On some properties of Nörlund ideal convergence of sequence in neutrosophic normed spaces

Vakeel A. Khan*
Department of Mathematics
Aligarh Muslim University
Aligarh–202002
India
vakhanmaths@gmail.com

Mohammad Arshad
Department of Mathematics
Aligarh Muslim University
Aligarh- 202002
India
mohammadarshad3828@gmail.com

Abstract. The purpose of this paper is to introduce the Nörlund ideal convergent sequence spaces with respect to these spaces $\mathscr{N}^f_{I_0(\mathcal{S})}$, $\mathscr{N}^f_{I(\mathcal{S})}$ and $\mathscr{N}^f_{I_\infty(\mathcal{S})}$. Also, we studied the Nörlund ideal Cauchy criterion in neutrosophic normed space and its properties. Also, we define an open ball $B(x, \epsilon, \gamma)$ and closed ball $B[x, \epsilon, \gamma]$ in neutrosophic norm space. Furthermore, we also look at some of these convergent sequence spaces' topological and algebraic properties.

Keywords: ideal convergent, ideal Cauchy, Nörland mean, Nörlund matrix, sequence space, Nörlund ideal convergent, Nörlund ideal Cauchy sequence and neutrosophic normed space.

1. Introduction

The fuzzy set was first developed in 1965 by Zadeh [27], and they have since been used in a variety of domains, including artificial intelligence, robotics, and control theory. According to him, a fuzzy set assigns a membership value from $[0, 1]$ to each element of a given crisp universe set.

Atanassov K.T. in [14], [13] introduced the intuitionistic fuzzy set (IFS) on a universe X as an extension of the fuzzy set. Coker [15] used this concept to develop intuitionistic fuzzy topological spaces. Saadati and Park [20] investigated these spaces and their extension, resulting in the idea of intuitionistic fuzzy normed space.

In 1998, Samarandache [3] presented the first philosophical point for neutrosophic set. The concept of classic set theory has been extended in the form of the neutrosophic set by adding an intermediate membership function. Examples

$*$. Corresponding author

of other generalizations are the Fuzzy set [27], and intuitionistic fuzzy set [14]. The actual definition of neutrosophic sets was given based on the independence of membership, non-membership, and hesitation function.

In 2006, F. Samarandache and W.B. Vasantha Kanasamy in [26] introduced the concept of neutrosophic algebraic structures.

Bera and Mahapatra [21] first introduced the neutrosophic soft linear space. Neutrosophic soft norm linear space, convexity, metric [34], and Cauchy sequence were examined by Bera and Mahapatra [22]. The purpose of the current paper is to change the intuitionistic fuzzy normed space of the structure into neutrosophic normed space. The Cauchy sequence has been studied on neutrosophic normed space in an attempt to investigate some beautiful results in this structure.

H. Fast [5] and I. J. Schoenberg [6] introduce the idea of statistical convergence, whereas J. C̃erveñanský [28] and J.S. Connor [29, 30] develop it. R.C. Buck [31, 32] and D.S. Mitrinović [33] include some examples of statistical convergence in mathematical analysis and number theory. The idea of statistical convergence with regard to the intuitionistic fuzzy norm was introduced by Mursaleen [16]. In neutrosophic normed space, statistical convergence was first investigated by Kirisci and Simsek [7]. The concept of "ideal convergence" is an extension of the notion of "statistical convergence", and it is dependent on the idea of the ideal of subsets of the set $\mathbb{N}$. Śalát et al. [23], [24], Filipów and Tryba [19], Khan and Nazreen [12], Khan et al. [11], Khan and Nazreen [12] and several more writers further investigated the concept of I-convergent from the perspective of sequence space and related it with the summability theory. To better understand the I-convergence in neutrosophic normed space, we have been inspired by this.

The purpose of this study is to define new neutrosophic sequence spaces using the Nörlund matrix and the neutrosophic norm. Also, we will study Nörlund I-convergent and Nörlund I-Cauchy in neutrosophic normed spaces, and by using the Nörlund matrix $\mathscr{N}^f$ and the notion of Nörlund I-convergent of sequence in neutrosophic normed space, we introduce some new spaces of Nörlund I-convergent sequence with regard to the neutrosophic norm $(\mathcal{U}, \mathcal{V}, \mathcal{W})$. We also investigate at some of these convergent sequence spaces' topological and algebraic properties, as well as some interesting connections between these spaces $\mathscr{N}^f_{I_0(\mathcal{S})}$, $\mathscr{N}^f_{I(\mathcal{S})}$ and $\mathscr{N}^f_{I_\infty(\mathcal{S})}$.

2. Preliminaries

Definition 2.1 ([9])**.** Let I be the power set of any set Z, where Z is the set. Then, I is called ideal, if:

(1) $\emptyset \in I$;

(2) $\vartheta_1, \vartheta_2 \in I \Rightarrow \vartheta_1 \cup \vartheta_2 \in I$, additive;

(3) $\vartheta_1 \in I, \vartheta_2 \subseteq \vartheta_1 \Rightarrow \vartheta_2 \in I$, hereditary.

If $I \neq 2^Z$ then $I \subseteq 2^Z$ is called nontrivial. If I contain every singleton subset of X. then nontrivial ideal $I \subseteq 2^Z$ is called admissible. If there are no non-trivial ideal $K \neq I$ then nontrivial ideal I is maximal such that $I \subset K$.

Definition 2.2 ([9])**.** Let $\mathscr{F}$ be the power set of any set Z, where Z is the set. Then, $\mathscr{F}$ is said to be filter. If: (1) $\emptyset \notin \mathscr{F}$;

(2) For $\vartheta_1, \vartheta_2 \in \mathscr{F}$; $\vartheta_1 \cap \vartheta_2 \in \mathscr{F}$;

(3) If $\vartheta_1 \in \mathscr{F}$ and $\vartheta_2 \supset \vartheta_1$ imply $\vartheta_2 \in \mathscr{F}$.

$\mathscr{F}(I)$ is the filter associated with each ideal$I$ of Z such that $\mathscr{F}(I) = \big\{A \subset Z : A^c \in I\big\}$ is true for each ideal of Z. Then, using the article, we present I as an admissible ideal.

Note. Class $\mathscr{F}(I) = \{\vartheta_1 \subset Z : \vartheta_1 = Z/\vartheta_2, \text{ for some } \ \vartheta_2 \in I\}$ is a filter on Z, where $I \subset P(Z)$ is a non-trivial ideal. $\mathscr{F}(I)$ is described as the filter associated with the ideal I.

Definition 2.3 ([8])**.** In any set Z, let I be a non trivial ideal subset of a power set $(P(Z))$. So, it is said that a sequence $x = (x_k)$ is ideally convergent to α, iff the set $\{k \in Z : |x_k - \alpha| \geq \epsilon\} \in I$ and we write it as $I - \lim x = \alpha$, for every $\epsilon > 0$.

Definition 2.4 ([8])**.** In any set Z, let I be a non trivial ideal subset of a power set $(P(Z))$. So, it is said that a number sequence $x = (x_k)$ is ideally Cauchy. If, for any $\epsilon > 0, \exists\ L = L(\epsilon)$, the set $\{k \in Z : |x_k - x_L| \geq \epsilon\} \in I$.

The Nörlund matrix $\mathscr{N}^f$ was initially used in the theory of sequence space by Wang [25]. Remember that $t = (t_k)$ is a non negative sequence of real numbers and $A_n = \sum_{k=0}^{n} t_k, \forall\ n \in \mathbb{N}$ with $t_0 > 0$. Then, with regard to the sequence $t = (t_k)$, the Norlund matrix $\mathscr{N}^f = (a^t_{nm})$ is defined as follows:

$$a^t_{nm} = \begin{cases} \dfrac{t_{n-m}}{A_n}, & \text{if } 0 \leq m \leq n \\ 0, & \text{if } m > n, \end{cases} \tag{1}$$

for all $n, m \in \mathbb{N}$. It is known that the Nörlund matrix $\mathscr{N}^f$ is regular iff $t_n/T_n \to 0$ as $n \to \infty$.

Let $t_0 = D_0 = 1$ and define L_n for $n \in \{1, 2, 3, ...\}$ by

$$D_n = \begin{bmatrix} t_1 & 1 & 0 & 0 & ...0 \\ t_2 & t_1 & 1 & 0 & ...0 \\ t_3 & t_2 & t_1 & 1 & ...0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & ...1 \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & ...t_1 \end{bmatrix} \tag{2}$$

Then, the inverse matrix $L^t = (l^t_{nm})$ of Nörland matrix $\mathscr{N}^f = (a^t_{nm})$ was define by Mears in [4], for all $n \in \mathbb{N}$, as follows

$$l_{nm} = \begin{cases} (-1)^{n-m} D_{n-m} T_k, & \text{if } (0 \leq m \leq n), \\ 0, & \text{otherwise,} \end{cases}$$

for all $n, m \in \mathbb{N}$.

One can refer to [4, 2, 1] for more background about Norland space.

In this paper, the natural and real number sets, respectively, are denoted by the letters $\mathbb{N}$ and $\mathbb{R}$. ω also represents for the linear space having all real sequences. The sequence spaces c_0, c and l_∞ represent the spaces of all null, convergent, and bounded sequences, respectively. We now define the Nörlund sequence space established by Wang in [25] as follows

$$\mathscr{N}^f = \left\{ x = (x_k) \in \omega = \sum_{n=0}^{\infty} |\frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} x_k|^p < \infty, 1 \leq p < \infty \right\},$$

where $A_n = \sum_{k=0}^{n} a_k$. All sequences whose Norlund transformations are in the spacel_∞ and l_p with $1 \leq p < \infty$ are contained in the spaces $l_\infty(\mathscr{N}^f)$ and $l_p(\mathscr{N}^f)$.

Motivated by [17], Khan [8] recently presented the sequence spaces $c_0^I(\mathscr{N}^f)$, $c^I(\mathscr{N}^f)$, and $l_\infty^I(\mathscr{N}^f)$ as the sets of all sequences whose $\mathscr{N}^f$ transformations are in spaces c_0, c, and l_∞, respectively. Khan did this by using the concept of Nörlund $I-$convergence, Nörlund $I-$ null and Nörlund $I-$ bounded sequence space,where I is an admissible ideal of subset of $\mathbb{N}$. For more details on these spaces, we refer to [18, 8]. Define

$$c_0^I(\mathscr{N}^f) := \left\{ y = (y_k) \in \omega : \{n \in \mathbb{N} : |\mathscr{N}_n^f(y)| \geq \epsilon\} \in I \right\},$$
$$c^I(\mathscr{N}^f) := \{y{=}(y_k){\in}\omega : \{n{\in}\mathbb{N} : |\mathscr{N}_n^f(y) - K|{\geq}\epsilon \text{ for some } K \in \mathbb{R}\}{\in}I\},$$
$$l_\infty^I(\mathscr{N}^f) := \left\{ y = (y_k) \in \omega : \exists M > 0 \ \text{ s.t } \{n \in \mathbb{N} : |\mathscr{N}_n^f(y)| \geq M\} \in I \right\},$$

where

$$\mathscr{N}_n^f(y) := \frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} y_k, \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

Definition 2.5 ([10, 7])**.** Given an binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-norm if:

(a) $*$ is commutative and associative;

(b) $*$ is continuous;

(c) $\vartheta * 1 = \vartheta \ \forall \ \vartheta \in [0,1]$;

(d) $\vartheta_1 * \vartheta_2 \leq \vartheta_3 * \vartheta_4$ whenever $\vartheta_1 \leq \vartheta_3$ and $\vartheta_2 \leq \vartheta_4$ for each $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in [0,1]$.

Example 2.1. For $\vartheta_1, \vartheta_2 \in [0,1]$, define $\vartheta_1 * \vartheta_2 = \vartheta_1 \vartheta_2$ or $\vartheta_1 * \vartheta_2 = \min\{\vartheta_1, \vartheta_2\}$, then $*$ is continuous t-norm.

Definition 2.6 ([10, 7])**.** Given an binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-conorm if:

(a) $\diamond$ is commutative and associative;

(b) $\diamond$ is continuous;

(c) $\vartheta \diamond 0 = \vartheta \ \forall\ \sigma \in [0,1]$;

(d) $\vartheta_1 \diamond \vartheta_2 \leq \vartheta_3 \diamond \vartheta_4$ whenever $\vartheta_1 \leq \vartheta_3$ and $\vartheta_2 \leq \vartheta_4$ for each $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in [0,1]$.

Example 2.2. Let $\vartheta_1, \vartheta_2 \in [0,1]$. Define $\vartheta_1 \diamond \vartheta_2 = \min\{\vartheta_1 + \vartheta_2, 1\}$ or $\vartheta_1 \diamond \vartheta_2 = \max\{\vartheta_1, \vartheta_2\}$, then $\diamond$ is continuous t-conorm.

Definition 2.7 ([20])**.** Take Z as a linear space and $\mathcal{S} = \{< x, \mathcal{U}(x), \mathcal{V}(x), \mathcal{W}(x) >: x \in Z\}$ be a normed space such that $\mathcal{S} : Z \times (0,\infty) \longrightarrow [0,1]$. Assume $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm respectively. The four-tuple $V = (Z, \mathcal{S}, *, \diamond)$ is said to be neutrosophic normed space (NNS) if the subsequent conditions are hold, for all $x, y, \in Z$ and $\gamma, \delta > 0$:

(1) $0 \leq \mathcal{U}(x,\gamma) \leq 1, 0 \leq \mathcal{V}(x,\gamma) \leq 1, 0 \leq \mathcal{W}(x,\gamma) \leq 1, \gamma \in \mathbb{R}^+$;

(2) $\mathcal{U}(x,\gamma) + \mathcal{V}(x,\gamma) + \mathcal{W}(x,\gamma) \leq 3$, for $\gamma \in \mathbb{R}^+$;

(3) $\mathcal{U}(x,\gamma) = 1$ for $\gamma > 0$ iff $x = 0$;

(4) $\mathcal{U}(\alpha x,\gamma) = \mathcal{U}\ (x, \frac{\gamma}{|\alpha|})$;

(5) $\mathcal{U}(x,\gamma) * \mathcal{U}(y,\delta) \leq \mathcal{U}(x+y, \gamma+\delta)$;

(6) $\mathcal{U}(x,*)$ is continuous nondecreasing function;

(7) $\lim_{\gamma\to\infty} \mathcal{U}(x,\gamma) = 1$;

(8) $\mathcal{V}(x,\gamma) = 0$ for $\gamma > 0$ iff $x = 0$;

(9) $\mathcal{V}(\alpha x,\gamma) = \mathcal{V}(x, \frac{\gamma}{|\alpha|})$;

(10) $\mathcal{V}(x,\gamma) \diamond \mathcal{V}(y,\delta) \geq \mathcal{V}(x+y, \gamma+\delta)$;

(11) $\mathcal{V}(x,\diamond)$ is continuous nonincreasing function;

(12) $\lim_{\gamma\to\infty} \mathcal{V}(x,\gamma) = 0$;

(13) $\mathcal{W}(x,\gamma) = 0$ for $\gamma > 0$ iff $x = 0$;

(14) $\mathcal{W}(\alpha x,\gamma) = \mathcal{W}(x, \frac{\gamma}{|\alpha|})$;

(15) $\mathcal{W}(x,\gamma) \diamond \mathcal{W}(y,\delta) \geq \mathcal{W}(x+y, \gamma+\delta)$;

(16) $\mathcal{W}(x,\diamond)$ is continuous nonincreasing function;

(17) $\lim_{\gamma\to\infty} \mathcal{W}(x,\gamma) = 0$;

(18) if $\gamma \leq 0$, then $\mathcal{U}(x,\gamma) = 0, \mathcal{V}(x,\gamma) = 1, \mathcal{W}(x,\gamma) = 1$.

In such case, $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$ is said to be neutrosophoic norm (NN).

Example 2.3 ([10])**.** Suppose $(Z, \|.\|)$ be a normed space. Using the $*$ and $\diamond$ operations, as t-norm $x * y = x.y$ and t-conorm $x \diamond y = x + y - xy$, for $\gamma > ||x||$ and $\gamma > 0$

$$\mathcal{U}(x,\gamma) = \frac{\gamma}{\gamma + \|\mu\|}\ ,\ \mathcal{V}(x,\gamma) = \frac{\|x\|}{\gamma + \|x\|} \text{ and } \mathcal{W}(x,\gamma) = \frac{\|x\|}{\gamma},$$

for all $x, y \in Z$. If we take $\gamma \leq \|x\|$, then $\mathcal{U}(x,\gamma)$=0, $\mathcal{V}(x,\gamma) = 1$ and $W(x,\gamma)$=1. Then, $(Z, \mathcal{S}, *, \diamond)$ is NNS in such a way that $\mathcal{S} : Z \times \mathbb{R}^+ \to [0,1]$.

Example 2.4. Suppose $(Z = \mathbb{R}, \|.\|)$ be a normed space, where $\|a\| = |a|, \forall\, a \in \mathbb{R}$. Using the $*$ and $\diamond$ operations, as t-norm $x * y = \min\{x, y\}$ and t-conorm $x \diamond y = \max\{x, y\}$, $\forall x, y \in [0, 1]$ and define

$$\mathcal{U}(x,\gamma) = \frac{\gamma}{\gamma + r\|x\|}\ ,\ \mathcal{V}(x,\gamma) = \frac{r\|x\|}{\gamma + \|x\|}\ \text{ and }\ \mathcal{W}(x,\gamma) = \frac{r\|x\|}{\gamma},$$

where $r > 0$ Then, $\mathcal{S} = \{(x,\gamma), \mathcal{U}(x,\gamma), \mathcal{V}(x,\gamma), \mathcal{W}(x,\gamma) : (x,\gamma) \in Z \times \mathbb{R}^+\}$ is a NN on Z.

Definition 2.8 ([7])**.** Let V be an NNS. A sequence $x = \{x_k\}$ is said to be convergent to α with respect to $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$, if for every $0 < \epsilon < 1$ and $\gamma > 0$, there exists $k \in \mathbb{N}$, such that $\mathcal{U}(x_k - \alpha, \gamma) > 1 - \epsilon$, $\mathcal{V}(x_k - \alpha, \gamma) < \epsilon$ and $\mathcal{W}(x_k - \alpha, \gamma) < \epsilon$. That is, for all $\gamma > 0$, we have

$$\lim_{k\to\infty} \mathcal{U}(x_k - \alpha, \gamma) = 1, \lim_{k\to\infty} \mathcal{V}(x_k - \alpha, \gamma) = 0 \text{ and } \lim_{k\to\infty} \mathcal{W}(x_k - \alpha, \gamma) = 0.$$

The convergent in NNS $V = (Z, \mathcal{S}, *, \diamond)$ is denoted by $\mathcal{S} - \lim x_k = \alpha$.

Definition 2.9 ([7])**.** Let V be an NNS. A sequence $x = \{x_k\}$ is Cauchy sequence with respect to $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$, if for every $0 < \epsilon < 1$ and $\gamma > 0$, there exists $K \in \mathbb{N}$, such that $\mathcal{U}(x_n - x_k, \gamma) > 1 - \epsilon$, $\mathcal{V}(x_n - x_k, \gamma) < \epsilon$ and $\mathcal{W}(x_n - x_k, \gamma) < \epsilon$, for all $n, k \in K$.

Definition 2.10 ([7])**.** Let V be an NNS. Then, open ball with center x and radius ϵ is defined as, for $0 < \epsilon < 1$, $x \in Z$ and $\gamma > 0$,

$$B(x, \epsilon, \gamma) = \{y \in Z : \mathcal{U}(x - y, \gamma)\ > 1 - \epsilon,\ \ \mathcal{V}(x - y, \gamma) < \epsilon,\ \ \mathcal{W}(x - y, \gamma) < \epsilon\}.$$

Definition 2.11 ([7])**.** Let V be an NNS and $Y \subseteq Z$. Then, Y is said to be open if for each $y \in Y$, there exist $\gamma > 0$, $0 < \epsilon < 1$ such that $B(y, \epsilon, \gamma) \subseteq Y$.

3. Main results (on the Nörlund sequence)

Throughout the article, we assume that the sequences $x = \{x_k\} \in \omega$ and $\mathscr{N}_n^f(x)$ are connected as shown in (3) and I is an admissible ideal of a subset of $\mathbb{N}$. In this section, by using a domain of the Nörlund matrix which is used in [8] and $I-$convergence w.r.t. neutrosophic norm $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$. As shown below, we define new Norlund sequence spaces:

$$\begin{aligned}\mathscr{N}^f_{I_0(\mathcal{S})} := \big\{x = \{x_n\} \in \omega : \{n \in \mathbb{N} : \mathcal{U}\big(\mathscr{N}_n^f(x), \gamma\big) \le 1 - \epsilon \\ \text{or } \mathcal{V}\big(\mathscr{N}_n^f(x), \gamma\big) \ge \epsilon, \mathcal{W}\big(\mathscr{N}_n^f(x), \gamma\big) \ge \epsilon\} \in I\big\}\end{aligned} \tag{4}$$

$$\begin{aligned}\mathscr{N}^f_{I(\mathcal{S})} := \big\{x{=}\{x_n\}{\in}\omega : \{n{\in}\mathbb{N} : \text{for some } \gamma \in \mathbb{R}\ ,\ \mathcal{U}\big(\mathscr{N}_n^f(x) - \alpha, \gamma\big){\le}1 - \epsilon \\ \text{or } \mathcal{V}\big(\mathscr{N}_n^f(x) - \alpha, \gamma\big) \ge \epsilon, \mathcal{W}\big(\mathscr{N}_n^f(x) - \alpha, \gamma\big) \ge \epsilon\} \in I\big\}\end{aligned} \tag{5}$$

$$\mathscr{N}^f_{I^\infty(\mathcal{S})} := \big\{x{=}\{x_n\}{\in}\omega : \{n{\in}\mathbb{N}, \exists\epsilon{\in}(0,1) \text{ s.t } \mathcal{U}\big(\mathscr{N}^f_n(x),\gamma\big){\leq}1-\epsilon$$
$$\text{or } \mathcal{V}\big(\mathscr{N}^f_n(x),\gamma\big) \geq \epsilon, \mathcal{W}\big(\mathscr{N}^f_n(x),\gamma\big) \geq \epsilon\} \in I\big\}. \tag{6}$$

We describe an open ball and a closed ball with a center at x and a radius $\gamma > 0$ with regard to the neutrosophic $\epsilon \in (0,1)$ parameter, indicated by $\mathscr{B}(x,\epsilon,\gamma)$ and $\mathscr{B}[x,\epsilon,\gamma]$, as follows:

$$\mathscr{B}(x,\epsilon,\gamma){=}\big\{z{=}\{z_k\}{\in}\omega : \{n \in \mathbb{N} : \mathcal{U}\big(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(z),\gamma\big){\leq}1{-}\epsilon$$
$$\text{or } \mathcal{V}\big(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(z),\gamma\big) \geq \epsilon, \mathcal{W}\big(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(z),\gamma\big) \geq \epsilon\} \in I\big\} \tag{7}$$

and

$$\mathscr{B}[x,\epsilon,\gamma]{=}\big\{z{=}\{z_k\}{\in}\omega : \{n{\in}\mathbb{N} : \mathcal{U}\big(\mathscr{N}^f_n(x){-}\mathscr{N}^f_n(z),\gamma\big){<}1-\epsilon$$
$$\text{or } \mathcal{V}\big(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(z),\gamma\big) > \epsilon, \mathcal{W}\big(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(z),\gamma\big) > \epsilon\} \in I\big\}. \tag{8}$$

In this case, we write $I_{(\mathcal{S})}{-}\lim(x) = \alpha$ since $\{x_n\}$ converges to some $\alpha \in \mathbb{C}$ represented by $x_n \xrightarrow{I_{(\mathcal{S})}} \alpha$ if $\{x_n\} \in \mathscr{N}^t_{I(\mathcal{S})}$.

Theorem 3.1. The inclusion relation $\mathscr{N}^f_{I_0(\mathcal{S})} \subset \mathscr{N}^f_{I(\mathcal{S})} \subset \mathscr{N}^f_{I^\infty(\mathcal{S})}$ holds.

Proof. We know that $\mathscr{N}^f_{I_0(\mathcal{S})} \subset \mathscr{N}^f_{I(\mathcal{S})}$. Then, we only show that $\mathscr{N}^f_{I(\mathcal{S})} \subset \mathscr{N}^f_{I^\infty(\mathcal{S})}$. Consider $x = \{x_n\} \in \mathscr{N}^f_{I(\mathcal{S})}$. Then, there exists $\alpha \in \mathbb{C}$, such that $I_{(\mathcal{S})}{-}\lim(x_k) = \alpha$. Thus, for any $0 < \epsilon < 1$ and $\gamma > 0$ the set

$$P = \Big\{n \in \mathbb{N} : \mathcal{U}\left(\mathscr{N}^f_n(x) - \alpha, \frac{\gamma}{2}\right) > 1 - \epsilon \text{ and } \mathcal{V}\left(\mathscr{N}^f_n(x) - \alpha, \frac{\gamma}{2}\right) < \epsilon,$$
$$\mathcal{W}\left(\mathscr{N}^f_n(x) - \alpha, \frac{\gamma}{2}\right) < \epsilon\Big\} \in \mathscr{F}(I).$$

Suppose $\mathcal{U}\big(\alpha, \frac{\gamma}{2}\big) = u$, $\mathcal{V}\big(\alpha, \frac{\gamma}{2}\big) = v$ and $\mathcal{W}\big(\alpha, \frac{\gamma}{2}\big) = w$, for all $\gamma > 0$. Since $u, v, w \in (0,1)$ and $0 < \epsilon < 1$, there exists $r_1, r_2, r_3 \in (0,1)$, such that $(1-\epsilon) * u > 1 - r_1$, $\epsilon \diamond v < r_2$ and $\epsilon \diamond w < r_3$, we have

$$\begin{aligned}
\mathcal{U}\big(\mathscr{N}^f_n(x),\gamma\big) &= \mathcal{U}\big(\mathscr{N}^f_n(x) - \alpha + \alpha, \gamma\big) \\
&\geq \mathcal{U}\left(\mathscr{N}^f_n(x) - \alpha, \frac{\gamma}{2}\right) * \mathcal{U}\left(\alpha, \frac{\gamma}{2}\right) \\
&> (1-\epsilon) * u \\
&> 1 - r_1,
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}\big(\mathscr{N}^f_n(x),\gamma\big) &= \mathcal{V}\big(\mathscr{N}^f_n(x) - \alpha + \alpha, \gamma\big) \\
&\leq \mathcal{V}\left(\mathscr{N}^f_n(x) - \alpha, \frac{\gamma}{2}\right) \diamond \mathcal{V}\left(\alpha, \frac{\gamma}{2}\right) \\
&< \epsilon \diamond v \\
&< r_2
\end{aligned}$$

and

$$\begin{aligned}\mathcal{W}\big(\mathscr{N}_n^f(x),\gamma\big) =&\mathcal{W}\big(\mathscr{N}_n^f(x)-\alpha+\alpha,\gamma\big)\\ \leq&\mathcal{W}\Big(\mathscr{N}_n^f(x)-\alpha,\frac{\gamma}{2}\Big)\diamond\mathcal{W}\Big(\alpha,\frac{\gamma}{2}\Big)\\ <&\epsilon\diamond w\\ <&r_3.\end{aligned}$$

Taking $r=\max\{r_1,r_2,r_3\}$, then $\big\{n\in\mathbb{N},\exists\ r\in(0,1):\mathcal{U}\big(\mathscr{N}_n^f(x),\gamma\big)>1-r$ and $\mathcal{V}\big(\mathscr{N}_n^f(x),\gamma\big)<r,\mathcal{W}\big(\mathscr{N}_n^f(x),\gamma\big)<r\big\}\in\mathscr{F}(I)\implies x=\{x_k\}\in\mathscr{N}^f_{I^\infty(\mathcal{S})}$ implies $\mathscr{N}^f_{I(\mathcal{S})}\subset\mathscr{N}^f_{I^\infty(\mathcal{S})}$. □

The contrary of an inclusion relation does not hold. To defend our claim, consider the following examples.

Example 3.1. Suppose $(\mathbb{R},\|.\|)$ be a normed space such that $\|x\|=\sup_k|x_k|$, and $\vartheta_1*\vartheta_2=\min\{\vartheta_1,\vartheta_2\}$ and $\vartheta_1\diamond\vartheta_2=\max\{\vartheta_1,\vartheta_2\}$, $\forall\vartheta_1,\vartheta_2\in(0,1)$. Now, define norms $\mathcal{S}=(\mathcal{U},\mathcal{V},\mathcal{W})$ on $\mathbb{R}^2\times(0,\infty)$ as follows

$$\mathcal{U}(x,\gamma)=\frac{\gamma}{\gamma+\|x\|}\quad,\ \mathcal{V}(x,\gamma)=\frac{\|x\|}{\gamma+\|x\|}\text{ and }\mathcal{W}(x,\gamma)=\frac{\|x\|}{\gamma}.$$

Then, $(\mathbb{R},\mathcal{S},*,\diamond)$ is a NNS. Consider the sequence (x_k)=$\{1\}$. It can be easily seen that $(x_k)\in\mathscr{N}^f_{I(\mathcal{S})}$ and $x_k\xrightarrow{I(\mathcal{S})}1$, but $x_k\notin\mathscr{N}^f_{I_0(\mathcal{S})}$.

Theorem 3.2. The spaces $\mathscr{N}^f_{I_0(\mathcal{S})}$ and $\mathscr{N}^f_{I(\mathcal{S})}$ are linear spaces.

Proof. We know that $\mathscr{N}^f_{I_0(\mathcal{S})}\subset\mathscr{N}^f_{I(\mathcal{S})}$. Then, we'll illustrate the outcome for $\mathscr{N}^f_{I(\mathcal{S})}$ The proof of linearity of the space $\mathscr{N}^f_{I_0(\mathcal{S})}$ follows similarly. Suppose sequences $x=\{x_k\},y=\{y_k\}\in\mathscr{N}^f_{I(\mathcal{S})}$. Then, there exist $\alpha_1,\alpha_2\in\mathbb{C}$, such that $\{x_k\}$ and $\{y_k\}$ I–converge to α_1 and α_2 respectively.

We will show that the sequence $\mu x_k+\nu y_k$ I–converges to $\mu\alpha_1+\nu\alpha_2$ for any scalars μ and ν. Consider the following sets for c and d

$$\mathscr{P}_1=\Bigg\{n\in\mathbb{N}:\mathcal{U}\Bigg(\mathscr{N}_n^f(x)-\alpha_1,\frac{\gamma}{2|\mu|}\Bigg)\leq1-\epsilon\text{ or }\mathcal{V}\Bigg(\mathscr{N}_n^f(x)-\alpha_1,\frac{\gamma}{2|\mu|}\Bigg)\geq\epsilon,\\ \mathcal{W}\Bigg(\mathscr{N}_n^f(x)-\alpha_1,\frac{\gamma}{2|\mu|}\Bigg)\geq\epsilon\Bigg\}\in I,$$

$$\mathscr{P}_2=\Bigg\{n\in\mathbb{N}:\mathcal{U}\Bigg(\mathscr{N}_n^f(y)-\alpha_2,\frac{\gamma}{2|\nu|}\Bigg)\leq1-\epsilon\text{ or }\mathcal{V}\Bigg(\mathscr{N}_n^f(y)-\alpha_2,\frac{\gamma}{2|\nu|}\Bigg)\geq\epsilon,\\ \mathcal{W}\Bigg(\mathscr{N}_n^f(y)-\alpha_2,\frac{\gamma}{2|\nu|}\Bigg)\geq\epsilon\Bigg\}\in I.$$

Now, we take the complement of $\mathscr{P}_1$ and $\mathscr{P}_2$

$$\mathscr{P}_1^c = \Big\{ n \in \mathbb{N} : \mathcal{U}\Big(\mathscr{N}_n^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\Big) > 1-\epsilon \text{ and } \mathcal{V}\Big(\mathscr{N}_n^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\Big) < \epsilon, \\ \mathcal{W}\Big(\mathscr{N}_n^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\Big) < \epsilon \Big\} \in F(I),$$

$$\mathscr{P}_2^c = \Big\{ n \in \mathbb{N} : \mathcal{U}\Big(\mathscr{N}_n^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\Big) > 1-\epsilon \text{ and } \mathcal{V}\Big(\mathscr{N}_n^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\Big) < \epsilon, \\ \mathcal{W}\Big(\mathscr{N}_n^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\Big) < \epsilon \Big\} \in F(I).$$

Consequently, set $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ produces $\mathscr{P} \in I$. Thus, $\mathscr{P}^c$ is a set that is not empty in $\mathcal{F}(I)$. We'll illustrate this for each $\{x_k\}, \{y_k\} \in \mathscr{N}_{I(\mathcal{S})}^f$

$$\mathscr{P}^c \subset \Big\{ n \in \mathbb{N} : \mathcal{U}\Big(\mathscr{N}_n^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma\Big) > 1-\epsilon \\ \text{and } \mathcal{V}\Big(\mathscr{N}_n^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma\Big) < \epsilon, \\ \mathcal{W}\Big(\mathscr{N}_i^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma\Big) < \epsilon \Big\}.$$

Let $i \in \mathscr{P}^c$. In this case,

$$\mathcal{U}\Big(\mathscr{N}_i^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\Big) > 1-\epsilon \text{ and } \mathcal{V}\Big(\mathscr{N}_i^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\Big) < \epsilon, \\ \mathcal{W}\Big(\mathscr{N}_i^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\Big) < \epsilon, \\ \mathcal{U}\Big(\mathscr{N}_i^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\Big) > 1-\epsilon \text{ and } \mathcal{V}\Big(\mathscr{N}_i^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\Big) < \epsilon, \\ \mathcal{W}\Big(\mathscr{N}_i^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\Big) < \epsilon.$$

Consider

$$\begin{aligned} &\mathcal{U}\Big(\mathscr{N}_i^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma\Big) \\ &\geq \mathcal{U}\Big(\mu\mathscr{N}_i^f(x) - \mu\alpha_1, \frac{\gamma}{2}\Big) * \mathcal{U}\Big(\nu\mathscr{N}_i^f(y) - \nu\alpha_2, \frac{\gamma}{2}\Big) \\ &= \mathcal{U}\Big(\mathscr{N}_i^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\Big) * \mathcal{U}\Big(\mathscr{N}_i^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\Big) \\ &> (1-\epsilon) * (1-\epsilon) \\ &> 1-\epsilon. \end{aligned}$$

$$\implies \mathcal{U}\bigg(\mathscr{N}_i^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma\bigg) > 1 - \epsilon$$

$$\begin{aligned}\mathcal{V}\bigg(\mathscr{N}_i^f(\mu x+\nu y)-(\mu\alpha_1+\nu\alpha_2),\gamma\bigg) &\leq \mathcal{V}\bigg(\mu\mathscr{N}_i^f(x)-\mu\alpha_1,\frac{\gamma}{2}\bigg)\diamond\mathcal{V}\bigg(\nu\mathscr{N}_i(y)-\nu\alpha_2,\frac{\gamma}{2}\bigg)\\ &= \mathcal{V}\bigg(\mathscr{N}_i^f(x)-\alpha_1,\frac{\gamma}{2|\mu|}\bigg)\diamond\mathscr{V}\bigg(\mathscr{N}_i^f(y)-\alpha_2,\frac{\gamma}{2|\nu|}\bigg)\\ &< \epsilon\diamond\epsilon\\ &< \epsilon.\end{aligned}$$

$$\implies \mathcal{V}\bigg(\mathscr{N}_i^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma\bigg) < \epsilon \text{ and}$$

$$\begin{aligned}&\mathcal{W}\bigg(\mathscr{N}_i^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma\bigg)\\ &\leq \mathcal{W}\bigg(\mu\mathscr{N}_i^f(x) - \mu\alpha_1, \frac{\gamma}{2}\bigg) \diamond \mathcal{W}\bigg(\mu\mathscr{N}_i(y) - \nu\alpha_2, \frac{\gamma}{2}\bigg)\\ &= \mathcal{W}\bigg(\mathscr{N}_i^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\bigg) \diamond \mathcal{W}\bigg(\mathscr{N}_i^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\bigg)\\ &< \epsilon \diamond \epsilon\\ &< \epsilon.\end{aligned}$$

$\implies \mathcal{W}(\mathscr{N}_i^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma) < \epsilon$. Thus, $\mathscr{P}^c \subset \{n \in \mathbb{N} : \mathcal{U}(\mathscr{N}_n^f(\mu x+\nu y)-(\mu\alpha_1+\nu\alpha_2),\gamma) > 1-\epsilon$ and $\mathcal{V}(\mathscr{N}_n^f(\mu x+\nu y)-(\mu\alpha_1+\nu\alpha_2),\gamma) < \epsilon, \mathcal{W}(\mathscr{N}_n^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma) < \epsilon\}$. Since $\mathscr{P}^c \in \mathscr{F}(I)$.

By the properties of $\mathscr{F}(I)$, we have $\{n \in \mathbb{N} : \mathcal{U}(\mathscr{N}_n^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma) > 1 - \epsilon$ and $\mathcal{V}(\mathscr{N}_n^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma) < \epsilon, \mathcal{W}(\mathscr{N}_n^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma) < \epsilon\} \in \mathscr{F}(I)$. It indicates that the sequence $(\mu x_k + \nu y_k)$ I–converge to $\mu\alpha_1 + \nu\alpha_2$. Therefore, $(\mu x_k + \nu y_k) \in \mathscr{N}_{I(\mathcal{S})}^f$. Hence, $\mathscr{N}_{I(\mathcal{S})}^f$ is linear space. □

Theorem 3.3. Each open ball in neutrosophic $0 < \epsilon < 1$ with centre at x and radius $0 < \jmath < 1$, i.e., $\mathscr{B}(x, \gamma, \epsilon)$ is an open set in $\mathscr{N}_{I(\mathcal{S})}^f$, where $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$ is a neutrosophic norm.

Proof. Suppose that $\mathscr{B}(x, \gamma, \epsilon)$ is an open ball with a radius of $\gamma > 0$ and a neutrosophic $0 < \epsilon < 1$ parameter, with its centre at $x = (x_k) \in \mathscr{N}_{I(\mathcal{S})}^f$

$$\begin{aligned}\mathscr{B}(x, \gamma, \epsilon) = \big\{y = (y_k) \in \omega : \{n \in \mathbb{N} : \mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma\big) \leq 1 - \epsilon\\ \text{or } \mathcal{V}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma\big) \geq \epsilon, \mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma\big) \geq \epsilon\} \in I\big\}.\end{aligned}$$

Then

$$\begin{aligned}\mathscr{B}^c(x, \gamma, \epsilon) = \big\{y = (y_k) \in \omega : \{n \in \mathbb{N} : \mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma\big) > 1 - \epsilon \text{ and}\\ \mathcal{V}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma\big) < \epsilon, \mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma\big) < \epsilon\} \in F(I)\big\}.\end{aligned}$$

Suppose $y = (y_k) \in \mathscr{B}^c(x, \gamma, \epsilon)$. Then, for $\mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma\big) > 1 - \epsilon$, $\mathcal{V}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma\big) < \epsilon$ and $\mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma\big) < \epsilon$ so, there exists $\gamma_0 \in (0, \gamma)$ such that $\mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma_0\big) > 1 - \epsilon$, $\mathcal{V}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma_0\big) < \epsilon$ and $\mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma_0\big) < \epsilon$.

Putting $\epsilon_0 = \mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma_0\big)$, we have $\epsilon_0 > 1 - \epsilon$. Then, $\exists\, p \in (0, 1)$ such that $\epsilon_0 > 1 - p > 1 - \epsilon$. For $\epsilon_0 > 1 - p$, we can have $\epsilon_1, \epsilon_2, \epsilon_3 \in (0, 1)$, such that $\epsilon_0 * \epsilon_1 > 1 - p$, $(1 - \epsilon_0) \diamond (1 - \epsilon_2) < p$. and $(1 - \epsilon_0) \diamond (1 - \epsilon_3) < p$. Let $\epsilon_4 = \max\{\epsilon_1, \epsilon_2, \epsilon_3\}$.

Now, consider the open ball $\mathscr{B}^c(y, \gamma - \gamma_0, 1 - \epsilon_4)$. We shall show that $\mathscr{B}^c(y, \gamma - \gamma_0, 1 - \epsilon_4) \subset \mathscr{B}^c(x, \gamma, \epsilon)$.

Let $z = \{z_k\} \in \mathscr{B}^c(y, \gamma - \gamma_0, 1 - \epsilon_4)$, then $\mathcal{U}\big(\mathscr{N}_n^f(y) - \mathscr{N}_n^f(z), \gamma - \gamma_0\big) > \epsilon_4$ and $\mathcal{V}\big(\mathscr{N}_n^f(y) - \mathscr{N}_n^f(z), \gamma - \gamma_0\big) < 1 - \epsilon_4$, $\mathcal{W}\big(\mathscr{N}_n^f(y) - \mathscr{N}_n^f(z), \gamma - \gamma_0\big) < 1 - \epsilon_4$. Therefore,

$$\begin{aligned} \mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(z), \gamma\big) &\geq \mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma_0\big) * \mathcal{U}\big(\mathscr{N}_n^f(y) - \mathscr{N}_n^f(z), \gamma - \gamma_0\big) \\ &\geq \epsilon_0 * \epsilon_4 \geq \epsilon_0 * \epsilon_1 \\ &> (1 - p) \\ &> (1 - \epsilon) \end{aligned}$$

$$\begin{aligned} \mathcal{V}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(z), \gamma\big) &\leq \mathcal{V}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma_0\big) \diamond \mathcal{V}\big(\mathscr{N}_n^f(y) - \mathscr{N}_n^f(z), \gamma - \gamma_0\big) \\ &\leq (1 - \epsilon_0) \diamond (1 - \epsilon_4) \leq \epsilon_0 \diamond \epsilon_2 \\ &< p \\ &< \epsilon \end{aligned}$$

and

$$\begin{aligned} \mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(z), \gamma\big) &\leq \mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma_0\big) \diamond \mathcal{W}\big(\mathscr{N}_n^f(y) - \mathscr{N}_n^f(z), \gamma - \gamma_0\big) \\ &\leq \epsilon_0 \diamond \epsilon_4 \leq \epsilon_0 \diamond \epsilon_3 \\ &< p \\ &< \epsilon \end{aligned}$$

Therefore, the set $\big\{n \in \mathbb{N} : \mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(z), \gamma\big) > 1 - \epsilon$ and $\mathcal{V}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(z), \gamma\big) < \epsilon$, $\mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(z), \gamma\big) < \epsilon\big\} \in \mathscr{F}(I)$.

$\implies z = (z_k) \in \mathscr{B}^c(x, \gamma, \epsilon)$,

$\implies \mathscr{B}^c\big(y, \gamma - \gamma_0, 1 - \epsilon_4\big) \subset \mathscr{B}^c\big(x, \gamma, \epsilon\big)$. □

Remark 3.1. The spaces $\mathscr{N}_{I(\mathcal{S})}^f$ and $\mathscr{N}_{I_0(\mathcal{S})}^f$ are Nörland I-convergent and Nörland I-null in NNS with respect to neutrosophic norms $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$.

Now, define a collection $\tau_{I(\mathcal{S})}^{\mathscr{N}^f}$ of a subset of $\mathscr{N}_{I(\mathcal{S})}^f$ as follows: $\tau_{I(\mathcal{S})}^{\mathscr{N}^f} = \{P \subset \mathscr{N}_{I(\mathcal{S})}^f$: for every $x = (x_k) \in P\ \exists\ \gamma > 0$ and $\epsilon \in (0, 1)$ s.t $\mathscr{B}(x, \gamma, \epsilon) \subset P\}$. Then, $\tau_{I(\mathcal{S})}^{\mathscr{N}^f}$ constructs a topology on sequence space $\mathscr{N}_{I(\mathcal{S})}^f$. The collection described by $\mathcal{B} = \{\mathscr{B}(x, \gamma, \epsilon) : b \in \mathscr{N}^f_{I(\mathcal{S})}, r > 0 \text{ and } \epsilon \in (0,1)\}$ is the topology's base $\tau^{\mathscr{N}^f}_{I(\mathcal{S})}$ on the space $\mathscr{N}^f_{I(\mathcal{S})}$.

Theorem 3.4. The topology $\tau^{\mathscr{N}^f}_{I(\mathcal{S})}$ on the space $\mathscr{N}^f_{I_0(\mathcal{S})}$ is first countable.

Proof. For every $x = \{x_k\} \in \mathscr{N}^f_{I(\mathcal{S})}$, consider the set $\mathcal{B} = \{\mathscr{B}(x, \frac{1}{n}, \frac{1}{n})\} : n = 1, 2, 3, 4, ...\}$, which is a local countable basis at $x = (x_k)$. As a result, the topology $\tau^{\mathscr{N}^f}_{I(\mathcal{S})}$ on the space $\mathscr{N}^f_{I_0(\mathcal{S})}$ is first countable. □

Theorem 3.5. The spaces $\mathscr{N}^f_{I(\mathcal{S})}$ and $\mathscr{N}^f_{I_0(\mathcal{S})}$ are Hausdorff spaces.

Proof. We know that $\mathscr{N}^f_{I_0(\mathcal{S})} \subset \mathscr{N}^f_{I(\mathcal{S})}$.

We will only show the solution for $\mathscr{N}^f_{I(\mathcal{S})}$. Suppose $x = (x_k), y = (y_k) \in \mathscr{N}^f_{I(\mathcal{S})}$ as well as $x \neq y$. Then, for any $n \in \mathbb{N}$ and $\gamma > 0$, implies $0 < \mathcal{U}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma) < 1$, $0 < \mathcal{V}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma) < 1$ and $0 < \mathcal{W}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma) < 1$.

Putting $\epsilon_1 = \mathcal{U}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma)$, $\epsilon_2 = \mathcal{V}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma)$, $\epsilon_3 = \mathcal{W}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma)$ and $\epsilon = \max\{\epsilon_1, 1-\epsilon_2, 1-\epsilon_3\}$. Then, for each $\epsilon_0 \in (\epsilon, 1)$ there exist $\epsilon_4, \epsilon_5, \epsilon_6 \in (0,1)$, such that $\epsilon_4 * \epsilon_4 \geq \epsilon_0$, $(1-\epsilon_5) \diamond (1-\epsilon_5) \leq (1-\epsilon_0)$ and $(1-\epsilon_6) \diamond (1-\epsilon_6) \leq (1-\epsilon_0)$. Once again putting $\epsilon_7 = \max\{\epsilon_4, 1-\epsilon_5, 1-\epsilon_6, \}$, think about the open balls. $\mathscr{B}(x, 1-\epsilon_7, \frac{\gamma}{2})$ and $\mathscr{B}(y, 1-\epsilon_7, \frac{\gamma}{2})$ respectively centred at x and y. Then, it is obvious that $\mathscr{B}^c(x, 1-\epsilon_7, \frac{\gamma}{2}) \cap \mathscr{B}^c(y, 1-\epsilon_7, \frac{\gamma}{2}) = \phi$.

If possible let $x = \{x_k\} \in \mathscr{B}^c(x, 1-\epsilon_7, \frac{\gamma}{2}) \cap \mathscr{B}^c(y, 1-\epsilon_7, \frac{\gamma}{2})$. Then, we have

$$\begin{aligned} \epsilon_1 &= \mathcal{U}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma) \\ &\geq \mathcal{U}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(z), \frac{\gamma}{2}) * \mathcal{U}(\mathscr{N}^f_n(z) - \mathscr{N}^f_n(y), \frac{\gamma}{2}) \\ &> \epsilon_7 * \epsilon_7 \\ &\geq \epsilon_4 * \epsilon_4 \\ &\geq \epsilon_0 \\ &> \epsilon_1, \end{aligned} \tag{9}$$

$$\begin{aligned} \epsilon_2 &= \mathcal{V}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma) \\ &\leq \mathcal{V}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(z), \frac{\gamma}{2}) \diamond \mathcal{V}(\mathscr{N}^f_n(z) - \mathscr{N}^f_n(y), \frac{\gamma}{2}) \\ &< (1-\epsilon_7) \diamond (1-\epsilon_7) \\ &\leq (1-\epsilon_5) \diamond (1-\epsilon_5) \\ &\leq (1-\epsilon_0) \\ &< \epsilon_2 \end{aligned} \tag{10}$$

and

$$\begin{aligned}\epsilon_3 &= \mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}^f(y), \gamma\big)\\ &\leq \mathcal{W}(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(z), \frac{\gamma}{2}) \diamond \mathcal{W}(\mathscr{N}_n^f(z) - \mathscr{N}_n^f(y), \frac{\gamma}{2})\\ &< (1-\epsilon_7) \diamond (1-\epsilon_7)\\ &\leq (1-\epsilon_6) \diamond (1-\epsilon_6)\\ &\leq (1-\epsilon_0)\\ &< \epsilon_3.\end{aligned} \tag{11}$$

We have a contradiction from equations (9), (10) and (11). Therefore, $\mathscr{B}^c(x, 1-\epsilon_7, \frac{\gamma}{2}) \cap \mathscr{B}^c(y, 1-\epsilon_7, \frac{\gamma}{2}) = \phi$. Hence, the space $\mathscr{N}^f_{I(\mathcal{S})}$ is a Hausdorff space. □

Theorem 3.6. Suppose $\tau^{\mathscr{N}^f}_{I(\mathcal{S})}$ be a topology on a neutrosophic norm spaces $\mathscr{N}^f_{I(\mathcal{S})}$, then a sequence $x = \{x_k\} \in \mathscr{N}^f_{I(\mathcal{S})}$, such that $(x_k) \longrightarrow \alpha$, iff $\mathcal{U}(\mathscr{N}_n^f(x) - \alpha) \longrightarrow 1, \mathcal{V}(\mathscr{N}_n^f(x) - \alpha) \longrightarrow 0$ and $\mathcal{W}(\mathscr{N}_n^f(x) - \alpha) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof. Consider a sequence $\{x_k\} \to \alpha$, and Fix $\gamma_0 > 0$, then for $\gamma \in (0,1)$, $\exists$ $n_0 \in \mathbb{N}$ s.t. $\{x_k\} \in \mathscr{B}(x, \gamma, \epsilon)$, $\forall\ k \geq n_0$, then for a $\gamma > 0$, $\mathscr{B}(x, \gamma, \epsilon) = \{x = (x_k) \in \omega : \mathcal{U}(\mathscr{N}_n^f(x) - \alpha, \gamma) \leq 1-\epsilon$ or $\mathcal{V}(\mathscr{N}_n^f(x) - \alpha, \gamma) \geq \epsilon$, $\mathcal{W}(\mathscr{N}_n^f(x) - \alpha, \gamma) \geq \epsilon\} \in I$, such that $\mathscr{B}^c(x, \gamma, \epsilon) \in \mathscr{F}(I)$ then

$$1 - \mathcal{U}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon, \mathscr{V}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon, \mathcal{W}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon.$$

Hence, $\mathcal{U}(\mathscr{N}_n^f(x) - \alpha, \gamma) \to 1, \mathcal{V}(\mathscr{N}_n^f(x) - \alpha, \gamma) \to 0,$ and $\mathcal{W}(\mathscr{N}_n^f(x) - \alpha, \gamma) \to 0$ as $n \to \infty$. Conversely, if $\forall\ \gamma > 0, \mathcal{U}(\mathscr{N}_n^f(x) - \alpha, \gamma) \to 1, \mathcal{V}(\mathscr{N}_n^f(x) - \alpha, \gamma) \to 0,$ and $\mathcal{W}(\mathscr{N}_n^f(x) - \alpha, \gamma) \to 0$ as $n \to \infty$. Then, for each $\epsilon \in (0,1)$, $\exists\ n_0 \in \mathbb{N}$ s.t. $1 - \mathcal{U}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon, \mathcal{V}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon, \mathcal{W}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon\ \forall\ n \geq n_0$. Hence, we have

$$\mathcal{U}(\mathscr{N}_n^f(x) - \alpha, \gamma) > 1-\epsilon, \mathcal{V}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon, \mathcal{W}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon, \forall\ n \geq n_0.$$

Thus, $\{x_k\} \in \mathscr{B}^c(x, \gamma, \epsilon)$, $\forall\ k \geq n_0$ and hence $\{x_k\} \to \alpha$. □

Now, we establish results about the relationship between Nörlund I-convergent and Nörlund I-Cauchy sequence in NNS.

Definition 3.1. In an NNS V. A sequence $x = \{x_n\} \in V$ is said to be Nörlund I-convergent to $\alpha \in \mathbb{C}$ with regard to neutrosophic norms $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$, denoted by $x_n \to \alpha$, if for every $\epsilon \in (0,1)$ and $\gamma > 0$, where

$$\begin{aligned}N_1 = \Big\{n \in \mathbb{N} : \mathcal{U}\Big(\mathscr{N}_n^f(x) - \alpha, \gamma\Big) \leq 1-\epsilon\\ \text{or } \mathcal{V}\Big(\mathscr{N}_n^f(x) - \alpha, \gamma\Big) \geq \epsilon, \mathcal{W}\Big(\mathscr{N}_n^f(x) - \alpha, \gamma\Big) \geq \epsilon\Big\} \in I\end{aligned}$$

and we write $I_{\mathcal{S}}\text{–}\lim(x_n) = \alpha$.

Definition 3.2. A sequence $x = \{x_n\} \in V$ is said to Nörlund I-Cauchy with respect to neutrosophic norms $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$, if for every $\epsilon \in (0,1)$ and $\gamma > 0$, $\exists\ k \in \mathbb{N}$, such that

$$N_2 = \Big\{ n \in \mathbb{N} : \mathcal{U}\Big(\mathscr{N}_n^f(x) - \mathscr{N}_k^f(x), \gamma\Big) \leq 1 - \epsilon \\ \text{or } \mathcal{V}\Big(\mathscr{N}_n^f(x) - \mathscr{N}_k^f(x), \gamma\Big) \geq \epsilon, \mathcal{W}\Big(\mathscr{N}_n^f(x) - \mathscr{N}_k^f(x), \gamma\Big) \geq \epsilon \Big\} \in I.$$

Theorem 3.7. Let $\mathscr{N}_{I(\mathcal{S})}^f$ be an NNS. If a sequence $x = \{x_k\} \in$ is Nörlund I–convergent w.r.t NN $\mathcal{S}$, then the $I_{(\mathcal{S})}$–$lim(x)$ is unique.

Proof. Let $x = \{x_k\}$ is Nörlund I–convergent in NNS. Let on contrary that α_1 and α_2 are two distinct elements, thus $I_{(\mathcal{S})}-\lim(x_k) = \alpha_1$ and $I_{(\mathcal{S})}-\lim(x_k) = \alpha_2$. For a given $\epsilon > 0$, choose $p > 0$ such that $(1-p) * (1-p) > 1 - \epsilon$, $p \diamond p < \epsilon$ and $p \diamond p < \epsilon$, for $\gamma > 0$.

We show that $\alpha_1 = \alpha_2$. We define $P_1 = \{n \in \mathbb{N} : \mathcal{U}(\mathscr{N}_n^f(x) - \alpha_1, \gamma) \leq 1 - \epsilon\}$, $P_2 = \{n \in \mathbb{N} : \mathcal{V}(\mathscr{N}_n^f(x) - \alpha_1, \gamma) \geq \epsilon\}$, $P_3 = \{n \in \mathbb{N} : \mathcal{W}(\mathscr{N}_n^f(x) - \alpha_1, \gamma) \geq \epsilon\}$, $Q_1 = \{n \in \mathbb{N} : \mathcal{U}(\mathscr{N}_n^f(x) - \alpha_2, \gamma) \leq 1-\epsilon\}$, $Q_2 = \{n \in \mathbb{N} : \mathcal{V}(\mathscr{N}_n^f(x) - \alpha_2, \gamma) \geq \epsilon\}$, $Q_3 = \{n \in \mathbb{N} : \mathcal{W}(\mathscr{N}_n^f(x) - \alpha_2, \gamma) \geq \epsilon\}$, where $A = (P_1 \cup Q_1) \cap (P_2 \cup Q_2) \cap (P_3 \cup Q_3)$ sets $P_1, P_2, P_3, Q_1, Q_2, Q_3$ and A must be belongs to I, since $\{x_k\}$ has two distinct I-limits with regard to neutrosophic norm $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$, *i.e.* α_1, α_2. As a result, $A^c \in \mathscr{F}(I)$ implies that A^c is not empty. Let us write some $n_0 \in A^c$ then either $n_0 \in P_1^c \cap Q_1^c$ or $n_0 \in P_2^c \cap Q_2^c$ or $n_0 \in P_3^c \cap Q_3^c$.

If $n_0 \in P_1^c \cap Q_1^c$, it follows that

$$\mathcal{U}\Big(\mathscr{N}_{n_0}^f(x) - \alpha_1, \frac{\gamma}{2}\Big) > 1 - p \text{ and } \mathcal{U}\Big(\mathscr{N}_{n_0}^f(x) - \alpha_2, \frac{\gamma}{2}\Big) > 1 - p.$$

Hence,

$$\begin{aligned} \mathcal{U}\Big(\alpha_1 - \alpha_2, \gamma\Big) &\geq \mathcal{U}\Big(\mathscr{N}_{n_0}^f(x) - \alpha_1, \frac{\gamma}{2}\Big) * \mathcal{U}\Big(\mathscr{N}_{n_0}^f(x) - \alpha_2, \frac{\gamma}{2}\Big) \\ &> (1-p) * (1-p) \\ &> (1-\epsilon). \end{aligned}$$

Because $\epsilon > 0$ was arbitrary, $\mathcal{U}(\alpha_1 - \alpha_2, \gamma) = 1$ was given to all $\gamma > 0$. Thus, we have $\alpha_1 = \alpha_2$, which is a contradiction.

If $n_0 \in P_2^c \cap Q_2^c$, it follows that

$$\mathcal{V}\Big(\mathscr{N}_{n_0}^f(x) - \alpha_1, \frac{\gamma}{2}\Big) < p \text{ and } \mathcal{V}\Big(\mathscr{N}_{n_0}^f(x) - \alpha_2, \frac{\gamma}{2}\Big) < p.$$

Hence,

$$\begin{aligned}\mathcal{V}\Big(\alpha_1-\alpha_2,\gamma\Big) &\le \mathcal{V}\Big(\mathscr{N}^f_{n_0}(x)-\alpha_1,\frac{\gamma}{2}\Big)\diamond\mathcal{V}\Big(\mathscr{N}^f_{n_0}(x)-\alpha_2,\frac{\gamma}{2}\Big)\\ &< p\diamond p\\ &< \epsilon.\end{aligned}$$

Because $\epsilon>0$ was arbitrary, $\mathcal{V}(\alpha_1-\alpha_2,\gamma)=0$ was given to all $\gamma>0$. Thus, we have $\alpha_1=\alpha_2$, which is a contradiction.

If $n_0\in P_3^c\cap Q_3^c$, it follows that

$$\mathcal{W}\Big(\mathscr{N}^f_{n_0}(x)-\alpha_1,\frac{\gamma}{2}\Big)<p \text{ and } \mathcal{W}\Big(\mathscr{N}^f_{n_0}(x)-\alpha_2,\frac{\gamma}{2}\Big)<p.$$

Hence,

$$\begin{aligned}\mathcal{W}\Big(\alpha_1-\alpha_2,\gamma\Big) &\le \mathcal{W}\Big(\mathscr{N}^f_{n_0}(x)-\alpha_1,\frac{\gamma}{2}\Big)\diamond\mathcal{W}\Big(\mathscr{N}^f_{n_0}(x)-\alpha_2,\frac{\gamma}{2}\Big)\\ &< p\diamond p\\ &< \epsilon.\end{aligned}$$

Because $\epsilon>0$ was arbitrary, $\mathcal{W}(\alpha_1-\alpha_2,\gamma)=0$ was given to all $\gamma>0$. Thus, we have $\alpha_1=\alpha_2$, which is a contradiction.

As an outcome, in all cases, α_1=α_2, implying that the $I_{(\mathcal{S})}$–limit is unique. □

Now, we establish results about the relationship between Nörlund I-convergent and Nörlund I-Cauchy sequence in NNS.

Theorem 3.8. A sequence $x=\{x_k\}\in\mathscr{N}^f_{I(\mathcal{S})}$ is I–convergent with regard to neutrosophic norms $\mathcal{S}=(\mathcal{U},\mathcal{V},\mathcal{W})$ if and olny if it is I–Cauchy with respect to the same norms.

Proof. Let $x=(x_k)$ is Nörlund I–convergent with regard to neutrosophic norms $(\mathcal{S})$ such that $I_{(\mathcal{S})}$–$\lim(x_k)=\alpha$. For given $\epsilon\in(0,1)$ there exists $p_1\in(0,1)$, such that $(1-p_1)*(1-p_1)>1-\epsilon$ and $p_1\diamond p_1<\epsilon$. Since $I_{(\mathcal{S})}$–$\lim(x_k)=\alpha$ therefore, for all $\gamma>0$, $A_1=\big\{n\in\mathbb{N}:\mathcal{U}\big(\mathscr{N}^f_n(x)-\alpha,\gamma\big)\le 1-p_1$ or $\mathcal{V}\big(\mathscr{N}^f_n(x)-\alpha,\gamma\big)\ge p_1,\mathcal{W}\big(\mathscr{N}^f_n(x)-\alpha,\gamma\big)\ge p_1\big\}\in I$, that implies $A_1^c=\big\{n\in\mathbb{N}:\mathcal{U}\big(\mathscr{N}^f_n(x)-\alpha,\gamma\big)>1-p_1$ and $\mathcal{V}\big(\mathscr{N}^f_n(x)-\alpha,\gamma\big)<p_1,\mathcal{W}\big(\mathscr{N}^f_n(x)-\alpha,\gamma\big)<p_1\big\}\in\mathscr{F}(I)$. Let a natural number $J\in A_1^c$, we have $\mathcal{U}\big(\mathscr{N}^f_J(x)-\alpha,\gamma\big)>1-p_1$ and $\mathcal{V}\big(\mathscr{N}^f_J(x)-\alpha,\gamma\big)<p_1,\mathcal{W}\big(\mathscr{N}^f_J(x)-\alpha,\gamma\big)<p_1$.

Now, we show that for $x\in\mathscr{N}^f_{I(\mathcal{S})}$ $\exists$ a natural number $J=J(x,\epsilon,\gamma)$ s.t. the set $A_2=\big\{n\in\mathbb{N}:\mathcal{U}\big(\mathscr{N}^f_n(x)-\mathscr{N}^f_J(x),\gamma\big)\le 1-\epsilon$ or $\mathcal{V}\big(\mathscr{N}^f_n(x)-\mathscr{N}^f_J(x),\gamma\big)\ge\epsilon,\mathcal{W}\big(\mathscr{N}^f_n(x)-\mathscr{N}^f_J(\vartheta),\gamma\big)\ge\epsilon\big\}\in I$. For this, we need prove that $A_2\subset A_1$., Let on contrary that $A_2 \nsubseteq A_1$. Then, $\exists\, l \in A_2$, but not in A_1 we have $\mathcal{U}\big(\mathcal{N}_l^f(x) - \mathcal{N}_J^f(x), \gamma\big) \leq 1-\epsilon$. Then, $\mathcal{U}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) > 1-p_1$.

In particular, $\mathcal{U}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) > 1-p_1$. Then

$$\begin{aligned} 1-\epsilon &\geq \mathcal{U}\big(\mathcal{N}_l^f(x) - \mathcal{N}_J^f(x), \gamma\big)\\ &\geq \mathcal{U}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) * \mathcal{U}\big(\mathcal{N}_J^f(x) - \alpha, \frac{\gamma}{2}\big)\\ &> (1-p_1) * (1-p_1)\\ &> (1-\epsilon) \end{aligned}$$

which is a contradiction.

$\implies \mathcal{U}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) \leq 1-p_1$.

Similarly, consider $\mathcal{V}\big(\mathcal{N}_l^f(x) - \mathcal{N}_J^f(x), \gamma\big) \geq \epsilon$. Then, $\mathcal{V}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) < p_1$. In particular, $\mathcal{V}\big(\mathcal{N}_n^f(x) - \alpha, \frac{\gamma}{2}\big) < p_1$. Then

$$\begin{aligned} \epsilon &\leq \mathcal{V}\big(\mathcal{N}_l^f(x) - \mathcal{N}_J^f(x), \gamma\big)\\ &\leq \mathcal{V}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) \diamond \mathcal{V}\big(\mathcal{N}_J^f(x) - \alpha, \frac{\gamma}{2}\big)\\ &< p_1 \diamond p_1\\ &< \epsilon \end{aligned}$$

which is a contradiction.

$\implies \mathcal{V}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) \geq p_1$ and similarly consider $\mathcal{W}\big(\mathcal{N}_l^f(x) - \mathcal{N}_J^f(x), \gamma\big) \geq \epsilon$. Then, $\mathcal{W}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) < p_1$.

In particular $\mathcal{W}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) < p_1$. Then

$$\begin{aligned} \epsilon &\leq \mathcal{W}\big(\mathcal{N}_l^f(x) - \mathcal{N}_J^f(x), \gamma\big)\\ &\leq \mathcal{W}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) \diamond \mathcal{W}\big(\mathcal{N}_J^f(x) - \alpha, \frac{\gamma}{2}\big)\\ &< p_1 \diamond p_1\\ &< \epsilon \end{aligned}$$

which is again a contradiction.

$\implies \mathcal{W}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) \geq p_1$.

Therefore, for $l \in A_2$, we have $\mathcal{U}\big(\mathcal{N}_l^f(x) - \alpha, \gamma\big) \leq 1-p_1$ or $\mathcal{V}\big(\mathcal{N}_l^f(x) - \alpha, \gamma\big) \geq p_1, \mathcal{W}\big(\mathcal{N}_l^f(x) - \alpha, \gamma\big) \geq p_1$.

$\implies l \in A_1$. Hence, $A_2 \subset A_1$. Since $A_1 \in I$, so $A_2 \in I$. Consequently, the sequence $x = \{x_k\}$ is Nörlund I–Cauchy with regard to norms $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$.

Conversely, suppose the sequence $x = \{x_k\}$ is Nörlund I–Cauchy with regard to the norms $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$. Then, $\exists\, j \in \mathbb{N}$ such that $B_1 = \big\{n \in \mathbb{N} : \mathcal{U}\big(\mathcal{N}_n^f(x) - \mathcal{N}_j^f(x), \gamma\big) \leq 1-\epsilon$ or $\mathcal{V}\big(\mathcal{N}_n^f(x) - \mathcal{N}_j^f(x), \gamma\big) \geq \epsilon, \mathcal{W}\big(\mathcal{N}_n^f(x) - \mathcal{N}_j^f(x), \gamma\big) \geq \epsilon\big\} \in I$. But, on the other hand, the sequence $x = (x_k)$ is not Nörlund I–convergent denoted by B_2,

$$B_2 = \Big\{ n \in \mathbb{N} : \mathcal{U}\Big(\mathscr{N}_n^f(x) - \alpha, \frac{\gamma}{2}\Big) > 1 - p_1 \text{ or } \mathcal{V}\Big(\mathscr{N}_n^f(x) - \alpha, \frac{\gamma}{2}\Big) < p_1, \\ \mathcal{W}\Big(\mathscr{N}_n^f(x) - \alpha, \frac{\gamma}{2}\Big) < p_1 \Big\} \in I,$$

$\implies$

$$\begin{aligned} 1 - \epsilon &\geq \mathcal{U}\Big(\mathscr{N}_n^f(x) - \mathscr{N}_j^f(x), \gamma\Big) \\ &\geq \mathcal{U}\Big(\mathscr{N}_n^f(x) - \alpha, \frac{\gamma}{2}\Big) * \mathcal{U}\Big(\mathscr{N}_j^f(x) - \alpha, \frac{\gamma}{2}\Big) \\ &> (1 - p_1) * (1 - p_1) \\ &> 1 - \epsilon \end{aligned}$$

which is a contradiction. Now,

$$\begin{aligned} \epsilon &\leq \mathcal{V}\Big(\mathscr{N}_n^f(x) - \mathscr{N}_j^f(x), \gamma\Big) \\ &\leq \mathcal{V}\Big(\mathscr{N}_n^f(x) - \alpha, \frac{\gamma}{2}\Big) \diamond \mathcal{V}\Big(\mathscr{N}_j^f(x) - \alpha, \frac{\gamma}{2}\Big) \\ &< p_1 \diamond p_1 \\ &< \epsilon \end{aligned}$$

which is again a contradiction and

$$\begin{aligned} \epsilon &\leq \mathcal{W}\Big(\mathscr{N}_n^f(x) - \mathscr{N}_j^f(x), \gamma\Big) \\ &\leq \mathcal{W}\Big(\mathscr{N}_n^f(x) - \alpha, \frac{\gamma}{2}\Big) \diamond \mathcal{W}\Big(\mathscr{N}_j^f(x) - \alpha, \frac{\gamma}{2}\Big) \\ &< p_1 \diamond p_1 \\ &< \epsilon. \end{aligned}$$

This again contradicts it. Therefore, $B_2 \in \mathscr{F}(I)$, and hence $x = \{x_k\}$ is Nörlund I–convergent. □

The following theorems are easy to prove.

Theorem 3.9. In NNS V, a sequence $x = \{x_k\} \in V$ is Nörlund Cauchy with regard to NN $\mathcal{S}$. and $\mathscr{N}_{I(\mathcal{S})}^f$ cluster to α in Z then $\{x_k\}$ is Nörlund I-convergent to α with regard to same NN S.

Theorem 3.10. In NNS V, a sequence $x = \{x_k\} \in V$ is Nörlund Cauchy with regard to NN $\mathcal{S}$ then it is Nórlund I-Cauchy with regard to NN $\mathcal{S}$.

Now, follows the notations:

The space of all sequences whose N^f- transform is neutrosophic bounded sequence is denoted as $l^{\infty}_{(\mathcal{S})}(\mathcal{N}^f)$.

$\mathcal{N}^f_{I^{\infty}_{(\mathcal{S})}}$ indicates the space containing all sequences with neutrosophic bounded N^f- transforms and neutrosophic Norland ideal convergent sequences.

Theorem 3.11. Space $\mathcal{N}^f_{I^{\infty}_{(\mathcal{S})}}$ is closed linear space of $l^{\infty}_{(\mathcal{S})}(\mathcal{N}^f)$.

Proof. The given space is a subspace of $l^{\infty}_{(\mathcal{S})}(\mathcal{N}^f)$, as we are aware. Now, that $\mathcal{N}^f_{I^{\infty}_{(\mathcal{S})}}$ must be proved to be closed, we demonstrate that $\overline{\mathcal{N}^f_{I^{\infty}_{(\mathcal{S})}}} = \mathcal{N}^f_{I^{\infty}_{(\mathcal{S})}}$. (where $\overline{\mathcal{N}^f_{I^{\infty}_{(\mathcal{S})}}}$ denoted the closure of $\mathcal{N}^f_{I^{\infty}_{(\mathcal{S})}}$).

It is clear that $\mathcal{N}^f_{I^{\infty}_{(\mathcal{S})}} \subset \overline{\mathcal{N}^f_{I^{\infty}_{(\mathcal{S})}}}$.

Conversely, we show that $\overline{\mathcal{N}^f_{I^{\infty}_{(\mathcal{S})}}} \subset \mathcal{N}^f_{I^{\infty}_{(\mathcal{S})}}$.

Let $x \in \overline{\mathcal{N}^f_{I^{\infty}_{(\mathcal{S})}}}$ then , $\mathscr{B}(x,\gamma,\epsilon) \cap \mathcal{N}^f_{I^{\infty}_{(\mathcal{S})}} \neq \phi$, for evey open ball $\mathscr{B}(x,\gamma,\epsilon)$ of any radius $\gamma > 0$ and $\epsilon > 0$ centred at x. So, let $x \in \mathscr{B}(x,\gamma,\epsilon) \cap \mathcal{N}^f_{I^{\infty}_{(\mathcal{S})}}$ and $0 < p < 1$ and $\gamma > 0$, choose $\epsilon \in (0,1)$ s.t. $(1-p) * (1-p) > 1-\epsilon$ and $p \diamond p < \epsilon$.

Since $y \in \mathscr{B}(x,\gamma,\epsilon) \cap \mathcal{N}^f_{I^{\infty}_{(\mathcal{S})}}$ so, there exits a subset A of $\mathbb{N}$ s.t $A \in \mathscr{F}(I)$ and $\forall n \in A$, we have $\mathcal{U}(\mathcal{N}^f_n(x) - \mathcal{N}^f_n(y), \frac{\gamma}{2}) > 1-p$ and $\mathcal{V}(\mathcal{N}^f_n(x) - \mathcal{N}^f_n(y), \frac{\gamma}{2}) < p$, $\mathcal{W}(\mathcal{N}^f_n(x) - \mathcal{N}^f_n(y), \frac{\gamma}{2}) < p$ and $\mathcal{U}(\mathcal{N}^f_n(y) - \alpha, \frac{\gamma}{2}) > 1-p$ and $\mathcal{V}(\mathcal{N}^f_n(y) - \alpha, \frac{\gamma}{2}) < p$, $\mathcal{W}(\mathcal{N}^f_n(y) - \alpha, \frac{\gamma}{2}) < p$.

Hence, $\forall n \in A$, we obtain

$$\begin{aligned}
\mathcal{U}(\mathcal{N}^f_n(x) - \alpha, \frac{\gamma}{2}) &= \mathcal{U}(\mathcal{N}^f_n(x) - \mathcal{N}^f_n(y) + \mathcal{N}^f_n(y) - \alpha, \gamma) \\
&\geq \mathcal{U}(\mathcal{N}^f_n(x) - \mathcal{N}^f_n(y), \frac{\gamma}{2}) * \mathcal{U}(\mathcal{N}^f_n(y) - \alpha, \frac{\gamma}{2}) \\
&> (1-p) * (1-p) \\
&> 1 - \epsilon,
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}(\mathcal{N}^f_n(x) - \alpha, \frac{\gamma}{2}) &= \mathcal{V}(\mathcal{N}^f_n(x) - \mathcal{N}^f_n(y) + \mathcal{N}^f_n(y) - \alpha, \gamma) \\
&\leq \mathcal{U}(\mathcal{N}^f_n(x) - \mathcal{N}^f_n(y), \frac{\gamma}{2}) \diamond \mathcal{V}(\mathcal{N}^f_n(y) - \alpha, \frac{\gamma}{2}) \\
&< p \diamond p \\
&< \epsilon
\end{aligned}$$

and

$$\begin{aligned}\mathcal{W}(\mathscr{N}_n^f(x)-\alpha,\frac{\gamma}{2}) &= \mathcal{W}(\mathscr{N}_n^f(x)-\mathscr{N}_n^f(y)+\mathscr{N}_n^f(y)-\alpha,\gamma)\\ &\le \mathcal{W}(\mathscr{N}_n^f(x)-\mathscr{N}_n^f(y),\frac{\gamma}{2})\diamond\mathcal{W}(\mathscr{N}_n^f(y)-\alpha,\frac{\gamma}{2})\\ &< p\diamond p\\ &< \epsilon.\end{aligned}$$

Thus, $A\subset\{n\in\mathbb{N}:\mathcal{U}(\mathscr{N}_n^f(x)-\alpha,\gamma)>1-\epsilon$ and $\mathcal{V}(\mathscr{N}_n^f(x)-\alpha,\gamma)<\epsilon\ ,\ \mathcal{W}(\mathscr{N}_n^f(x)-\alpha,\gamma)<\epsilon\}$.

As $A\in\mathscr{F}(I)$, which implies that $\{n\in\mathbb{N}:\mathcal{U}(\mathscr{N}_n^f(x)-\alpha,\gamma)>1-\epsilon$ and $\mathcal{V}(\mathscr{N}_n^f(x)-\alpha,\gamma)<\epsilon\ ,\ \mathcal{W}(\mathscr{N}_n^f(x)-\alpha,\gamma)<\epsilon\}\in\mathscr{F}(I)$. Therefore, $x\in\mathscr{N}^f_{I^\infty_{(\mathcal{S})}}$. Hence, $\overline{\mathscr{N}^f_{I^\infty_{(\mathcal{S})}}}\subset\mathscr{N}^f_{I^\infty_{(\mathcal{S})}}$. □

Theorem 3.12. Let $x=\{x_k\}\in\omega$ be a sequence. If $\exists$ a sequence $y=\{y_k\}\in\mathscr{N}^f_{I(\mathcal{S})}$, such that $\mathscr{N}_n^f(x)=\mathscr{N}_n^f(y)$ for almost all n relative to neutrosophic I, then $x\in\mathscr{N}^f_{I(\mathcal{S})}$.

Proof. Consider $\mathscr{N}_n^f(x)=\mathscr{N}_n^f(y)$ for almost all n relative to I. Then $\{n\in\mathbb{N}:\mathscr{N}_n^f(x)\neq\mathscr{N}_n^f(y)\}\in I$. This implies $\{n\in\mathbb{N}:\mathscr{N}_n^f(x)=\mathscr{N}_n^f(y)\}\in\mathscr{F}(I)$. Therefore, for $n\in\mathscr{F}(I)\ \forall\ \gamma>0$, $\mathcal{U}(\mathscr{N}_n^f(x)-\mathscr{N}_n^f(y),\frac{\gamma}{2})=1$, $\mathcal{V}(\mathscr{N}_n^f(x)-\mathscr{N}_n^f(y),\frac{\gamma}{2})=0$ and $\mathcal{W}(\mathscr{N}_n^f(x)-\mathscr{N}_n^f(y),\frac{\gamma}{2})=0$. Since $\{y_k\}\in\mathscr{N}^f_{I(\mathcal{S})}$, let $I_{(\mathcal{S})}-\lim(y_k)=\alpha$. Then, for any $\epsilon\in(0,1)$ and $\gamma>0$,

$$\begin{aligned}A_1=\Big\{n\in\mathbb{N}:&\,\mathcal{U}(\mathscr{N}_n^f(y)-\alpha,\frac{\gamma}{2})>1-\epsilon\text{ and }\mathcal{V}\Big(\mathscr{N}_n^f(y)-\alpha,\frac{\gamma}{2}\Big)<\epsilon,\\ &\mathcal{W}(\mathscr{N}_n^f(y)-\alpha,\frac{\gamma}{2})<\epsilon\Big\}\in\mathscr{F}(I).\end{aligned}$$

Consider the set $A_2=\{n\in\mathbb{N}:\mathcal{U}(\mathscr{N}_n^f(x)-\alpha,\gamma)>1-\epsilon$ and $\mathcal{V}(\mathscr{N}_n^f(x)-\alpha,\gamma)<\epsilon,\mathcal{W}(\mathscr{N}_n^f(x)-\alpha,\gamma)<\epsilon\}$.

We show that $A_1\subset A_2$. So, for $n\in A_1$ we have

$$\begin{aligned}\mathcal{U}(\mathscr{N}_n^f(x)-\alpha,\gamma) &\ge \mathcal{U}(\mathscr{N}_n^f(x)-\mathcal{N}_n^f(y),\frac{\gamma}{2})*\mathcal{U}(\mathscr{N}_n^f(y)-\alpha,\frac{\gamma}{2})\\ &> 1*(1-\epsilon)\\ &= 1-\epsilon,\end{aligned}$$

$$\begin{aligned}\mathcal{V}(\mathscr{N}_n^f(x)-\alpha,\gamma) &\le \mathcal{V}(\mathscr{N}_n^f(x)-\mathscr{N}_n^f(y),\frac{\gamma}{2})\diamond\mathcal{V}(\mathscr{N}_n^f(y)-\alpha,\frac{\gamma}{2})\\ &< 0\diamond\epsilon\\ &= \epsilon\end{aligned}$$

and

$$\begin{aligned}\mathcal{W}\big(\mathcal{N}_n^f(x)-\alpha,\gamma\big) &\leq \mathcal{W}\big(\mathcal{N}_n^f(x)-\mathcal{N}_n^f(y),\frac{\gamma}{2}\big)\diamond \mathcal{W}\big(\mathcal{N}_n^f(y)-\alpha,\frac{\gamma}{2}\big)\\ &< 0\diamond \epsilon\\ &= \epsilon.\end{aligned}$$

This implies that $n \in A_2$ and hence $A_1 \subset A_2$. Since $A_1 \in \mathscr{F}(I)$, therefore $A_2 \in \mathscr{F}(I)$. Hence, $x = \{x_k\} \in \mathcal{N}^f_{I(\mathcal{S})}$. □

Conclusion

In this research, we investigated the ideal convergence of extended Nörlund sequences in NNS and defined a new type of sequence space $\mathcal{N}^f_{I_0(\mathcal{S})}$, $\mathcal{N}^f_{I(\mathcal{S})}$ and $\mathcal{N}^f_{I_\infty(\mathcal{S})}$ utilising the previously studied Nörlund matrix $\mathcal{N}^f$. In NNS, the concepts of Nörlund ideal convergence and Nörlund ideal Cauchy sequence are examined, and significant findings are established. We may also investigate the topological properties of these spaces, which will give a better technique for dealing with ambiguity and inexactness in numerous fields of science, engineering, and economics.

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References

[1] B. C. Tripathy, P. J. Dowari, *Nórlund, and Riesz mean of a sequence of complex uncertain variables*, Filomat, 32 (2018), 2875-2881.

[2] B. C. Tripathy, A. Baruah, *Nórlund and Riesz mean of sequences of fuzzy real numbers*, Applied Mathematics Letters, 23 (2010), 651-655.

[3] F. Samarandache, *Neutrosophic set - A generalization of the intuitionistic fuzzy set*, Inter. J. Pure Appl. Math, 24 (2005), 287-297.

[4] F. M. Mears, *The inverse nórlund mean,* Annals of Mathematics, (1943), 401-410.

[5] H. Fast, *Sur la convergence statistique*, Colloq. Math., 2 (1951), 241-244.

[6] I. J. Schoenberg, *The integrability of certain function abd related summability methods*, Amer. Math. Monthly, 66 (1959), 361-375.

[7] M. Kirisci, N. Simsek, *Neutrosophic normed spaces and statistical convergence*, The Journal of Analysis, 28 (2020), 1059-1073.

[8] V.A. Khan, S.A.A. Abdullah, K.M.A.S. Alshlool, *A study of Nórlund ideal convergent sequence spaces,* Yugoslav Journal of Operations Research, DOI: https://doi.org/10.2298/YJOR200716044K

[9] V.A. Khan, I.A. Khan, *Spaces of intuitionistic fuzzy Nörlund $I-$convergent sequences,* Afrika Matematika, 33 (2022).

[10] V. A. Khan, M. Arshad, M.D. Khan, *Some results of neutrosophic normed space VIA Tribonacci convergent sequence spaces*, J. Inequal. Appl, 42 (2022), 1-27.

[11] V.A Khan. R.K.A Rababah, K.M.A. Alshlool, S.A.A Abdullah, A. Ahmad, *On ideal convergence Fibonacci difference sequence spaces*, Advances in Difference Equations, 2018(1):199, 2018.

[12] V.A. Khan, N. Khan, Y. Khan, *On Zweier paranorm I-convergent double sequence spaces*, Cogent Mathematics, 3 (2016).

[13] K.T. Atanassov, *More on intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 33 (1989), 37-45.

[14] K.T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20 (1986), 87-96.

[15] S. Özçağ, Ç. Doğan, *On connectedness in intuitionistic fuzzy special topological spaces*, Internat. J. Math. and Math. Sci., 21 (1998), 33-40.

[16] M. Mursaleen, S.A. Mohiuddine, *Statistical convergence of double sequence in intuitionistic fuzzy normed space*, Chaos, Solitons & Fractals, 41 (2009), 2414-2421.

[17] O. Tug, F. Basar, *On the spaces of Norlund null and Norlund convergent sequences,* This Journal of Pure and Applied Mathematics, 7 (2016), 76-87.

[18] O. Tug, *On some generalized Nörlund ideal convergent sequence spaces,* ZANCO Journal of Pure and Applied Sciences, 28 (2016), 97-103.

[19] R. Filipów, J. Tryba, *Ideal convergence versus matrix summability*, Studia Math., 245 (2019), 101-127.

[20] R. Saadati, J.H. Park, *On the intuitionistic fuzzy topological spaces*, Chaos, Solitons & Fractals, 27 (2006), 331-334.

[21] T. Bera, N.K. Mahapatra, *On neutrosophic soft linear space*, Fuzzy Information and Engineering, 9 (2017), 299-324.

[22] T. Bera, N.K. Mahapatra, *Neutrosophic soft normed linear spaces*, Neutrosophic Sets and Systems, 23 (2018), 1-6.

[23] T. Śalát, B.C. Tripathy, Miloś, *On some properties of I-convergence*, Tatra Mt. Math. Publ, 28 (2004), 274-286.

[24] T. Śalát, B.C. Tripathy, Miloś, *On the I-convergence field*, Ital. J. Pure Appl. Math., 17 (2005), 1-8.

[25] C.S. Wang, *On Nörlund sequence space*, Tamkang J. Math., 9 (1978), 269-274.

[26] W.B.V. Kandasamy, F. Samarandache, *Neutrosophic rings*, Published by Hexis, Phoenix, Arizona (USA) in 2006 (2006).

[27] L.A. Zadeh, *Fuzzy sets*, Inf. Control, 8 (1965), 338-353.

[28] J. Cervenansky, *Statistical convergence and statistical continuity*, Zbornik Vedeckych Prac M_tFSTU, 6 (1998), 207-212.

[29] J.S. Connor, *The statistical and strong p-Cesaro convergence of sequence*, Analysis, 8 (1988), 47-63.

[30] J.S. Connor, *Two valued measures and summability*, Analysis, 10 (1990), 373-385.

[31] R.C. Buck, *The measure theoretic approach to density*, Amer. J. Math, 68 (1946), 560-580.

[32] R.C. Buck, *Generalized asymptotic density*, Amer. J. Math, 75 (1953), 335-346.

[33] D.S. Mitrinović, J. Sandor, B. Crstici, *Handbook of number Theory*, Kluwer Acad. Publ., Dordrecht-Boston-London, 1996.

[34] M. Kirisci, N. Simsek, *Neutrosophic metric spaces*, Mathematical Sciences, Math. Sci., 14 (2020), 241-248.

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