sigma}}, \overline{\overline{\mu}})$. By the definition of the complement fuzzy graph, we have $\overline{\overline{V}} = \overline{V} = V$ and $\overline{\overline{\sigma}} = \overline{\sigma} = \sigma$. It suffices to prove that $\overline{\overline{\mu}}(uv) = \mu(uv)$ for all $u, v \in V$. We have

$$\begin{aligned}&\overline{\overline{\mu}}(uv) = \overline{\sigma}(u) \wedge \overline{\sigma}(v) - \overline{\mu}(uv)\\ &= \sigma(u) \wedge \sigma(v) - \overline{\mu}(uv) = \sigma(u) \wedge \sigma(v) - (\sigma(u) \wedge \sigma(v) - \mu(uv)) = \mu(uv). \quad \square\end{aligned}$$

Definition 2.2. *Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs such that $V_1 \cap V_2 = \varnothing$. Union of two fuzzy graphs G_1 and G_2 is denoted by $G_1 \cup G_2 = (V, \sigma, \mu)$ such that $V = V_1 \cup V_2$,*

$$\sigma(v) = \begin{cases} \sigma_1(v), & v \in V_1 \\ \sigma_2(v), & v \in V_2 \end{cases} \quad \textit{and} \quad \mu(uv) = \begin{cases} \mu_1(uv), & u, v \in V_1 \\ \mu_2(uv), & u, v \in V_2 \\ 0, & o.w \end{cases}$$

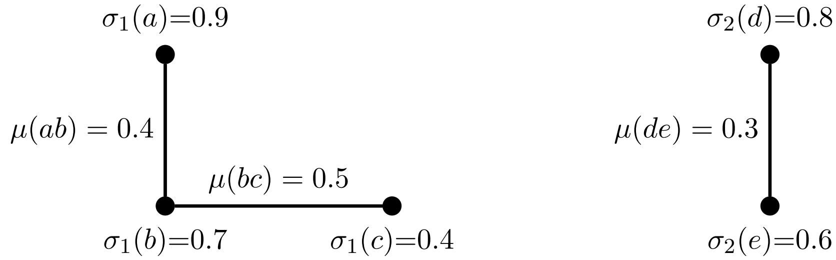
It is easy to see $G_1 \cup G_2$ is a fuzzy graph.

Definition 2.3. *Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs such that $V_1 \cap V_2 = \phi$. Sum of two fuzzy graphs G_1 and G_2 is denoted by $G_1 + G_2 = (V, \sigma, \mu)$ such that $V = V_1 \cup V_2$,*

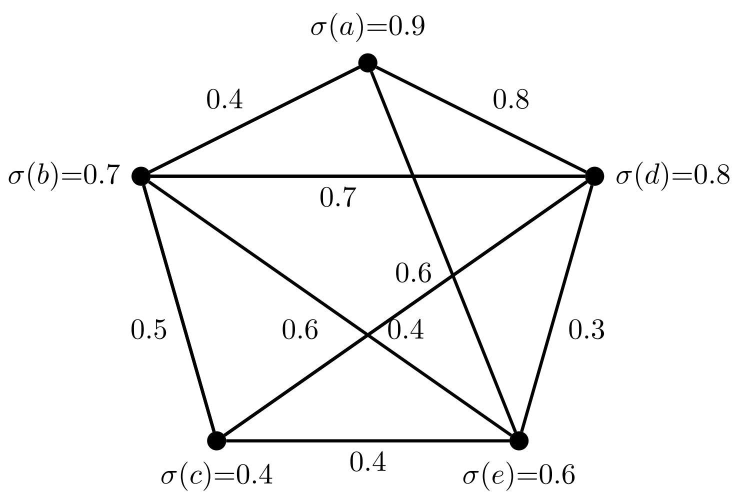
$$\sigma(v) = \begin{cases} \sigma_1(v), & v \in V_1 \\ \sigma_2(v), & v \in V_2 \end{cases} \quad \textit{and} \quad \mu(uv) = \begin{cases} \mu_1(uv), & u, v \in V_1 \\ \mu_2(uv), & u, v \in V_2 \\ \sigma_1(u) \wedge \sigma_2(v), & u \in V_1,\ v \in V_2 \end{cases}.$$

Example 2.2. Let $G_1 = (V_1, \sigma_1, \mu_1)$ be a fuzzy graph such that $V_1 = \{a, b, c\}$ and $\sigma_1 : V_1 \to [0, 1]$ and $\mu_1 : V_1 \times V_1 \to [0, 1]$ be maps such that $\sigma_1(a) = 0.9$, $\sigma_1(b) = 0.7$, $\sigma_1(c) = 0.4$, $\mu_1(ab) = 0.4$ and $\mu_1(bc) = 0.5$. Also, let $G_2 = (V_2, \sigma_2, \mu_2)$ be a fuzzy graph such that $V_2 = \{d, e\}$ and $\sigma_2 : V_2 \to [0, 1]$ and $\mu_2 : V_2 \times V_2 \to [0, 1]$ be maps such that $\sigma_2(d) = 0.8$, $\sigma_2(e) = 0.6$, $\mu_2(de) = 0.3$.

The fuzzy graphs G_1 and G_2 are drawn as follows, respectively:



By the definition of the sum of two graphs, $\mu(ab) = 0.4$, $\mu(ad) = 0.8$, $\mu(ae) = 0.6$, $\mu(de) = 0.3$, $\mu(be) = 0.6$, $\mu(bc) = 0.5$, $\mu(bd) = 0.7$, $\mu(ce) = 0.4$, $\mu(dc) = 0.4$ and the diagram of the fuzzy graph $G_1 + G_2$ is as follows:



Lemma 2.2. *Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs. Then*

a) $\overline{G_1 \cup G_2} = \overline{G_1} + \overline{G_2}$;

b) $\overline{G_1 + G_2} = \overline{G_1} \cup \overline{G_2}$.

Proof. Suppose that $G_1 \cup G_2 = (V, \sigma, \mu)$, $\overline{G_1 \cup G_2} = (\overline{V}, \overline{\sigma}, \overline{\mu})$, $\overline{G_1} = (\overline{V_1}, \overline{\sigma_1}, \overline{\mu_1})$, $\overline{G_2} = (\overline{V_2}, \overline{\sigma_2}, \overline{\mu_2})$ and $\overline{G_1} + \overline{G_2} = (V', \sigma', \mu')$. By the definition of the union and sum of two graphs, we have $V = \overline{V} = V_1 \cup V_2 = \overline{V_1} \cup \overline{V_2} = V'$ and $\overline{\sigma(v)} = \sigma(v)$ for all $v \in V$. It suffices to prove that $\mu_{\overline{G_1 \cup G_2}}(uv) = \mu'_{\overline{G_1} + \overline{G_2}}(uv)$ for all $uv \in \mathcal{E}$. We have

$$\mu_{\overline{G_1 \cup G_2}}(uv) = \sigma(u) \wedge \sigma(v) - \mu(uv) = \sigma(u) \wedge \sigma(v) - \begin{cases} \mu_1(uv), & u, v \in V_1 \\ \mu_2(uv), & u, v \in V_2 \\ 0, & o.w \end{cases}$$

$$= \begin{cases} \sigma_1(u) \wedge \sigma_1(v) - \mu_1(uv), & u, v \in V_1 \\ \sigma_2(u) \wedge \sigma_2(v) - \mu_2(uv), & u, v \in V_2 \\ \sigma_1(u) \wedge \sigma_2(v), & u \in V_1,\, v \in V_2 \end{cases}$$

$$= \begin{cases} \overline{\mu_1}(uv), & u, v \in V_1 \\ \overline{\mu_2}(uv), & u, v \in V_2 \\ \sigma_1(u) \wedge \sigma_2(v), & u \in V_1,\, v \in V_2 \end{cases} = \mu'_{\overline{G_1}+\overline{G_2}}(uv).$$

The other conclusion is proved similarly. □

Definition 2.4. *Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs. The Cartesian product of graphs G_1 and G_2 is denoted by $G_1 \times G_2 = (V, \sigma, \mu)$ is a fuzzy graph such that $V = V_1 \times V_2$,*

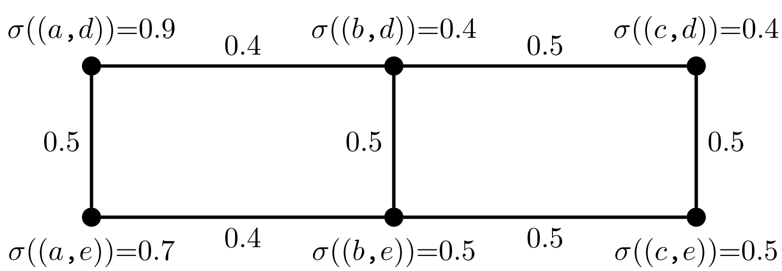
$$\sigma((u, v)) = \sigma_1(u) \vee \sigma_2(v),$$

where $\vee$ is denoted maximum and

$$\mu((u, v)(u', v')) = \begin{cases} \mu_2(vv'), & \text{if } u = u' \\ \mu_1(uu'), & \text{if } v = v' \\ 0, & o.w \end{cases}.$$

It is easy to show that $d_{G_1 \times G_2}((u, v)) = d_{G_1}(u) + d_{G_2}(v)$.

Example 2.3. Let G_1 and G_2 be the fuzzy graphs of Example 2.2. We have the following diagram for the fuzzy graph $G_1 \times G_2$.



Let $G = (V, \sigma, \mu)$ be a fuzzy graph the neighbor of vertex v is denoted by $N_G(v)$ and is defined as follows:

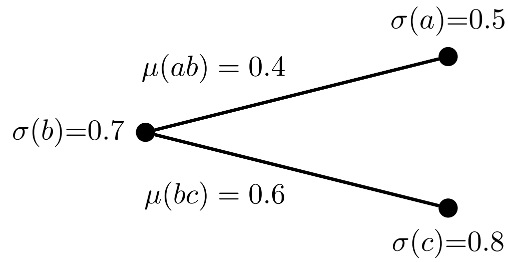
$$N_G(v) = \{u \in V \mid \mu(uv) > 0\}.$$

Definition 2.5. *The fuzzy common neighborhood graph or briefly fuzzy congraph of $G = (V, \sigma, \mu)$ is a fuzzy graph as $con(G) = (V, \omega, \lambda)$ such that $\omega(x) = \sigma(x)$ and*

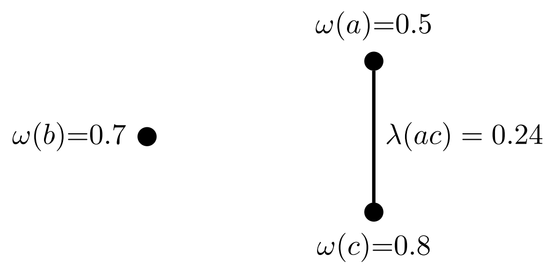
$$\lambda(uv) = \min_{x \in H}\{\mu(ux).\mu(vx)\},$$

where $H = N_G(u) \cap N_G(v)$.

Example 2.4. Let $V = \{a, b, c\}$ and $\sigma : V \to [0, 1]$ be a map such that $\sigma(a) = 0.5$, $\sigma(b) = 0.7$, $\sigma(c) = 0.8$. Also, let $\mu : V \times V \to [0, 1]$ be a map such that $\mu(ab) = 0.4$ and $\mu(bc) = 0.6$. We have the following diagram for the fuzzy graph $G = (V, \sigma, \mu)$.

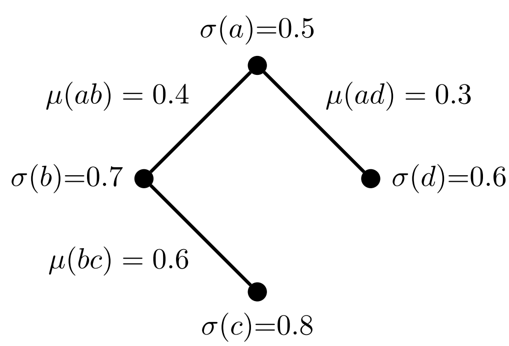


By using the definition of the fuzzy congraph, we have $\lambda(ab) = 0$, $\lambda(bc) = 0$, $\lambda(ac) = 0.24$ and $con(G) = (V, \omega, \lambda)$ is as follows:



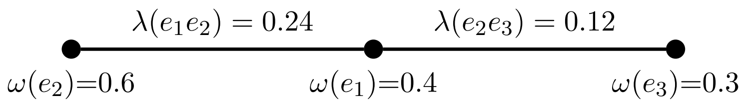
Definition 2.6. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The fuzzy line graph of G is a fuzzy graph as $L(G) = (\mathcal{E}, \omega, \lambda)$ such that $\omega(e) = \mu(uv)$ for all $e = uv \in \mathcal{E}$ and $\lambda(e_1e_2) = \omega(e_1).\omega(e_2)$ for all $e_1 = uv_1$, $e_2 = uv_2$ in $\mathcal{E}$.*

Example 2.5. Let $V = \{a, b, c, d\}$ and $\sigma : V \to [0, 1]$ be a map such that $\sigma(a) = 0.5$, $\sigma(b) = 0.7$, $\sigma(c) = 0.8$ and $\sigma(d) = 0.6$. Also, let $\mu : V \times V \to [0, 1]$ be a map such that $\mu(e_1) = \mu(ab) = 0.4$, $\mu(e_2) = \mu(bc) = 0.6$ and $\mu(e_3) = \mu(ad) = 0.3$. We have the following diagram for the fuzzy graph $G = (V, \sigma, \mu)$.



By using the definition of the fuzzy line graph, we have $\omega(e_1) = 0.4$, $\omega(e_2) = 0.6$, $\omega(e_3) = 0.3$, $\lambda(e_1e_2) = 0.24$, $\lambda(e_2e_3) = 0.12$ and the diagram of $L(G) = (\mathcal{E}, \omega, \lambda)$

is as follows:



Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $V = \{v_1, v_2, ..., v_p\}$, $\mathcal{E} = \{e_1, e_2, ..., e_q\}$ the vertex set and the edge set of G, respectively.

The *adjacency matrix* of fuzzy graph G is the $p \times p$ matrix $A_F = A_F(G)$ whose (i, j) entry denoted by a_{ij}, is defined by $a_{ij} = \mu(v_iv_j)$.

The (vertex-edge) *incidence matrix* of fuzzy graph G is the $p \times q$ matrix M_F, with rows indexed by the vertices and columns indexed by the edges, whose (i, j) entry denoted by m_{ij}, is defined as follows:

$$m_{ij} = \begin{cases} \mu(e_j), & \text{if } v_i \text{ is an endpoint of edge } e_j \\ 0, & \text{o. w} \end{cases}$$

The *fuzzy degree matrix* of G is the $p \times p$ matrix D_F whose (i, j) entry denoted by d_{ij}, is defined as follows:

$$d_{ij} = \begin{cases} \sum\limits_{v_k \in V} \mu^2(v_iv_k), & \text{if } i = j \\ 0, & \text{o. w} \end{cases}$$

The *edge matrix* of fuzzy graph G is the $q \times q$ matrix E_F whose (i, j) entry denoted by e_{ij}, is defined as follows:

$$e_{ij} = \begin{cases} \mu(e_i), & \text{if } i = j \\ 0, & \text{o. w} \end{cases}$$

Definition 2.7. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrix of size $m \times n$. Then we define $C = A \odot B$ is the $m \times n$ matrix whose (i, j) entry denoted by $a_{ij} \times b_{ij}$.*

Theorem 2.1. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph such that A_F, M_F and D_F are the adjacency, incidence and fuzzy degree matrices of G, respectively. Then*

$$M_F \times M_F^T = A_F \odot A_F + D_F.$$

Proof. Let $A_F = [a_{ij}]_{p \times p}$, $M_F = [m_{ij}]_{p \times q}$, $D_F = [d_{ij}]_{p \times p}$, $A_F \odot A_F = [t_{ij}]_{p \times p}$ and $M_F \times M_F^T = [b_{ij}]_{p \times p}$. First, let $i \neq j$. Then we get

$$\begin{aligned} b_{ij} &= \sum_{k=1}^{q} m_{ik}.m_{kj}^T = \sum_{k=1}^{q} m_{ik}.m_{jk} \\ &= \begin{cases} m_{ik'}.m_{jk'}, & \text{if } v_i \text{ and } v_j \text{ are endpoints of edge } e_{k'} \\ 0, & \text{o. w} \end{cases}, \end{aligned}$$

for some $1 \leq k' \leq q$. It follows that

$$\begin{aligned} b_{ij} &= \begin{cases} \mu^2(e_{k'}), & \text{if } v_i \text{ and } v_j \text{ are endpoints of edge } e_{k'} \\ 0, & \text{o. w} \end{cases} \\ &= a_{ij}.a_{ij} = t_{ij} = t_{ij} + 0 = t_{ij} + d_{ij}, \end{aligned}$$

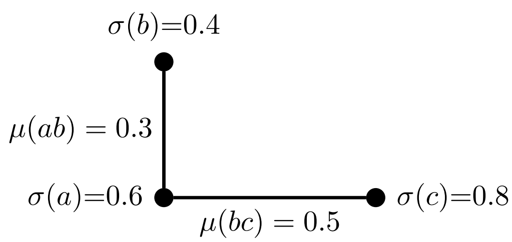
which proves our assertion.

If $i = j$, then

$$b_{ii} = \sum_{k=1}^{q} m_{ik}.m_{ki}^T = \sum_{k=1}^{q} m_{ik}.m_{ik} = \sum_{e_k = v_i v_t \in \mathcal{E}} \mu^2(e_k) = \sum_{v_t \in V} \mu^2(v_i v_t) = d_{ii}.$$

Then $b_{ii} = 0 + d_{ii} = a_{ii}.a_{ii} + d_{ii}$, which completes the proof. $\square$

Example 2.6. Let $G = (V, \sigma, \mu)$ be the following fuzzy graph:



By the definitions of the adjacency, incidence, and fuzzy degree matrix in the fuzzy graph, we have:

$$A_F = \begin{bmatrix} 0 & 0.3 & 0.5 \\ 0.3 & 0 & 0 \\ 0.5 & 0 & 0 \end{bmatrix}, M_F = \begin{bmatrix} 0.3 & 0.5 \\ 0.3 & 0 \\ 0 & 0.5 \end{bmatrix}, D_F = \begin{bmatrix} 0.34 & 0 & 0 \\ 0 & 0.09 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}.$$

It is easy to see that $M_F \times M_F^T = A_F \odot A_F + D_F$.

In the fuzzy graph $G = (V, \sigma, \mu)$, if for every $v \in V$ set $\sigma(v) = 1$ and for every edge e set $\mu(e) = 1$, then we can assume that every crisp graph is a fuzzy graph. Therefore, we can obtain similar results for crisp graphs. So, we have the following result which is well-known in the graph theory [3].

Corollary 2.1. *Let G be a graph and A, M and D be the adjacency, incidence and degree matrix of G, respectively. Then*

$$M \times M^T = A + D.$$

The next theorem characterized the degree of every vertex in the fuzzy con-graph.

Theorem 2.2. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $con(G) = (V, \omega, \lambda)$ the fuzzy congraph of G. If G has no cycles of size 4, then*

$$d_{con(G)}(v) = \sum_{u \in V} \mu(uv) \times d_G(u) - \sum_{u \in V} \mu^2(uv), \qquad v \in V.$$

Proof.

$$d_{con(G)}(v) = \sum_{u \in V} \lambda(uv) = \sum_{u \in V} \min_{x \in H}\{\mu(vx) \times \mu(xu)\},$$

where $H = N_G(u) \cap N_G(v)$. Since G has no cycle of size 4, it follows that $H \subseteq \{w\}$ for some $w \in V$. Therefore,

$$\begin{aligned} d_{con(G)}(v) &= \sum_{vw, wu \in \mathcal{E}(G)} \mu(vw) \times \mu(wu) \\ &= \sum_{vw \in \mathcal{E}(G)} \mu(vw) \times \sum_{u \in V} \mu(uw) - \sum_{w \in V} \mu^2(vw) \\ &= \sum_{w \in V} \mu(vw) \times d_G(w) - \sum_{w \in V} \mu^2(vw) \\ &= \sum_{u \in V} \mu(vu) \times d_G(u) - \sum_{u \in V} \mu^2(vu). \qquad \square \end{aligned}$$

From the above theorem, one can immediately deduce the following corollary, which has proved in [13].

Corollary 2.2. *Let $G = (V, E)$ be a graph. If G has no cycles of size 4, then*

$$d_{con(G)}(v) = \sum_{u \in N_G(v)} d_G(u) - d_G(v), \qquad v \in V.$$

The next theorem characterize the degree of every vertex in the fuzzy line graph.

Theorem 2.3. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $L(G) = (\mathcal{E}, \omega, \lambda)$ its fuzzy line graph. Then*

$$d_{L(G)}(e) = \mu(v_i v_j)(d_G(v_i) + d_G(v_j) - 2\mu(v_i v_j)), \qquad e = v_i v_j \in \mathcal{E}(G).$$

Proof. For an arbitrary edge $e = v_i v_j \in \mathcal{E}(G)$, set $e' = v_s v_t$, where $s \neq i$ and $t \neq j$. We have

$$d_{L(G)}(e) = \sum_{e \neq e'} \lambda(ee') = \sum_{e \neq e' = v_i v_t \in \mathcal{E}(G)} \lambda(ee') + \sum_{e \neq e' = v_j v_s \in \mathcal{E}(G)} \lambda(ee')$$

$$
\begin{aligned}
&= \sum_{v_t \neq v_i, t\neq j} \mu(v_iv_j)\mu(v_iv_t) + \sum_{v_s \neq v_j, s\neq i} \mu(v_iv_j)\mu(v_jv_s) \\
&= \mu(v_iv_j) \sum_{v_t \neq v_i, t\neq j} \mu(v_iv_t) + \mu(v_iv_j) \sum_{v_s \neq v_j, s\neq i} \mu(v_jv_s) \\
&= \mu(v_iv_j)(\sum_{v_t\in V} \mu(v_iv_t) - \mu(v_iv_j)) + \mu(v_iv_j)(\sum_{v_s\in V} \mu(v_jv_s) - \mu(v_iv_j)) \\
&= \mu(v_iv_j)(d_G(v_i) + d_G(v_j) - 2\mu(v_iv_j)). \qquad \square
\end{aligned}
$$

From the above theorem, we can conclude the following result, which is trivial in the line graph.

Corollary 2.3. *Let $G = (V, E)$ be a graph and $L(G) = (E, W)$ the line graph of G. Then*

$$d_{L(G)}(e) = d_G(u) + d_G(v) - 2, \qquad e \in E.$$

Theorem 2.4. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph with the incidence and edge matrix M_F and E_F, respectively. Suppose that $L(G) = (\mathcal{E}, \omega, \lambda)$ is the fuzzy line graph of G with the adjacency matrix L_F. Then*

$$M_F^T \times M_F = L_F + 2E_F \odot E_F.$$

Proof. Let $M_F = [m_{ij}]_{p\times q}$, $L_F = [l_{ij}]_{q\times q}$, $E_F = [e_{ij}]_{q\times q}$ and $M_F^T \times M_F = [b_{ij}]_{q\times q}$. For $i \neq j$, we get

$$
\begin{aligned}
b_{ij} &= \sum_{k=1}^{p} m_{ik}^T.m_{kj} = \sum_{k=1}^{p} m_{ki}.m_{kj} \\
&= \begin{cases} m_{k'i}.m_{k'j}, & \text{if } v_{k'} \text{ is an endpoint of edges } e_i \text{ and } e_j \\ 0, & \text{otherwise} \end{cases},
\end{aligned}
$$

for some $1 \leq k' \leq p$. Hence

$$
b_{ij} = \begin{cases} \mu(e_i).\mu(e_j), & \text{if } v_{k'} \text{ is an endpoint of edges } e_i \text{ and } e_j \\ 0, & \text{otherwise;} \end{cases}
$$

thus

$$
\begin{aligned}
b_{ij} &= \begin{cases} \lambda(e_ie_j), & \text{if } v_{k'} \text{ is an endpoint of edges } e_i \text{ and } e_j \\ 0, & \text{otherwise} \end{cases} \\
&= l_{ij} = l_{ij} + 0 = l_{ij} + 2e_{ij}.e_{ij}.
\end{aligned}
$$

which proves our assertion.

Now, suppose that $i = j$ and $e_i = v_tv_s$. We have

$$b_{ii} = \sum_{k=1}^{p} m_{ik}^T.m_{kj} = \sum_{k=1}^{p} m_{ki}.m_{ki} = m_{ti}^2 + m_{si}^2 = 2\mu^2(e_i) = 2e_{ii}.e_{ii} = l_{ii} + 2e_{ii}.e_{ii}.$$

Therefore, the proof is complete. $\square$

Example 2.7. Let G be the fuzzy graph of example 2.5. By the definitions of the incidence, edge matrix, and the adjacency matrix of the line graph of G, we have the following matrices:

$$M_F = \begin{bmatrix} 0.4 & 0 & 0.3 \\ 0.4 & 0.6 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, L_F = \begin{bmatrix} 0 & 0.24 & 0.12 \\ 0.24 & 0 & 0 \\ 0.12 & 0 & 0 \end{bmatrix}, E_F = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}.$$

It is easy to check that $M_F^T \times M_F = L_F + 2E_F \odot E_F$.

From the above theorem, we deduce the following result, which has proved in [33].

Corollary 2.4. *Let G be a graph with the incidence matrix M and the adjacency matrix line graph L. Then*

$$M^T \times M = L + 2I_{q\times q}.$$

Theorem 2.5. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph with the adjacency and fuzzy degree matrix A_F and D_F, respectively. Suppose that $con(G) = (V, \omega, \lambda)$ is the fuzzy congraph of G with the adjacency matrix B_F. If G has no cycles of size 4, then*

$$A_F^2 = B_F + D_F.$$

Proof. Let $A_F = [a_{ij}]_{p\times p}$, $B_F = [b_{ij}]_{p\times p}$, $D_F = [d_{ij}]_{p\times p}$ and $A_F^2 = [c_{ij}]_{p\times p}$. For $i \neq j$, we have

$$c_{ij} = \sum_{k=1}^{p} a_{ik}.a_{kj} = \begin{cases} a_{it}.a_{tj}, & \text{if the vertices } v_i \text{ and } v_j \text{ are connected to } v_t \\ 0, & \text{o. w} \end{cases}$$

for some $1 \leq t \neq i, j \leq p$. It follows that

$$c_{ij} = \begin{cases} \mu(v_i v_t).\mu(v_t v_j), & \text{if the vertices } v_i \text{ and } v_j \text{ are connected to } v_t \\ 0, & \text{o. w.} \end{cases}$$

Since G has no cycles of size 4, then

$$c_{ij} = \begin{cases} \lambda(v_i v_j), & \text{if the vertices } v_i \text{ and } v_j \text{ are connected to } v_t \\ 0, & \text{o. w.} \end{cases}$$

Thus, $c_{ij} = b_{ij} + 0 = b_{ij} + d_{ij}$, which proves our assertion in this case.

If $i = j$, then

$$c_{ii} = \sum_{k=1}^{p} a_{ik}.a_{ki} = \sum_{k=1}^{p} a_{ik}^2 = \sum_{v_k \in V} \mu^2(v_i v_k) = d_{ii} = b_{ii} + d_{ii}.$$

This completes the proof of the theorem. □

Example 2.8. Let G be the fuzzy graph of Example 2.4. By the definitions of the adjacency matrix of G, $con(G)$ and the fuzzy degree matrix of G, we have the following matrices:

$$A_F = \begin{bmatrix} 0 & 0.4 & 0 \\ 0.4 & 0 & 0.6 \\ 0 & 0.6 & 0 \end{bmatrix}, \quad B_F = \begin{bmatrix} 0 & 0 & 0.24 \\ 0 & 0 & 0 \\ 0.24 & 0. & 0 \end{bmatrix}, \quad D_F = \begin{bmatrix} 0.16 & 0 & 0.24 \\ 0 & 0.52 & 0 \\ 0 & 0. & 0.36 \end{bmatrix}.$$

It is easy to see that $A_F^2 = B_F + D_F$.

Corollary 2.5. *Let G be a graph such that A and B are the adjacency matrices of G and $con(G)$, respectively. If G has no cycles of size 4, then $A^2 = B + D$, where D is the degree matrix of G.*

3. Conclusion

It is well known that fuzzy graphs are among the most ubiquitous models of both natural and humman-made structures. They can be used to model many types of relations and process dynamics in computer science, biological, social systems and physical. Theoretical concepts of fuzzy graphs are highly utilized by computer science applications. Especially in research areas of computer science such as data mining, image segmentation, clustering, image capturing and networking. So, in this paper, some new fuzzy graphs are presented and some properties of them are studied. As a consequence of our results, some well-known assertions in the graph theory are given. in our future work, we will introduce cubic vague fuzzy graphs and define new operations such as strong product, direct product, lexicographic product, union, and composition on it.

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