Some aspects of the vertex-order graph

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Abstract. The vertex-order graph of the finite cyclic group G is based on its components C_d of the vertex-order graph $\Im(G)$, whose vertices are of order $'d'$ as the divisors of the order of the group G. The important properties of the vertex-order graph and its complements namely girth, radius, diameter, clique number, independence number and rank are derived. Further, the complement $\overline{\Im(G)}$ of the vertex-order graph is proved as a complete t-partite graph and shown with an example. Later, we compute the first, second and third Zagreb indices of the graph $\Im(G)$, $\overline{\Im(Z_p)}$ and $\overline{\Im(Z_{pq})}$.
Keywords: vertex-order graph, complete t-partite, Zagreb index.

1. Introduction

Group theoretical facts with disconnected graph will yield the finest application in the real world problems like protein-protein interaction, genetically disorders, existence of new virus with pandemic potential, handling drug discovery situation etc., in the medical science field. Over the past four decades, researchers developed enormous amount of applications in the area of algebraic graph theory, especially algebraic facts with connected graphs [2, 3, 13].

A graph H is said to be connected if there exists a path between every pair of vertices; Otherwise, the graph is disconnected. A disconnected graph consists of two or more connected subgraphs of H. Each of these connected subgraphs are called component of H.

A clique C of H is a subgraph of a graph H such that all vertices in the subgraph are completely connected with each other. The clique number of the graph H, denoted by $Clique(H)$, is the number of vertices in the maximal clique of H. An independent set or stable set in a graph H is a set of pairwise non

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adjacent vertices of H. The independence number of a graph H, denoted by $\alpha(H)$, is the maximum size of an independent set of vertices. The girth of a graph H with a cycle is the length of the shortest cycle. The eccentricity of a vertex u, denoted by $e(u)$, is the greatest distance from u to all other vertices in the graph H. That is,

$$e(u) = \max_{x \in V(G)} d(u, x).$$

The radius of the graph H, denoted by $rad(H)$, is the value of smallest eccentricity. The diameter of the graph H, denoted by $diam(H)$, is the value of greatest eccentricity.

The eigenvalues of a graph H are defined to be the eigenvalues of its adjacency matrix. The rank of the graph H, denoted by $\rho(H)$, is defined as the number of non zero eigenvalues of its adjacency matrix.

Kiruthika and Kalamani [15] found generalization of the vertex partition and edge partition of the power graph of the finite abelian group of an order pq if $p < q$, where p and q are distinct primes. They found some types of topological indices of graphs related to the groups. Jahandideh, Sarmin and Omer computed the various types of indices like Szeged index, edge Wiener index, first Zagreb index and the second Zagreb index for the non commuting graph [5]. Ramanae, Gundloor and Jummannaver [18] investigated the third Zagreb index, forgotten index and coindices of cluster graphs. Veylaki, Nikmehr and Tavalla [19] explained some basic mathematical properties for the third and hyper Zagreb coindices of graph operations.

Topological descriptors are based on the graph impression of the molecule and can also encode chemical information concerning atom type and bond multiplicity. It plays a vital role in the area of Quantitative Structure Activity Relation(QSAR), Quantitative Structure Property Relation(QSPR) and Fuzzy Lattice Neural Network(FLNN) [14]. One of the classical topological index is the familiar Zagreb index that was first introduced in [11] where Gutman and Trinajstic[10] examined the dependence of total-electron energy on molecular structure and this was elaborated in [8, 9].

The first and second Zagreb indices of a graph H, are defined as

$$M_1(H) = \sum_{uv \in E(H)} [d(u) + d(v)],$$
$$M_2(H) = \sum_{uv \in E(H)} [d(u)d(v)],$$

respectively.

Another Zagreb index called the third Zagreb index of a graph H, denoted by $M_3(H)$, is defined by Fath-Tabar [4] as

$$M_3(H) = \sum_{uv \in E(H)} |d(u) - d(v)|.$$

The Zagreb indices [8, 12, 16, 17] play a very important key role in the past, present and future research developments.

In this research, we newly defined the vertex-order graph $\Im(G)$ of the finite cyclic group G. The study of certain properties of the vertex-order graph of the finite cyclic group G is the main outcome and is presented in this research. The complement of the vertex-order graph is also defined with simple proofs.

Throughout this paper, we follow the terminologies and notations of [6] for groups and [20, 7] for graphs.

2. Some theoretical properties of the vertex-order graph

In this section, some simple characteristics of the graph $\Im(G)$ are studied with theorems and examples.

Definition 2.1. *A* ***vertex-order*** *graph of a finite cyclic group G is a simple graph whose vertices are elements of the group G and there is an edge between any two distinct vertices iff its orders are equal and is denoted by $\Im(G)$.*

Example 2.1. The vertex-order graph $\Im(Z_9)$ of the finite cyclic group G is shown in Figure 1.

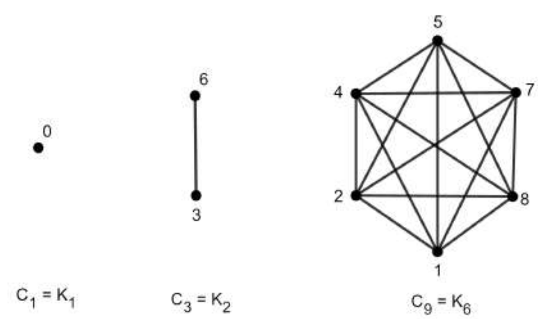

Figure 1: The vertex-order graph $\Im(Z_9)$.

Theorem 2.1. *The girth $gr(\Im(G))$ of the vertex-order graph $\Im(G)$ is given by*

$$gr(\Im(G)) = \begin{cases} 3, & \text{if } \phi(n) \geq 3 \\ \infty, & \text{otherwise.} \end{cases}$$

where ϕ is the euler totient function.

Proof. Let $\Im(G)$ be the vertex-order graph of the finite cyclic group G of order n.

It is noted that the length of the shortest cycle of $\Im(G)$ is the minimum length of the cycles of all the components C_d where d is the divisor of n. The graph

$\Im(G)$ is disconnected and the components C_d are all complete. The complete subgraph is denoted by K_m where $m = \phi(d)$ and each K_m is $(m-1)$ regular.

If $\phi(n) \geq 3$ then the vertex-order graph contains the complete graph K_m and $m \geq 3$. In this case, the length of the shortest cycle is 3.

In all other cases, the graph does not contain any cycle.

Hence,

$$gr(\Im(G)) = \begin{cases} 3, & \text{if } \phi(n) \geq 3 \\ \infty, & \text{otherwise.} \end{cases} \quad \square$$

Theorem 2.2. *For any vertex-order graph $\Im(G)$, the eccentricity of the vertex v is $e(v) = \infty$*

Proof. Let $e(v)$ be the eccentricity of the vertex v of the vertex-order graph $\Im(G)$.

The distance between the any two vertices v_i, v_j is ∞, if v_i, v_j are the vertices of two different components C_d of the vertex order graph. Since each component C_d is complete, the distance between v_i and v_j is 1 if v_i, v_j are vertices of the same component. Thus, $e(v) = \max_j d(v,\ v_j) = \infty$. Therefore, the eccentricity of the vertex v of the vertex-order graph is ∞. $\square$

Lemma 2.1. *Let G be the finite cyclic group of order n. Then, the following holds:*

(*i*) *diam*$(\Im(G)) = \infty$.

(*ii*) *rad*$(\Im(G)) = \infty$.

Theorem 2.3. *The independence number of the vertex-order graph denoted by $\alpha(\Im(G))$ is always t where t is the number of components of the graph.*

Proof. Let $\alpha(\Im(G))$ be the independence number of the vertex-order graph. It is clear to see that the independence number of a complete graph is 1. Since each component C_d is a clique, the independence number of complete graph is one i.e., $\alpha(C_d) = 1$ and is denoted by I_d. Also the the vertex-order graph is the disjoint union of its components C_d. Thus

$$\begin{aligned} \alpha(\Im(G)) &= \sum_d \alpha(C_d) \\ &= Number\ of\ components\ of\ the\ vertex\ order\ graph = t. \end{aligned}$$

$\therefore$ Independence number of the vertex-order graph is t. $\square$

Corollary 2.1. *The independence number $\alpha(\Im(Z_n))$ of the vertex-order graph is four if $n = pq$, where p and q are any two distinct primes.*

Example 2.2. The independence number of the vertex-order graph $\Im(Z_{15})$ is four, $\alpha[\Im(Z_{15})] = 4$ which is shown in Figure 3.

Theorem 2.4. *Let $\Im(G)$ be the vertex-order graph of the finite cyclic group G. Then*

$$\rho(\Im(G)) = \begin{cases} n-1, & \text{if } n \text{ is odd,} \\ n-2, & \text{if } n \text{ is even,} \end{cases}$$

where $\rho(\Im(G))$ is the rank of the vertex-order graph.

Proof. Let λ_1, λ_2, λ_3,...λ_n be the eigenvalues of the vertex-order graph of the finite cyclic group G order n.

If λ_i, $i = 1, 2, ...m$ are the eigenvalues of the complete graph K_m where $m = \phi(d)$ and d is the divisor of n, then for $m \neq 1$,

$$\lambda_i = \begin{cases} -1, & \text{if } i = 1, 2, 3....m-1, \\ m-1, & \text{if } i = m. \end{cases}$$

If $m = 1$, the eigenvalue of K_m is zero.

The set of eigenvalues of the vertex-order graph $\Im(G)$ is the union of all the eigenvalues of the complete graph K_m for all m.

So, the number of non-zero eigenvalues value of the vertex-order graph is $n-1$ if n is odd and $n-2$ if n is even, since the number of isolated vertices is 1 if n is odd and 2 if n is even.

Hence, the rank of the vertex-order graph

$$\rho(\Im(G)) = \begin{cases} n-1, & \text{if } n \text{ is} \quad \text{odd}, \\ n-2, & \text{if } n \text{ is} \quad \text{even}. \end{cases}$$ □

3. Properties of the complement of the vertex-order graph

Let $\overline{\Im(G)}$ be the complement of the vertex-order graph of the finite cyclic group G. In this section, some important properties of the complement of the vertex-order graph are discussed.

Theorem 3.1. *The complement $\overline{\Im(G)}$ of the vertex-order graph is a complete t-partite graph.*

Proof. Let t be the number of connected component of the vertex-order graph $\Im(G)$ of the finite cyclic group G of order n.

Each component C_d of the vertex-order graph is a complete subgraph K_m and the vertex-order graph $\Im(G)$ is the disjoint union of the complete subgraphs K_m where $m = \phi(d)$ and d is the divisors of n.

It is clear that no two vertices in the same component C_d of the vertices are adjacent in their complements. This implies that each component C_d of $\Im(G)$ is independent in their complement $\overline{\Im(G)}$. Hence, the complement graph $\overline{\Im(G)}$ is the complete t-partite graph where t is the number of components of the vertex-order graph. □

Example 3.1. Let $\overline{\Im(Z_{555})}$ be the complement of the the vertex-order graph of the group Z_{555}. Let C_1, C_3, C_5, C_{15}, C_{37}, C_{111}, C_{185}, C_{555} are the 8 components of the graph $\Im(Z_{555})$ where 1, 3, 5, 15, 37, 111, 185, 555 are the divisors of 555. Then, the number of vertices in the each component C_d of the vertex-order graph $\overline{\Im(Z_{555})}$ is given below:

$$\begin{aligned}
|C_1| &= \phi(1) = 1,\\
|C_3| &= \phi(3) = 2,\\
|C_5| &= \phi(5) = 4,\\
|C_{15}| &= \phi(15) = 8,\\
|C_{37}| &= \phi(37) = 36,\\
|C_{111}| &= \phi(111) = 72,\\
|C_{185}| &= \phi(185) = 144,\\
|C_{555}| &= \phi(555) = 288.
\end{aligned}$$

Hence, K_1, K_2, K_4, K_8, K_{36}, K_{72}, K_{144}, K_{288} are the complete subgraphs of $\Im(Z_{555})$. The set of vertices in K_m are independent in their complement $\overline{\Im(Z_{555})}$ for any m. Thus, $\overline{\Im(Z_{555})}$ is the complete 8-partite graph and is denoted by $K_{1,2,4,8,36,72,144,288}$ which is shown in Figure 2.

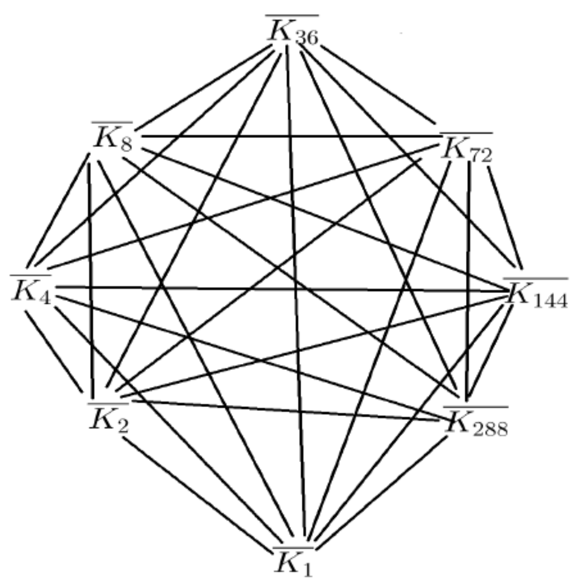

Figure 2: The edge adjacency of the complement of the Vertex-order graph $\overline{\Im(Z_{555})}$.

Lemma 3.1. *The complement* $\overline{\Im(G|e)}$ *of the vertex-order graph is complete bi-partite if* n *is the square of the prime number.*

Proof. Let C_d be the component of the vertex-order graph. If $n = p^2$, then $\Im(G)$ has exactly three distinct components namely C_1, C_p, C_{p^2} where p is a prime. Each of these are complete which is shown in Figure 1. By omitting the identity element, there are only two components C_p and C_{p^2} which are independent in their complement. Hence, $\overline{\Im(G|e)}$ is a complete bi-partite graph. □

Theorem 3.2. *The complement* $\overline{\Im(G)}$ *of the vertex-order graph has a clique as the number of independent set.*

Proof. Let $\overline{\Im(G)}$ be the complement of the vertex-order graph of the finite cyclic group G of order n.

Let $Clique(\overline{\Im(G)})$ be the clique number of the complement of the vertex-order graph. From Theorem 3.1, the complement of the vertex-order graph $\Im(G)$ is the complete t-partite graph. From this it is clear that the largest complete subgraph $\overline{\Im(G)}$ contains t vertices

$$\therefore \quad Clique(\overline{\Im(G)}) = t. \square$$

Corollary 3.1. *The independence number of the complement of the vertex-order graph* $\Im(G)$ *is the number of the generators of the finite cyclic group* G*, i.e.,* $\alpha(\overline{\Im(G)}) = \phi(n)$*.*

Theorem 3.3. *The girth of the complement of the vertex order graph* $\Im(G)$ *is given by*

$$gr\overline{(\Im(G))} = \begin{cases} \infty, & \textit{if } n = p, \\ 3, & \textit{if } n \neq p. \end{cases}$$

Proof. Let $\Im(G)$ be the vertex-order graph of order n. Let $gr\overline{(\Im(G))}$ be the girth of the complement of the vertex-order graph $\Im(G)$.

If $n = p$, a prime, then the complement graph does not contain cycle since $\overline{\Im(G)}$ is a star graph.

In this case, the girth of the complement of the vertex-order graph is ∞.

If $n \neq p$ where p is a prime, then $\overline{\Im(G)}$ is the complete t-partite graph for every $t \geq 3$ and the complement graph contains the cycle of length 3.

In this case, the girth of the complement of the graph $\Im(G)$ is 3. Thus,

$$gr\overline{(\Im(G))} = \begin{cases} \infty, & \text{if } n = p, \\ 3, & \text{if } n \neq p. \end{cases} \quad \square$$

Theorem 3.4. *Let* $\overline{\Im(G)}$ *be the complement of the vertex-order graph of the finite cyclic group* G*. Then, the following holds:*

(i) $rad(\overline{\Im(G)}) = 1$;

(ii) $diam(\overline{\Im(G)}) = 2$.

Proof. Let $\overline{\Im(G)}$ be the vertex-order graph associated with finite cyclic group G of order n. It is noticed that $\overline{\Im(G)}$ is connected, since $\Im(G)$ is disconnected. The minimum and maximum eccentricities are 1 and 2 respectively.

Henceforth, the proof follows $diam(\overline{\Im(G)}) = 2$ and $rad(\overline{\Im(G)}) = 1$. □

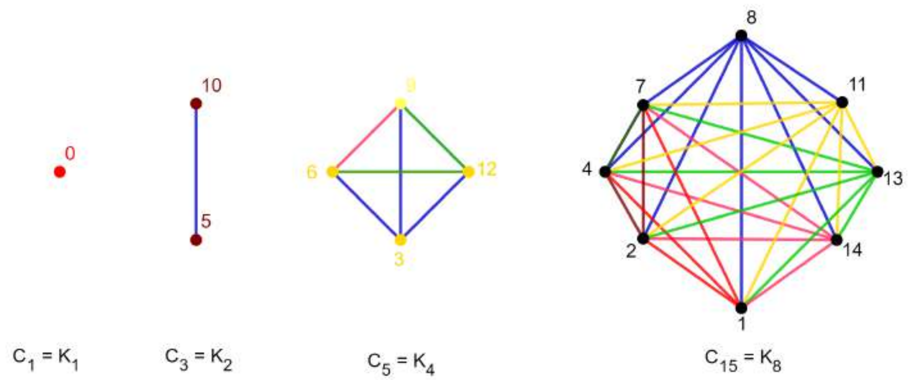

Figure 3: The vertex-order graph $\Im(Z_{15})$ with its four components.

Corollary 3.2. *The edge set of the complement of the vertex-order graph $\Im(G)$ is partitioned into tC_2 edge sets which is equal to the number of independent set of the graph $\Im(G)$.*

Example 3.2. The number of independent set of the graph $\Im(Z_{15})$ is 4 which is shown in Figure 3. From Corollary 3.2, the number of edge sets of the complement of the vertex-order graph $\Im(G)$ is 6 which is shown in Figure 4.

Then, the number of edges

$$\begin{aligned} |E(\overline{\Im(Z_{15})})| &= |E_1| + |E_2| + |E_3| + |E_4| + |E_5| + |E_6| \\ &= 4 + 2 + 8 + 8 + 32 + 16 \\ &= 70. \end{aligned}$$

Lemma 3.2. *The complement $\overline{\Im(G|e)}$ of the vertex-order graph is a null graph iff n is prime.*

Proof. If $n = p$, a prime, then $\overline{\Im(G)}$ is the star graph in which identity element e is an universal vertex. By omitting the identity element e, the star graph $K_{1,n-1}$ becomes null graph. Hence, $\overline{\Im(G|e)}$ is the null graph. □

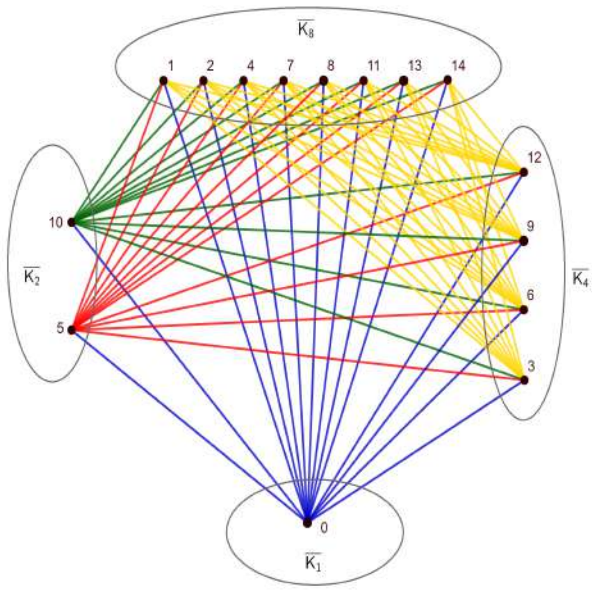

Figure 4: The complement of the Vertex-order graph $\Im(Z_{15})$ with its four independent sets.

Corollary 3.3. *The complement $\overline{\Im(G|i_v)}$ of the vertex-order graph is uni-cyclic if n is 6 where i_v is the isolated vertices of the graph $\Im(G)$.*

Theorem 3.5. *For any vertex-order graph $\Im(G)$, the rank of the complement of the vertex-order graph $\Im(G)$ is $\rho(\overline{\Im(G)}) = t$.*

Proof. Let $\rho(\overline{\Im(G)})$ be the rank of the complement of the vertex-order graph. From [1] the vertex-order graph $\Im(G)$ of rank t has clique number at most t; equality holds if and only if $\Im(G)$ is a complete t-partite graph. Thus, it is found that the rank of the complement of the vertex-order graph $\overline{\Im(G)}$ is the maximum clique of the graph $\overline{\Im(G)}$, since the every component C_d of $\Im(G)$ contain clique which is complete.

Rank of the complement of the vertex-order graph is the clique number of the complement of the vertex-order graph. From Theorem 3.2, the rank of the complement $\overline{\Im(G)}$ of the vertex-order graph is the number of connected components t $\rho(\overline{\Im(G)}) = Clique(\overline{\Im(G)}) = t$. □

Example 3.3. The rank of the complement of the vertex-order graph $\Im(Z_8)$ is four which is shown in Figure 5. $\rho(\overline{\Im(Z_8)}) = 4$.

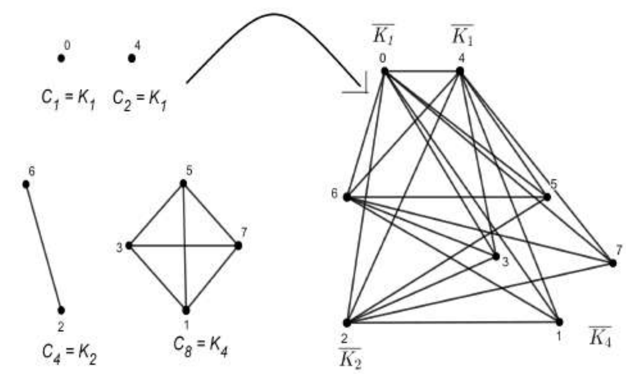

Figure 5: The vertex-order graph $\Im(Z_8)$ with its four components and transformation of its complement.

4. Computation of Zagreb indices of the vertex-order graph

In this section, we derive some Zagreb indices of the vertex-order graph.

Theorem 4.1. *The first Zagreb index of the vertex-order graph* $M_1(\Im(G))$ *is* $\sum_m m(m-1)^2$*, where* $m = \phi(d)$ *.*

Proof. Let $\Im(G)$ be the vertex-order graph of the finite cyclic group G. Since $\Im(G)$ is a disconnected graph, there are finite number of connected components C_d, each of which is a complete graph K_m where $m = \phi(d)$ and d is the divisor of the order of the group G. Then, the number of edges and vertices in K_m for $m \neq 1$ are mC_2 and m respectively. If $m = 1$, there is no edge. Thus, the first Zagreb index

$$\begin{aligned} M_1(\Im(G)) &= \sum_{uv \in E(G)} [d(u) + d(v)] \\ &= \sum_m mC_2[(m-1) + (m-1)] \\ &= \sum_m m(m-1)^2. \end{aligned}$$

□

Example 4.1. The first Zagreb index of the vertex-order graph $M_1(\Im(Z_{15}))$ is 430.

By the definition of vertex-order graph $\Im(Z_{15})$, the connected components are given by K_1, K_2, K_4, K_8 which is shown in Figure 3. Then, the first Zagreb index

$$M_1(\Im(Z_{15}) = \sum_m m(m-1)^2 = 430.$$

Theorem 4.2. *The second Zagreb index of the vertex-order graph* $M_2(\Im(G))$ *is* $\sum_m \frac{m(m-1)^3}{2}$ *where* $m = \phi(d)$.

Proof. Let $\Im(G)$ be the vertex-order graph of the finite cyclic group G. Since $\Im(G)$ is a disconnected graph, there are finite number of connected components C_d, each of which is a complete graph K_m where $m = \phi(d)$ and d is the divisor of the order of the group G. Then, the number of edges and vertices in K_m for $m \neq 1$ are mC_2 and m respectively. If $m = 1$ there is no edge. Thus, the second Zagreb index

$$\begin{aligned} M_2(\Im(G)) &= \sum_{uv \in E(G)} d(u)d(v) \\ &= \sum_m mC_2[(m-1)(m-1)] \\ &= \sum_m \frac{m(m-1)^3}{2}. \end{aligned}$$

□

Example 4.2. The second Zagreb index of the vertex-order graph $M_2(\Im(Z_{18}))$ is 752.

By the definition of vertex-order graph $\Im(Z_{18})$, the connected components are given by K_1,K_1, K_2, K_2, K_6, K_6. Then, the second Zagreb index $M_2(\Im(Z_{18}))$ is given by

$$M_2(\Im(Z_{18})) = \sum_m \frac{m(m-1)^3}{2} = 752.$$

Lemma 4.1. *Let* $\Im(G)$ *be the vertex-order graph of the finite cyclic group* G*. Then, the third Zagreb index of the vertex-order graph* $M_3(\Im(G))$ *is zero, since the order of the vertices of all the components* C_d *are equal.*

5. Computation of Zagreb indices of the complement of the vertex-order graph

In this section, some Zagreb indices of the complement of the vertex-order graph are derived with its generalizations.

Theorem 5.1. *Let* $\overline{\Im(G)}$ *be the complement of the vertex-order graph of the finite cyclic group* G *of order* pq *where* p *and* q *are any distinct primes. Then, its Zagreb indices are given by*

(1) $M_1(\overline{\Im(G)}) = p^2q^2(p+q+1) - pq(p^2+q^2+p+q-1)$;

(2) $M_3(\overline{\Im(G)}) = p^2q^2(p+q-5) - pq[3(p^2+q^2+5) - 9p - 11q] + p + q - 1) + 2(p^3+q^3) - 4p(p-1) - 6q(q-1)$.

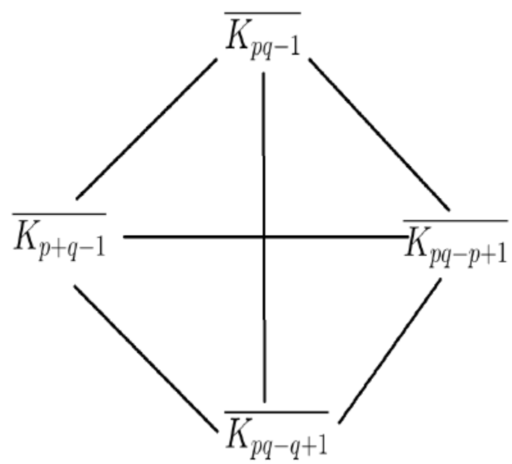

Figure 6: The edge adjacency of the complement of the vertex-order graph $\overline{\Im(Z_{pq})}$.

Proof. Consider the complement of the vertex-order graph $\Im(G)$ of order n where $n = pq$, p and q are any two distinct primes. The total number of vertices and edges of $\overline{\Im(G)}$ is given by pq and $p^2q + pq^2 - p^2 - q^2 - 2pq + 2p + 2q - 2$ respectively. Then, the vertex set can be divided into 1 vertex of degree $pq-1$, $p-1$ vertices of degree $pq-p+1$, $q-1$ vertices of degree $pq-q+1$ and $pq-p-q+1$ vertices of degree $p+q-1$. Let $d_{\overline{\Im(Z_{pq})}}(u)$ and $d_{\overline{\Im(Z_{pq})}}(v)$ be the degrees of the end vertices u and v respectively.

The edge set $E(\overline{\Im(G)})$ can be divided into two edge partitions based on the degrees of end vertices. These can easily done by using the four independent sets which is shown in Figure 6.

The first partition of edges $E_1(\overline{\Im(Z_{pq})})$ contains $p-1$ edges uv, where $d_{\overline{\Im(Z_{pq})}}(u) = pq-1$, $d_{\overline{\Im(Z_{pq})}}(v) = pq-p+1$, the second partition of edges $E_2(\overline{\Im(Z_{pq})})$ contains $q-1$ edges uv, where $d_{\overline{\Im(Z_{pq})}}(u) = pq-1$, $d_{\overline{\Im(Z_{pq})}}(v) = pq-q+1$, the third partition of edges $E_3(\overline{\Im(Z_{pq})})$ contains $pq-p-q+1$ edges uv, where $d_{\overline{\Im(Z_{pq})}}(u) = pq-1$, $d_{\overline{\Im(Z_{pq})}}(v) = p+q-1$.

The fourth partition of edges $E_4(\overline{\Im(Z_{pq})})$ contains $pq-p-q+1$ edges uv, where $d_{\overline{\Im(Z_{pq})}}(u) = pq-p+1$, $d_{\overline{\Im(Z_{pq})}}(v) = pq-q+1$.

The fifth partition of the edges $E_5(\overline{\Im(Z_{pq})})$ contains $(p-1)(pq-p-q+1)$ edges uv, where $d_{\overline{\Im(Z_{pq})}}(u) = pq-p+1$, $d_{\overline{\Im(Z_{pq})}}(v) = p+q-1$.

The sixth partition of the edges $E_6(\overline{\Im(Z_{pq})})$ contains $(q-1)(pq-p-q+1)$ edges uv, where $d_{\overline{\Im(Z_{pq})}}(u) = pq-q+1$, $d_{\overline{\Im(Z_{pq})}}(v) = p+q-1$. Then, the required results for the graph $\overline{\Im(Z_{pq})}$ by using the number of its edge partition as follows:

(1) The first Zagreb index

$$\begin{aligned}
M_1(\overline{\Im(Z_{pq})}) &= \sum_{uv \in E(\overline{\Im(Z_{pq})})} [d(u) + d(v)] \\
&= (p-1)[pq-1+pq-p+1] + (q-1)[pq-1+pq-q+1] \\
&+ (pq-p-q+1)[pq-1+p+q-1] \\
&+ (pq-p-q+1)[pq-p+1+pq-q+1] \\
&+ (p-1)(pq-p-q+1)[pq-p+1+p+q-1] \\
&+ (q-1)(pq-p-q+1)[pq-p+1+p+q-1] \\
&= p^2q^2(p+q+1) - pq(p^2+q^2+p+q-1).
\end{aligned}$$

(2) The third Zagreb index

$$\begin{aligned}
M_3(\overline{\Im(Z_{pq})}) &= \sum_{uv \in E(\overline{\Im(Z_{pq})})} |d(u) - d(v)| \\
&= (p-1)[pq-1-pq+p-1] + (q-1)[pq-1-pq+q-1] \\
&+ (pq-p-q+1)[pq-1-p-q+1] \\
&+ (pq-p-q+1)[pq-p+1-pq+q-1] \\
&+ (p-1)(pq-p-q+1)[pq-p+1-p-q+1] \\
&+ (q-1)(pq-p-q+1)[pq-p+1-p-q+1] \\
&= p^2q^2(p+q-5) - pq[3(p^2+q^2+5) - 9p - 11q] \\
&+ 2(p^3+q^3) - 4p(p-1) - 6q(q-1).
\end{aligned}$$

Similarly, we can generalize the second Zagreb index of the graph $\overline{\Im(Z_{pq})}$. □

Example 5.1. Let $\overline{\Im(Z_{15})}$ be the complement of the vertex-order graph of the finite cyclic group Z_{15}. Then, its Zagreb indices are given by

(1) $M_1(\overline{\Im(Z_{15})}) = 1410$;

(2) $M_3(\overline{\Im(Z_{15})}) = 310$.

Using Theorem 5.1 the results obtained for $\Im(Z_{15})$ are as follows:

(1) The first Zagreb index

$$\begin{aligned}
M_1(\overline{\Im(Z_{15})}) &= p^2q^2(p+q+1) - pq(p^2+q^2+p+q-1) \\
&= 3^2 5^2(3+5+1) - 3.5(3^2+5^2+3+5-1) \\
&= 225(9) - 15(41) \\
&= 1410.
\end{aligned}$$

(2) The third Zagreb index

$$\begin{aligned}M_3(\overline{\Im(Z_{15})}) &= p^2q^2(p+q-5) - pq[3(p^2+q^2+5) - 9p - 11q] \\ &+ 2(p^3+q^3) - 4p(p-1) - 6q(q-1) \\ &= 310.\end{aligned}$$

Similarly, we can enumerate the second Zagreb index of the graph $\overline{\Im(Z_{15})}$.

Theorem 5.2. *Let $\overline{\Im(G)}$ be the complement of the vertex-order graph of the finite cyclic group G of prime order p. Then, its Zagreb indices are given by*

(1) $M_1(\overline{\Im(G)}) = p(p-1)$;

(2) $M_2(\overline{\Im(G)}) = (p-1)^2$;

(3) $M_3(\overline{\Im(G)}) = (p-1)(p-2)$.

Proof. Let $\overline{\Im(Z_p)}$ be the complement of the vertex-order graph of the finite cyclic group G. Since $\Im(G)$ is a disconnected graph of complete graphs K_1, K_{p-1}, the total number of vertices and edges of $\overline{\Im(Z_p)}$ (or) $K_{1,p-1}$ are given by p and $p-1$ respectively. Then, the vertex set can be partitioned into 1 vertices of degree $p-1$, and $p-1$ vertices of degree 1. Thus, the only one edge set which is given by $E_{1,p-1} = p-1$. (1) The first Zagreb index

$$M_1(\overline{\Im(Z_p)}) = \sum_{uv \in E(\overline{\Im(Z_p)})} [d(u)+d(v)] = p(p-1).$$

(2) The second Zagreb index

$$\begin{aligned}M_2(\overline{\Im(Z_p)}) &= \sum_{uv \in E(\overline{\Im(Z_p)})} d(u)d(v) \\ &= (p-1)[(p-1)(1)] \\ &= (p-1)^2.\end{aligned}$$

(3) The third Zagreb index

$$M_3(\overline{\Im(Z_p)}) = \sum_{uv \in E(\overline{\Im(Z_p)})} |d(u)-d(v)| = (p-1)(p-2).$$ □

Example 5.2. The first, second and third Zagreb indices of the complement of the vertex-order graph $\Im(Z_{19})$ of the finite cyclic group Z_{19} are 342, 324, 306 respectively.

6. Conclusion

In this paper, the graph theoretical properties of the vertex-order graph and its complements are interpreted with their proofs. Also the some Zagreb indices of the vertex-order graph $\Im(G)$ and the complement of the vertex-order graph $\Im(Z_{pq})$, $\Im(Z_p)$ are derived with their examples.

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