Lebesgue's theorem and Egoroff's theorem for complex uncertain sequences

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Abstract. In this paper, within framework uncertain theory, we investigate Lebesgue's theorem, Egoroff's theorem and Riesz's theorem for complex uncertain sequences.
Keywords: uncertain theory, strongly order continuous, Lebesgue type theorems, Riesz type theorem.

1. Introduction

Uncertainty theory was initiated by Liu [2] in 2007 and advanced by Liu [3] in 2011 which based on an uncertain measure which supplies normality, duality, subadditivity, and product axioms. Recently, uncertainty theory has effectively been applied to uncertain programming (see, e,g., Liu [4], Liu and Chen [5]), uncertain risk analysis (see, e.g., Liu [6]), uncertain calculus (see, e.g., Liu [7]) and uncertain statistics (see, e.g., Tripathy and Nath [8]), etc.

Peng [9] proposed the notions of complex uncertain variables that are measurable functions from uncertainty spaces to the set of complex numbers. As convergence of sequences plays an essential role in the basic theory of mathematics, there are many mathematicians who have worked these in the field of uncertain measure. Liu [2] presented convergence in measure, convergence in mean, convergence almost surely (a.s.) and convergence in distribution in 2007. You [12] gave a kind type of convergence called convergence uniformly almost surely (u.a.s.) and proved the relationships among the convergence notions. Based on these concepts, the convergence of complex uncertain sequences was first worked by Chen, Ning and Wang [13]. Tripathy and Nath [8] investigated the statistical convergence concepts of complex uncertain sequences.

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Several kinds of convergence were investigated for sequence of measurable functions on a measure space, and fundamental relations between these types were examined [14]. Fuzzy measure theory is a generalisation of classical measure theory. This generalisation is acquired by exchanging the additivity axiom of classical measures with weak axioms of monotonicity and continuity [15]. As detailed in [16, 17, 18], several generalizations of Lebesgue's theorem, Egoroff's theorem and Riesz's theorem for sequence of measurable functions on classical measure spaces hold for fuzzy measures with the autocontinuity and finiteness.

This paper is devoted to presenting classical theorems such as Lebesgue's theorem, Egoroff's theorem and Riesz's theorem for complex uncertain sequences in uncertain theory.

2. Preliminaries

First, some basic notions and theorems in uncertainty theory are given, which are utilized in this paper.

Definition 2.1. *Assume that $\mathcal{L}$ be a σ-algebra on a non-empty set Γ. A set function $\mathcal{M}$ is named an uncertain measure if it supplies the subsequent axioms:*

(*i*) $\mathcal{M}\left\{\Gamma\right\} = 1$;

(*ii*) $\mathcal{M}\left\{\Lambda\right\} + \mathcal{M}\left\{\Lambda^c\right\} = 1$ *for any* $\Lambda \in \Gamma$

(*iii*) *For all countable sequence of* $\{\Lambda_p\} \subset \mathcal{L}$*, we obtain*

$$\mathcal{M}\left\{\bigcup_{p=1}^{\infty}\Lambda_p\right\} \leq \sum_{p=1}^{\infty}\mathcal{M}\left\{\Lambda_p\right\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is named an uncertainty space, and every element Λ in $\mathcal{L}$ is known as an event.

Definition 2.2. *A complex uncertain variable is a measurable function from the space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers, namely, for any Borel set of T of complex numbers, the set*

$$\{\zeta \in T\} = \{\gamma \in \Gamma : \zeta\left(\gamma\right) \in T\}$$

is an event.

Definition 2.3. *The sequence $\{\zeta_w\}$ is named to be convergent a.s. to ζ provided that there is an event Λ with $\mathcal{M}\left\{\Lambda\right\} = 1$ such that*

$$\lim_{w\to\infty} \|\zeta_w(\gamma) - \zeta(\gamma)\| = 0,$$

for every $\gamma \in \Lambda$.

Definition 2.4. *The sequence* $\{\zeta_w\}$ *is named to be convergent u.a.s. to* ζ *provided that there is a* $\{R_k\}$, $\mathcal{M}\{R_k\} \to 0$ *such that* $\{\zeta_w\}$ *converges uniformly to* ζ *in* $R_k^c = \Gamma - R_k$, *for any fixed* $k \in \mathbb{N}$.

Let T be an abstract space. $\mathcal{F}$ a σ-algebra of subsets of T, X a real normed space with the origin 0, $\mathcal{P}_0(X)$ the family of all nonvoid subsets of X; $\mathcal{P}_f(X)$ the family of closed, nonvoid sets of X and h the Hausdorff pseudometric on $\mathcal{P}_f(X)$ given by:

$$h(M;N) = \max\{e(M,N); e(N,M)\}, \text{ for every } M,N \in \mathcal{P}_f(X),$$

where $e(M,N) = \sup_{x\in X} d(x,N)$ is the excess of M over N.

Definition 2.5 ([1, 10, 11])**.** *A set multifunction* $\mu : \mathcal{F} \to \mathcal{P}_f(X)$ *is said to be:*

(*i*) *continuous from below if* $\lim_{n\to\infty} h(\mu(A_n), A) = 0$, *for each increasing sequence of sets* $(A_n)_n \subset \mathcal{F}$, *with* $A_n \nearrow A$.

(*ii*) *continuous from above if* $\lim_{n\to\infty} h(\mu(A_n), A) = 0$, *for each decreasing sequence of sets* $(A_n)_n \subset \mathcal{F}$, *with* $A_n \searrow A$.

(*iii*) *order continuous if* $\lim_{n\to\infty} |\mu(A_n)| = 0$, *for every sequence of sets* $(A_n)_n \subset \mathcal{F}$, *with* $A_n \searrow \emptyset$.

(*iv*) *strongly order continuous if* $\lim_{n\to\infty} |\mu(A_n)| = 0$, *for every sequence of sets* $(A_n)_n \subset \mathcal{F}$, *with* $A_n \searrow A$ *and* $\mu(A_n) = \{0\}$.

3. Main results

The aim of this study is to examine Lebesgue's theorem, Egoroff's theorem and Riesz's theorem in uncertain measure theory. Throughout the study, assume $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, Λ_w and Λ are both events in $\mathcal{L}$. Now, we give two notions of uncertain measure $\mathcal{M}$.

Definition 3.1. $\mathcal{M}$ *is named strongly order continuous, if it supplies that* $\lim_{w\to\infty} \mathcal{M}(\Lambda_w) = 0$ *whenever* $\Lambda_w \searrow \Lambda$ *and* $\mathcal{M}(\Lambda) = 0$.

Definition 3.2. $\mathcal{M}$ *is named strongly continuous at* Γ, *if it supplies that*

$$\lim_{w\to\infty} \mathcal{M}(\Lambda_w) = 1$$

whenever $\Lambda_w \nearrow \Lambda$ *and* $\mathcal{M}(\Lambda) = 1$.

Theorem 3.1 (Lebesgue's theorem)**.** *Assume that* $\{\zeta_w\}$ *be a complex uncertain sequence and* ζ *be a complex uncertain variable in* $(\Gamma, \mathcal{L}, \mathcal{M})$, *which supply the subsequent condition that* $\{\zeta_w\}$ *converges almost surely (a.s.) to* ζ. *Then,* $\{\zeta_w\}$ *converges in measure to* ζ *iff* $\mathcal{M}$ *is strongly order continuous.*

Proof. Presume that the sequence $\{\zeta_w\}$ converges to ζ a.s., and take H as the set of these points $\gamma \in \Gamma$ at which $\zeta_w(\gamma)$ does not convergence to $\zeta(\gamma)$, hen

$$H = \bigcup_{p=1}^{\infty}\bigcap_{w=1}^{\infty}\bigcup_{r=w}^{\infty}\left\{\gamma : \|\zeta_r(\gamma) - \zeta(\gamma)\| \geq \frac{1}{p}\right\}$$

and $\mathcal{M}(H) = 0$. In addition, we get

$$\mathcal{M}\left(\bigcap_{w=1}^{\infty}\bigcup_{r=w}^{\infty}\left\{\gamma : \|\zeta_r(\gamma) - \zeta(\gamma)\| \geq \frac{1}{p}\right\}\right) = 0$$

for any $p \geq 1$. If we accept

$$\Lambda_w^{(p)} = \bigcup_{r=w}^{\infty}\left\{\gamma : \|\zeta_r(\gamma) - \zeta(\gamma)\| \geq \frac{1}{p}\right\}$$

and

$$\Lambda^{(p)} = \bigcap_{w=1}^{\infty}\bigcup_{r=w}^{\infty}\left\{\gamma : \|\zeta_r(\gamma) - \zeta(\gamma)\| \geq \frac{1}{p}\right\}$$

for any $p \geq 1$, then

$$\bigcup_{r\geq w}^{\infty}\left\{\gamma : \|\zeta_r(\gamma) - \zeta(\gamma)\| \geq \frac{1}{p}\right\} \searrow \bigcap_{w=1}^{\infty}\bigcup_{r=w}^{\infty}\left\{\gamma : \|\zeta_r(\gamma) - \zeta(\gamma)\| \geq \frac{1}{p}\right\}, \ (w \to \infty)$$

and $\mathcal{M}(\Lambda^{(p)}) = 0$. According to strongly order continuity of $\mathcal{M}$, we can acquire $\lim_{w\to\infty} \mathcal{M}(\Lambda_w^{(p)}) = 0$ for any $p \geq 1$ and, so

$$\lim_{w\to\infty} \mathcal{M}\left(\left\{\gamma : \|\zeta_w(\gamma) - \zeta(\gamma)\| \geq \frac{1}{p}\right\}\right) \leq \lim_{w\to\infty} \mathcal{M}\left(\Lambda_w^{(p)}\right) = 0, \forall p \geq 1.$$

This demonstrates that $\{\zeta_r\}$ converges in measure to ζ. For any sequence $\{\Lambda_w\}_w$ of events with $\Lambda_w \searrow \Lambda$ and $\mathcal{M}(\Lambda) = 0$, we determine a complex uncertain sequence $\{\zeta_w\}$ by

$$\zeta_w(\gamma) = \begin{cases} 0, & \text{if } \gamma \in \Gamma - \Lambda_w, \\ 1, & \text{if } \gamma \in \Lambda_w \end{cases}$$

for any $w \geq 1$. It is easy to understand that $\{\zeta_w\}$ converges to 0 a.s. If $\{\zeta_w\}$ converges to 0 in measure, then we can acquire

$$\lim_{w\to\infty} \mathcal{M}(\Lambda_w) \leq \lim_{n\to\infty} \mathcal{M}\left(\left\{\gamma : \zeta_w(\gamma) \geq \frac{1}{2}\right\}\right) = 0.$$

As a result, $\mathcal{M}$ is strongly order continuous. □

Now, we generalize Egoroff's theorem in classical measure theory to uncertain measure theory.

Definition 3.3. *$\mathcal{M}$ is called to have feature (S), if for any sequence $\{\Lambda_w\}_w$ of events with $\lim_{w\to\infty}\mathcal{M}(\Lambda_w)=0$, there is a subsequence $\{\Lambda_{w_i}\}_i$ of $\{\Lambda_w\}_w$ such that $\mathcal{M}(\limsup\Lambda_{w_i})=0$.*

Theorem 3.2 (Egoroff's theorem)**.** *Assume that $\{\zeta_w\}$ be a complex uncertain sequence and ζ be a complex uncertain variable in $(\Gamma,\mathcal{L},\mathcal{M})$. If $\mathcal{M}$ is strongly order continuous and has feature (S), then*

$$\zeta_w\to\zeta(a.s.)\Rightarrow\zeta_w\to\zeta(u.a.s.)$$

Proof. Presume that $\mathcal{M}$ is strongly order continuous and has feature (S). Take H as the set of points $\gamma\in\Gamma$ whenever $\{\zeta_w\}$ does not convergence to ζ. Then, $\mathcal{M}(H)=0$ and $\{\zeta_w\}$ converges a.s. to ζ on $\Gamma-H$. If we indicate

$$H_w^{(r)}=\bigcap_{i=w}^{\infty}\left\{\gamma\in\Gamma:\|\zeta_i(\gamma)-\zeta(\gamma)\|<\frac{1}{r}\right\}$$

for any $r\geq 1$, then $H_w^{(r)}$ is increasing in w for all fixed r, and we obtain

$$\Gamma-H=\bigcap_{r=1}^{\infty}\bigcup_{w=1}^{\infty}H_w^{(r)}.$$

As for any fixed $r\geq 1$, $\Gamma-H\subseteq\bigcup_{w=1}^{\infty}H_w^{(r)}$, we get

$$\Gamma-H_w^{(r)}\searrow\bigcap_{w=1}^{\infty}\left(\Gamma-H_w^{(r)}\right).$$

Noting that $\bigcap_{w=1}^{\infty}(\Gamma-H_w^{(r)})\subset H$ for any fixed $r\geq 1$, so $\mathcal{M}(\bigcap_{w=1}^{\infty}(\Gamma-H_w^{(r)}))=0$ $(r=1,2,...)$. By utilizing the strong order continuity of $\mathcal{M}$, we get

$$\lim_{w\to\infty}\mathcal{M}\left(\Gamma-H_w^{(r)}\right)=0,\ \forall r\geq 1.$$

So, there is a subsequence $\{\Gamma-H_{w(r)}^{(r)}\}_r$ of $\{\Gamma-H_w^{(r)}:w,r\geq 1\}$ supplying

$$\mathcal{M}\left(\Gamma-H_{w(r)}^{(r)}\right)\leq\frac{1}{r},\ \forall r\geq 1$$

and so

$$\lim_{w\to\infty}\mathcal{M}\left(\Gamma-H_{w(r)}^{(r)}\right)=0.$$

By applying the feature (S) of $\mathcal{M}$ to the sequence $\{\Gamma-H_{w(r)}^{(r)}\}_r$, then there is a subsequence of $\{\Gamma-H_{w(r_i)}^{(r_i)}\}_i$ of $\{\Gamma-H_{w(r)}^{(r)}\}_r$ such that

$$\mathcal{M}\left(\overline{\lim_{i\to\infty}}\left(\Gamma-H_{w(r_i)}^{(r_i)}\right)\right)=0$$

and $r_1 < r_2 < ....$
At the same time, since

$$\left(\bigcup_{i=t}^{\infty}\left(\Gamma - H_{w(r_i)}^{(r_i)}\right)\right) \searrow \overline{\lim_{i\to\infty}}\left(\Gamma - H_{w(r_i)}^{(r_i)}\right)$$

so, by utilizing the strong order continuity of $\mathcal{M}$, we get

$$\lim_{t\to\infty} \mathcal{M}\left(\bigcup_{i=t}^{\infty}\left(\Gamma - H_{w(r_i)}^{(r_i)}\right)\right) = 0.$$

For any $\rho > 0$, we take t_0 such that $\mathcal{M}(\bigcup_{i=t_0}^{\infty}(\Gamma - H_{w(r_i)}^{(r_i)})) < \rho$, namely, $\mathcal{M}(\Gamma - \bigcap_{i=t_0}^{\infty} H_{w(r_i)}^{(r_i)}) < \rho$.

Take $H_\rho = \bigcap_{i=t_0}^{\infty} H_{w(r_i)}^{(r_i)}$, then $\mathcal{M}(\Gamma - H_\rho) < \rho$. Now, we need to demonstrate that $\{\zeta_w\}$ converges to ζ on H_ρ uniformly a.s. Since

$$H_\rho = \bigcap_{i=t_0}^{\infty}\bigcap_{j=w(r_i)}^{\infty}\left\{\gamma \in \Gamma : \|\zeta_i(\gamma) - \zeta(\gamma)\| < \frac{1}{r_i}\right\},$$

therefore, for any fixed $i \geq k_0$,

$$H_\rho \subset \bigcap_{j=w(r_i)}^{\infty}\left\{\gamma \in \Gamma : \|\zeta_j(\gamma) - \zeta(\gamma)\| < \frac{1}{r_i}\right\}.$$

For any given $\sigma > 0$, we take $i_0\,(\geq t_0)$ such that $\frac{1}{r_{i_0}} < \sigma$. Thus, as $j > w(r_{i_0})$, for any $\gamma \in H_\rho$, $\|\zeta_j(\gamma) - \zeta(\gamma)\| < \frac{1}{r_{i_0}} < \sigma$. This denotes that $\{\zeta_w\}$ converges to ζ on Γ_ρ uniformly a.s. The proof of the theorem is finalized. □

Definition 3.4. *$\mathcal{M}$ is named order continuous if it supplies that* $\lim_{w\to\infty}\mathcal{M}(\Lambda_w) = 0$ *whenever* $\Lambda_w \searrow \emptyset$.

Theorem 3.3. *Let $\mathcal{M}$ be an uncertain measure, assume that $\{\zeta_w\}$ be a complex uncertain sequence and ζ be a complex uncertain variable in $(\Gamma, \mathcal{L}, \mathcal{M})$. $\zeta_w \to \zeta(a.s.)$ implies $\zeta_w \to \zeta(u.a.s.)$, then $\mathcal{M}$ is strongly order continuous and hence order continuous.*

Proof. For any decreasing sequence $\{\Lambda_w\}_w$ of events with $\Lambda_w \searrow \Lambda$ and $\mathcal{M}(\Lambda) = 0$, we consider a complex uncertain sequence $\{\zeta_w\}$ as

$$\zeta_w(\gamma) = \begin{cases} 0, & \text{if } \gamma \in \Gamma - \Lambda_w, \\ 1, & \text{if } \gamma \in \Lambda_w \end{cases}$$

for any $w \geq 1$. It is easy to obtain that $\zeta_w \to 0$ $(a.s.)$. If $\zeta_w \to 0$ $(u.a.s.)$, then we can acquire for any $\sigma > 0$,

$$\lim_{w\to\infty} \mathcal{M}\{\gamma : \|\zeta_w(\gamma)\| \geq \sigma\} = 0.$$

As a result

$$\lim_{w\to\infty} \mathcal{M}\left(\Lambda_w\right) = \lim_{w\to\infty} \mathcal{M}\left\{\gamma : \zeta_w\left(\gamma\right) \geq \frac{1}{2}\right\} = 0.$$

This gives $\mathcal{M}$ is strongly order continuous and hence order continuous. □

Theorem 3.4 (Riesz's theorem)**.** *Assume that $\mathcal{M}$ be an uncertain measure with the feature (S). If $\{\zeta_w\}$ converges to ζ in measure, then there is a subsequence $\{\zeta_{w_r}\}_r$ of $\{\zeta_w\}_w$ such that $\zeta_{w_r} \to \zeta(a.s.)$.*

Proof. Let $\{\zeta_w\}$ converges to ζ in measure. Then

$$\lim_{w\to\infty} \mathcal{M}\left\{\gamma : \left\|\zeta_w\left(\gamma\right) - \zeta\left(\gamma\right)\right\| \geq \frac{1}{r}\right\} = 0,\ \forall r \geq 1.$$

If we take $\Lambda_w^{(r)} = \left\{\gamma : \left\|\zeta_w\left(\gamma\right) - \zeta\left(\gamma\right)\right\| \geq \frac{1}{r}\right\}$, then there is a subsequence $\{w_r\}_r$ such that $\mathcal{M}(\Lambda_{w_r}^{(r)}) \leq \frac{1}{r}$ for any $r \geq 1$. Since $\mathcal{M}$ has the feature (S), there is a subsequence $\{\Lambda_{w_{r_i}}^{(r_i)}\}$ of $\{\Lambda_{w_r}^{(r)}\}$ such that $\mathcal{M}(\overline{\lim}_{i\to\infty}\Lambda_{w_{r_i}}^{(r_i)}) = 0$. This gives that $\zeta_{w_{r_i}} \to \zeta(a.s.)$. □

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