Constructions of indecomposable representations of algebras via reflection functors

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Abstract. The main aim of this paper is to reform the Bernstein-Gelfand-Ponomarev theory in order to characterize representations of some (non-basic) artinian algebras. All non-isomorphic indecomposable projective and injective representations are constructed via Coxeter functors for a generalized path algebra of acyclic quiver and then for an artinian hereditary algebra of Gabriel-type with an admissible ideal. The methods given via natural quivers and reformed modulations are helpful for one to study some properties which are *not Morita-invariant* in representation theory.
Keywords: modulation, natural quiver, reflection functor, artinian algebra, generalized path algebra, indecomposable representation.

1. Introduction

Reflection functors were introduced into the representation theory of quivers by Bernstein, Gelfand and Ponomarev in their work on the 4-subspace problem [13] and on Gabriel's Theorem, e.g. [5, 2, 3]. Due to the latter result, one obtains the classifications of finite type and tame type of basic hereditary artinian algebras, that is, acyclic quiver algebras, over an algebraically closed field. Furthermore, there have been several generalizations, see [6, 11, 10, 4, 1, 9]. In [11, 10, 9], Bernstein-Gelfand-Ponomarev theory was generalized to hereditary tensor algebras of quivers over division rings. In [6], the authors gave an extension of

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the concept of reflection functors and some applications to quivers with relations (equivalently say, to some special basic non-hereditary artinian algebras). A special case of this theory has been developed by Marmaridis [25] and applied to certain quivers with relations. In [1], a theory of partial Coxeter functors was developed for a basic artin algebra with a simple projective noninjective module.

The fact that each finite dimensional basic algebra over an algebraically closed field is some quotient of path algebra plays an important role in algebraical representation theory, since it characterizes the structures of basic algebras and provides a method to give various examples of basic algebras using quivers. More importantly, it can be used to characterize finitely generated modules over an algebra. However, there are limitations to this approach. Firstly, the ground field has to be an algebraically closed field. Secondly, the characterization of representations of a finite dimensional algebra must be based on its corresponding basic algebra. But, some information of representations of the original algebra will be lost via its basic algebra. To solve this problem, Coelho and Liu[8] first introduced the concept of generalized path algebras, so as to have a more direct and new understanding for the structures and representations of algebras.

It is noted that artinian algebras having be studied in all former papers are basic. Although the module category of an artinian algebra and that of its corresponding basic algebra are equivalent which means the representation types of these two algebras are coherent, in usual it is difficult to consider the relation between the dimensions of their modules. It is the motivation for us to use the method of reflection functors to study non-basic artinian algebras and some data of their representations which are not Morita-invariant.

The main aim of this paper is to reform the Bernstein-Gelfand-Ponomarev theory to characterize the representation categories of some (non-basic) artinian algebras and to give a method for constructing indecomposable projective and injective representations via reflection functors and Coxeter functors. This makes it possible to compute the dimensions of indecomposable representations of a (non-basic) artinian algebra. The tool we use is the natural quiver of an artinian algebra.

In the classical setting, mathematicians dealt with the module theory of the path algebras of quivers. In this paper, we use the natural quivers of (non-basic) hereditary algebras and the reformed modulations via generalized path algebras which are isomorphic to hereditary algebras, see [15, 21, 8, 7], to solve the corresponding problems in modules over the generalized path algebras.

The natural quiver will have fewer arrows than the Ext-quiver when the algebra A is not basic. Natural quivers are not invariant under the Morita equivalence and much closer to reflect the structure of the algebra, rather than just its module category. There are numerous cases even in the representation theory that one needs the structure of the algebras, for example, the character values of finite groups in a block cannot be preserved through Morita equivalence.

We think natural quivers and generalized path algebras are valid to study some properties which are not Mortia invariant in representation theory.

When an artinian algebra A is of Gabriel-type [18], that is, A is isomorphic to some quotient of the generalized path algebra of its natural quiver Δ_A, then any representations of A can be induced directly from some representations of the generalized path algebra of Δ_A. From [18], we know that any artinian algebra splitting over its radical must be of Gabriel-type. It is more straightforward through representations of the generalized path algebra of Δ_A to set up an approach to representations of an artinian algebra.

Associated with any representation of a quiver is a dimension vector, and the dimension vectors of indecomposable modules are the positive roots of the quadratic form associated to the quiver (see e.g. [5, 11, 14]). Similar results seem to hold for certain quivers with relations. Some applications of reflection functors involve the study of the transformations of dimension vectors they induce. It turns out in [6] that there are applications of our functors which make use of the analogous transformations which is considered as a change of basis for a fixed root-system - a tilting of the axes relative to the roots which results in a different subset of roots lying in the positive cone.

For our need, for an artinian algebra, the dimension vectors of modules and the Cartan matrix are introduced in Section 2. First, some properties of dimension vectors are given, which are generalizations of the corresponding properties for a basic algebra. When the global dimension of an artinian algebra is finite, its Cartan matrix is invertible and can be computed through an integer matrix and two diagonal matrices. The Euler characteristic and the Euler quadratic form of an artinian algebra is defined from the Cartan matrix. On the other hand, the Euler form and quadratic form of a pre-modulation is defined. It was shown in [2] that the quadratic form and the Euler quadratic form coincide for a path algebra through the homological interpretation of the Euler characteristic. However, for a generalized path algebra, it is difficult to get the similar relation between its Euler quadratic form and the quadratic form from its corresponding pre-modulation in the reason that in the general case the homological interpretation of the Euler characteristic can not be computed via the inverse matrix of its Cartan matrix. So, in this paper, the homological interpretation of the Euler form, as well as the quadratic form, is characterized directly.

As analogue of the dimension vectors of indecomposable modules of quivers, it is interesting for one to discuss the relationship between the dimension vectors of indecomposable representations of artinian algebras and the positive roots of the quadratic forms associated to pre-modulations. Since the dimension vector and Cartan matrix of an artinian algebra are not invariant under the Morita equivalence, the mentioned relation above has only been a conjecture. This will be our further expectation for researching with this new method given via natural quivers and generalized modulations.

In Section 3, first, the reflection functors are given for the representation category $rep(\mathcal{M},\Omega)$ of a pre-modulation $\mathcal{M}$ with acyclic connected valued quiver and using of them as a pair of mutual invertible functors Δ_i^- and Δ_i^+, the categorical equivalence is obtained between the full subcategories $rep^{(i)}(\mathcal{M},\Omega)$ and $rep_{(i)}(\mathcal{M},\Omega)$ for $i=1,n$.

Moreover, we get the construction of all non-isomorphic indecomposable projective and injective representations of a generalized path algebra with acyclic quiver and then of an artinian hereditary algebra of Gabriel-type with admissible ideal.

At last, in Section 4, as application, we discuss the relationship between representation-type of a generalized path algebra and its natural quiver.

2. Dimension vectors of representations

2.1 Dimension vectors of modules over an artinian algebra

One attaches to each module of a basic algebra a vector with integral coordinates, called its dimension vector. This allows one to use methods of linear algebra when studying modules over a basic algebra. For example, an important application is in the famous Kac theorem which means the relation between dimension vectors of indecomposable modules and the so-called positive root system of a basic (hereditary) algebra. However, as we have known, the natural quiver is a tool to characterize an artinian (non-basic) algebra. In this paper, we try to give directly, but not through the theory of basic algebras, the description of the relationship between indecomposable modules of artinian (non-hereditary) algebras and the generalization of dimension vectors via natural quivers. Note that the dimension as a linear space and dimension vector defined below of a module are not Morita-invariant. This explains the validity of our discussion here.

Throughout this paper, we will always use k to be an algebraically closed field.

An artinian algebra A over k with Jacobson radical $r=r(A)$ is called *splitting over radical* if the natural homomorphism $A\to A/r$ is a splitting algebra homomorphism. In this case, A/r can be embedded into A as a subalgebra.

For two rings A and B, a finitely generated A-B-bimodule M, define $rk_{A,B}(M)$ to be the minimal number of generators of M as an A-B-bimodule among all genarating sets. Then we call $rk_{A,B}(M)$ the *rank* of M as A-B-bimodule.

The concept of generalized path algebra was introduced early in [8]. Here we review the different but equivalent definition which is given in [18].

Let $Q=(Q_0,Q_1)$ be a quiver. Given a collection of k-algebras $\mathcal{A}=\{A_i\mid i\in Q_0\}$ with the identity $e_i\in A_i$. Let $A_0=\prod_{i\in Q_0}A_i$ be the direct product k-algebra. Clearly, each e_i is an orthogonal central idempotent of A_0. For $i,j\in Q_0$, let $\Omega(i,j)$ be the subset of arrows in Q_1 from i to j. Write

$$_iM_j\overset{def}{=}A_i\Omega(i,j)A_j$$

be the free A_i-A_j-bimodule with basis $\Omega(i,j)$. This is the free $A_i \otimes_k A_j^{op}$-module over the set $\Omega(i,j)$. Thus,

$$M = \bigoplus_{(i,j)\in Q_0\times Q_0} A_i\Omega(i,j)A_j \tag{1}$$

is an A_0-A_0-bimodule. The *generalized path algebra*[8, 15, 18] is defined to the tensor algebra

$$T(A_0, M) = \bigoplus_{n=0}^{\infty} M^{\otimes_{A_0} n}.$$

Here $M^{\otimes_{A_0} n} = M \otimes_{A_0} M \otimes_{A_0} \ldots \otimes_{A_0} M$ and $M^{\otimes_{A_0} 0} = A_0$. We denote by $k(Q, \mathcal{A})$ the generalized path algebra. $k(Q, \mathcal{A})$ is called *(semi-)normal* if all A_i are (semi-)simple k-algebras.

Suppose that A is a left artinian k-algebra and $r = r(A)$ is its Jacobson radical. Write $A/r = A_1 \oplus \ldots \oplus A_s$, where A_i are two-sided simple ideals of A/r. Such a decomposition of A/r is also called *a block decomposition of the algebra A/r*. Then, r/r^2 is an A/r-bimodule. Let ${}_iM_j = A_i \cdot r/r^2 \cdot A_j$, which is finitely generated as an A_i-A_j-bimodule for each pair (i,j).

Now we introduce the concept of natural quiver and corresponding generalized path algebra of A.

Definition 2.1 ([18])**.** *Suppose that A is a left artinian k-algebra and $r = r(A)$ is its Jacobson radical. Write $A/r = A_1 \oplus \ldots \oplus A_s$, where A_i are two-sided simple ideals of A/r.*

(i) The natural quiver of A is defined by $\Delta_A = (\Delta_0, \Delta_1)$ with the vertex set Δ_0 to be the index set $\{1, 2, \ldots, s\}$ of the isomorphism classes of simple A-modules corresponding to the set of blocks of A/r; with the arrow set Δ_1 consisting of $t_{i,j}$ arrows from i to j for $i, j \in \Delta_0$ where $t_{i,j} = rk_{A_j,A_i}({}_jM_i)$. Obviously, there is no arrow from i to j if ${}_jM_i = 0$.

(ii) Denote $\mathcal{A} = \{A_i \mid i \in Q_0\}$. The generalized path algebra $k(\Delta_A, \mathcal{A})$ is called the corresponding generalized path algebra of A.

By Definition 2.1, the natural quiver of artinian algebra A is always finite.

In [18], we have known the following characterization of an artinian algebra A splitting over radical via its generalized path algebra.

Theorem 2.1 ([18])**.** *An artinian k-algebra A is splitting over radical if and only if there is an ideal I of the corresponding generalized path algebra $k(\Delta_A, \mathcal{A})$ of A and a positive integer s such that $A \cong k(\Delta_A, \mathcal{A})/I$ with $J^s \subset I \subset J$ where J is the ideal of $k(\Delta_A, \mathcal{A})$ generated by all $\mathcal{A}$-paths of length 1.*

This means that an artinian k-algebra splitting over radical is of Gabriel-type.

Definition 2.2. *Suppose that A is an artinian algebra splitting over radical r with ideal I satisfying $A \cong k(\Delta_A, \mathcal{A})/I$ due to Theorem 2.1. Write $(\Delta_A)_0 = \{1, 2, \ldots, s\}$. Let $A/r = A_1 \oplus \ldots \oplus A_s$ where A_i are simple ideals of A/r. For a right A-module M, the dimension vector of M is defined to be the vector*

$$\mathbf{dim} M = \begin{pmatrix} \frac{dim_k MA_1}{dim_k A_1} \\ \vdots \\ \frac{dim_k MA_s}{dim_k A_s} \end{pmatrix}$$

in $\mathbb{Q}^s$ for the field of rational numbers $\mathbb{Q}$, where A_i acts on M as subalgebras of A.

The notion of dimension vectors of modules of a basic algebra in [2] is in the special case of this definition. Clearly, dimension vector is not Morita-invariant.

Lemma 2.1. *Let A be an artinian k-algebra splitting over radical r such $A/r = A_1 \oplus \ldots \oplus A_s$ where A_i are simple ideals of A/r, and M be a right A-module. Embedding A_i into A, consider A_iA and A_iAA_i through the multiplication of A. Then, for any $i = 1, \ldots, s$,*

(i) the k-linear map

$$\theta_M^{(i)}: \ Hom_A(A_iA, M) \to MA_i \tag{2}$$

defined by the formula $\varphi \mapsto \varphi(1_{A_i}) = \varphi(1_{A_i})1_{A_i}$ for $\varphi \in Hom_A(A_iA, M)$, is an isomorphism of right A_iAA_i-modules, and it is functorial in M;

(ii) the isomorphism $\theta_{A_iA}^{(i)}: \ End(A_iA) \xrightarrow{\cong} A_iAA_i$ of right A_iAA_i-modules induces an isomorphism of k-algebras.

Proof. (i) For any $\bar{a}_i x \bar{b}_i \in A_iAA_i$,

$$\theta_M^{(i)}(\varphi \bar{a}_i x \bar{b}_i) = (\varphi \bar{a}_i x \bar{b}_i)(1_{A_i}) = \varphi(\bar{a}_i x \bar{b}_i) = \varphi(1_{A_i})\bar{a}_i x \bar{b}_i = (\theta_M^{(i)}(\varphi))\bar{a}_i x \bar{b}_i.$$

Then, $\theta_M^{(i)}$ is a homomorphism of right A_iAA_i-modules. And, $\theta_M^{(i)}$ is functorial in M from the following commutative diagram:

$$\begin{tikzcd} Hom_A(A_iA, M) \arrow[r, "\theta_M^{(i)}"] \arrow[d, "Hom_A(A_iA,\ f)"'] & MA_i \arrow[d, "f_{A_i}"'] \\ Hom_A(A_iA, N) \arrow[r, "\theta_N^{(i)}"] & NA_i \end{tikzcd}$$

where $f: M \to N$ is an A-homomorphism and f_{A_i} is the restriction of f on MA_i.

In order to prove $\theta_M^{(i)}$ is invertible, define a map $\zeta_M^{(i)}: \ MA_i \to Hom_A(A_iA, M)$ by the formula $\zeta_M^{(i)}(m\bar{a}_i)(\bar{b}_i x) = m\bar{a}_i\bar{b}_i x$ for $\bar{a}_i, \bar{b}_i \in A_i, x \in A$. It is easy to check that $\zeta_M^{(i)}(m\bar{a}_i): A_iA \to M$ is well-defined and is an A-homomorphism.

For any $m\bar{a}_i \in MA_i, \bar{b}_i x\bar{c}_i \in A_iAA_i, \bar{d}_i b \in A_iA$,

$$\zeta_M^{(i)}(m\bar{a}_i\bar{b}_ix\bar{c}_i)(\bar{d}_ib) = m\bar{a}_i\bar{b}_ix\bar{c}_i\bar{d}_ib = \zeta_M^{(i)}(m\bar{a}_i)(\bar{b}_ix\bar{c}_i\bar{d}_ib) = (\zeta_M^{(i)}(m\bar{a}_i)\bar{b}_ix\bar{c}_i)(\bar{d}_ib),$$

then $\zeta_M^{(i)}(m\bar{a}_i\bar{b}_ix\bar{c}_i) = \zeta_M^{(i)}(m\bar{a}_i)\bar{b}_ix\bar{c}_i$, which means $\zeta_M^{(i)}$ is a homomorphism of A_iAA_i-modules.

Moreover, for $f \in Hom_A(A_iA, M)$, $\bar{d}_ib \in A_iA$,

$$(\zeta_M^{(i)}\theta_M^{(i)})(f)(\bar{d}_ib) = (\zeta_M^{(i)}(\theta_M^{(i)}(f)))(\bar{d}_ib) = \theta_M^{(i)}(f)\bar{d}_ib = \theta_M^{(i)}(f\bar{d}_ib) = (f\bar{d}_ib)(1_{A_i}) = f(\bar{d}_ib)$$

then $\zeta_M^{(i)}\theta_M^{(i)} = id_{Hom_A(A_iA,M)}$. Similarly, $\theta_M^{(i)}\zeta_M^{(i)} = id_{MA_i}$. Hence, $\theta_M^{(i)}$ is an isomorphism.

(ii) This follows from (i) for $M = A_iA$. □

Lemma 2.2. *Let $A \cong k(\Delta_A, \mathcal{A})/I$ as in Definition 2.2. For each right A-module and $i \in \Delta_0$, the k-linear map (2) induces functorial isomorphisms of k-vector spaces*

$$Hom_A(P(i), M) \overset{\cong}{\to} MA_i \overset{\cong}{\to} DHom_A(M, I(i)).$$

where D is the standard duality $Hom_k(-, k)$, $P(i) = A_iA$ and $I(i) = Hom_k(AA_i, k)$.

Proof. The first isomorphism follows directly from Lemma 2.1 (i). The second isomorphism is the composition

$$\begin{aligned} DHom_A(M, I(i)) &= DHom_A(M, D(AA_i)) \cong DHom_A(D(D(M)), D(AA_i)) \\ &\cong DHom_{A^{op}}(AA_i, D(M)) \cong D(A_iD(M)) \text{ (by Lemma 2.1)} \\ &\cong Hom_k(A_iD(M), k) \cong Hom_k(D(M), k)A_i = D(D(M))A_i \\ &\cong MA_i. \quad \square \end{aligned}$$

This lemma yields $\mathbf{dim}M = \begin{pmatrix} \frac{dim_k Hom_A(P(1),M)}{dim_k A_1} \\ \vdots \\ \frac{dim_k Hom_A(P(s),M)}{dim_k A_s} \end{pmatrix} = \begin{pmatrix} \frac{dim_k Hom_A(M,I(1))}{dim_k A_1} \\ \vdots \\ \frac{dim_k Hom_A(M,I(s))}{dim_k A_s} \end{pmatrix}$.

When A is an artinian k-algebra splitting over radical r, i.e. $A = r + A/r$, we have $1_A = r_0 + 1_{A/r}$ for some $r_0 \in r$. Then, $1_{A/r} = 1_A 1_{A/r} = r_0 1_{A/r} + 1_{A/r}$, thus, $r_0 1_{A/r} = 0$. Similarly, $1_{A/r} r_0 = 0$. Then, $1_A = 1_A^2 = (r_0 + 1_{A/r})^2 = r_0^2 + 1_{A/r}$. Moreover, we can get $1_A = r_0^t + 1_{A/r}$ for any natural number t. But, r is nilpotent, so there is t such that $r_0^t = 0$. Hence,

$$1_A = 1_{A/r}.$$

For $A/r = A_1 + \ldots + A_s$, we have $A \supseteq A_1A + \ldots + A_sA \supseteq (A_1 + \ldots + A_s)A \supseteq 1_{A/r}A = 1_AA = A$. Therefore,

$$A = A_1A + \ldots + A_sA$$

which means that for all $i = 1, \ldots, s$, $P(i) = A_iA$ are projective right A-modules. It follows that $Hom_A(P(i), -)$ are exact functors for $i = 1, \ldots, s$.

Proposition 2.1. *Let $A \cong k(\Delta_A, \mathcal{A})/I$ as in Definition 2.2 and $0 \to L \to M \to N \to 0$ be a short exact sequence of right A-modules. Then,* $\mathbf{dim}M = \mathbf{dim}L + \mathbf{dimN}$.

Proof. Using of the exact functor $Hom_A(P(i), -)$ to the short exact sequence $0 \to L \to M \to N \to 0$, we get the exact sequence of k-vector spaces:

$$0 \to Hom_A(P(i), L) \to Hom_A(P(i), M) \to Hom_A(P(i), N) \to 0.$$

By Lemma 2.2, this short exact sequence becomes to the following:

$$0 \to LA_i \to MA_i \to NA_i \to 0.$$

Hence, for each $i \in (\Delta_A)_0$,

$$dim_k MA_i = dim_k LA_i + dim_k NA_i.$$

The statement follows from the definition of dimension vectors. □

Since A_i is isomorphic to the matrix algebra of order n_i over a division k-algebra D_i, in the sequel of this section we always *let n_i denote this notation of the order of the matrix algebra.* We know that there are primitive idempotents $e_{i1}, e_{i2}, \ldots, e_{in_i}$ of A_i such that $P(i) = A_i A = e_{i1}A \oplus e_{i2}A \oplus \ldots \oplus e_{in_i}A$ but $e_{i1}A \cong e_{i2}A \cong \ldots \cong e_{in_i}A$ as right A-modules. So, we can write $P(i) \cong \oplus n_i e_{i1} A$. Here, for $i = 1, \ldots, s$, $P_i = e_{i1}A$ are all indecomposable projective right A-modules. Moreover, $S_i = P_i/P_i r$, $i = 1, \ldots, s$, are all simple A-modules.

It is easy to see that $dim_k S_i = n_i dim_k D_i$ and $dim_k A_i = n_i^2 dim_k D_i$.

Since $A_i A_j = 0$ for $i \neq j$, we have $S_i A_j = \begin{cases} S_i, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$ Therefore, for $i = 1, \ldots, s$,

$$\mathbf{dim}S_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{n_i dim_k D_i}{n_i^2 dim_k D_i} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{n_i} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3}$$

which we denote as X_i. For an artinian k-algebra A, denote by $K_0(A)$ the Grothendieck group of A, $[M]$ the corresponding element in $K_0(A)$ for an A-module M.

Proposition 2.2. *Let $A \cong k(\Delta_A, \mathcal{A})/I$ as in Definition 2.2 and let $S_1, \ldots, S_s$ be a complete set of the isomorphism classes of simple right A-modules. Then,* *the Grothendieck group $K_0(A)$ of A is a free abelian group having as a basis the set $\{[S_1], \ldots, [S_s]\}$. Define $\mathbf{dim}[M] = \mathbf{dim}M$ as the dimension vector of $[M]$ for each A-module M and moreover $\mathbf{dim}(-[M]) = -\mathbf{dim}M$, then $\mathbf{dim}$ is a group homomorphism from $K_0(A)$ to $\mathbb{Q}^s$ and the set of dimension vectors is, i.e. the image of $\mathbf{dim}$,*

$$\mathbf{dim}K_0(A) = \{u_1X_1 + \ldots + u_sX_s : \ u_1, \ldots, u_s \in \mathbb{Z}\}.$$

Proof. Let M be a module in mod$A$ and let $0 = M_0 \subset M_1 \subset \ldots \subset M_t = M$ be a composition series for M. By the definition of $K_0(A)$, we have

$$\begin{aligned}[M] &= [M_t/M_{t-1}] + [M_{t-1}] = [M_t/M_{t-1}] + [M_{t-1}/M_{t-2}] + [M_{t-2}] = \ldots \\ &= \sum_{j=1}^{t}[M_j/M_{j-1}] = \sum_{i=1}^{s} c_i(M)[S_i]\end{aligned}$$

where $c_i(M)$ is the number of composition factors M_j/M_{j-1} of M that are isomorphic to S_i. Hence, $\{[S_1], \ldots, [S_s]\}$ generates the free abelian group $K_0(A)$.

Thus, by the definition of $\mathbf{dim}$ on $K_0(A)$ and Proposition 2.1, we know $\mathbf{dim}$ is a group homomorphism. $\square$

Since $K_0(A)$ of A is a free abelian group with rank s having as a basis the set $\{[S_1], \ldots, [S_s]\}$, it is also isomorphic to $\mathbb{Z}^s$ as groups, but not through $\mathbf{dim}$.

As a consequence, we show the relation between the dimension vector of a module M and the number of simple composition factors of M that are isomorphic to each simple modules S_i.

Corollary 2.1. *Let $A \cong k(\Delta_A, \mathcal{A})/I$ as in Definition 2.2 and let $S_1, \ldots, S_s$ be a complete set of the isomorphism classes of simple right A-modules. For any module M in mod$A$, let $c_i(M)$ be the number of composition factors M_j/M_{j-1} of M that are isomorphic to S_i and let $l(M)$ be the composition length of M. Then,*

$$c_i(M) = (dim_k MA_i)/(n_i dim_k D_i)$$

and thus $l(M) = \sum_{i=1}^{s}(dim_k MA_i)/(n_i dim_k D_i)$, where D_i is the division k-algebra such that A_i is isomorphic to the matrix algebra of order n_i over D_i for $A/r = A_1 \oplus \ldots \oplus A_s$ where A_i are simple ideals of A/r.

Proof. In the proof of Proposition 2.2, we have $[M] = \sum_{i=1}^{s} c_i(M)[S_i]$. Then, $\mathbf{dim}M = \mathbf{dim}[M] = \sum_{i=1}^{s} c_i(M)\mathbf{dim}[S_i] = \sum_{i=1}^{s} c_i(M)\mathbf{dim}S_i$. By (3), we get

$$dim_k MA_i = c_i(M) n_i dim_k D_i.$$

Thus, $l(M) = \sum_{i=1}^{s} c_i(M) = \sum_{i=1}^{s}(dim_k MA_i)/(n_i dim_k D_i)$. $\square$

Definition 2.3. *Let A be an artinian k-algebra splitting over radical r with $A/r = A_1 \oplus \ldots \oplus A_s$. The Cartan matrix of A is the $s \times s$ matrix*

$$C_A = \begin{pmatrix} c_{11} & \ldots & c_{1s} \\ \vdots & \ddots & \vdots \\ c_{s1} & \ldots & c_{ss} \end{pmatrix},$$

where $c_{ji} = dim_k A_i A A_j$ for $i, j = 1, \ldots, s$.

Let $e_1, \ldots, e_s$ be the complete set of primitive orthogonal idempotents. Then $A_i A \cong n_i e_i A$ as right A-modules where $n_i e_i A$ means the direct sum of n_i copies of $e_i A$, that is, $P(i) = n_i P_i$ for the indecomposable projective A-modules $P_i = e_i A$ $(i = 1, \ldots, s)$.

By Lemma 2.2,

$$\begin{aligned} A_i A A_j &\cong Hom_A(P(j), A_i A) \cong Hom_A(A_j A, A_i A) \\ &\cong Hom_A(n_j e_j A, n_i e_i A) \cong n_j n_i Hom_A(e_j A, e_i A). \end{aligned}$$

Thus, $dim_k Hom_A(P_j, P_i) = c_{ji}/(n_j n_i)$ for $i = 1, \ldots, s$.

On the other hand, by Lemma 2.2,

$$I(i) = Hom_k(AA_i, k) \cong Hom_k(n_i A e_i, k) \cong n_i Hom_k(Ae_i, k) = n_i I_i,$$

where $I_i = Ae_i$ $(i = 1, \ldots, s)$ are the indecomposable injective A-modules. Moreover, by Lemma 2.2,

$$\begin{aligned} Hom_A(P(j), P(i)) &\cong DHom_A(P(i), I(j)) \cong DDHom_A(I(j), I(i)) \\ &\cong Hom_A(I(j), I(i)) \cong n_i n_j Hom_A(I_j, I_i). \end{aligned}$$

Thus,

$$A_i A A_j \cong Hom_A(I(j), I(i)) \tag{4}$$

and $A_i A A_j \cong n_i n_j Hom_A(I_j, I_i)$. Hence, $dim_k Hom_A(I_j, I_i) = c_{ji}/(n_j n_i)$.

Therefore, through modulo $n_i n_j$ for each c_{ij}, the Cartan matrix of A records the numbers of linearly independent homomorphisms between the indecomposable projective A-modules and the numbers of linearly independent homomorphisms between the indecomposable injective A-modules.

Below we discuss some elementary facts on the Cartan matrix.

Proposition 2.3. *Let C_A be the Cartan matrix of an artinian algebra $A \cong k(\Delta_A, \mathcal{A})/I$ as in Definition 2.3. Then,*

(i) The i-th column of C_A is $\begin{pmatrix} n_1^2 dim_k D_1 & & \\ & \ddots & \\ & & n_s^2 dim_k D_s \end{pmatrix} \mathbf{dim} P(i)$ *and*

$$\begin{pmatrix} n_1^2 dim_k D_1 & & \\ & \ddots & \\ & & n_s^2 dim_k D_s \end{pmatrix} \mathbf{dim} P(i) = n_i C_A \mathbf{dim} S_i;$$

(ii) The i-th row of C_A is $(\mathbf{dim}I(i))^t \begin{pmatrix} n_1^2 dim_k D_1 & & \\ & \ddots & \\ & & n_s^2 dim_k D_s \end{pmatrix}$ *and*

$$\begin{pmatrix} n_1^2 dim_k D_1 & & \\ & \ddots & \\ & & n_s^2 dim_k D_s \end{pmatrix} \mathbf{dim}I(i) = n_i C_A^t \mathbf{dim}S_i.$$

Proof. (ii) $(\mathbf{dim}I(i))^t = (\frac{dim_k I(i)A_1}{dim_k A_1}, \dots, \frac{dim_k I(i)A_s}{dim_k A_s})$. By Lemma 2.2, we have $I(i)A_j \cong DHom_A(I(i), I(j))$. By (4), $Hom_A(I(i), I(j)) \cong A_j A A_i$. But,

$$dim_k DHom_A(I(i), I(j)) = dim_k Hom_A(I(i), I(j)).$$

Thus, $\frac{dim_k I(i)A_j}{dim_k A_j} = \frac{dim_k A_j A A_i}{dim_k A_j}$ for $j = 1, \dots, s$, which means the first result. From this and (3), the second result follows.

(i) Its proof is similar, since it is easy to be obtained from the definition of $\mathbf{dim}P(i)$ and (3). $\square$

Proposition 2.4. *Let A be an artinian algebra as in Definition 2.2 with $A \cong k(\Delta_A, \mathcal{A})/I$. Suppose the global dimension of A is finite. Then, the Cartan matrix C_A is invertible and there exists $B \in \mathcal{M}_s(\mathbb{Z})$ such that*

$$C_A^{-1} = \begin{pmatrix} \frac{1}{n_1^3 dim_k D_1} & & \\ & \ddots & \\ & & \frac{1}{n_s^3 dim_k D_s} \end{pmatrix} B \begin{pmatrix} n_1 & & \\ & \ddots & \\ & & n_s \end{pmatrix}$$

where $\mathcal{M}_s(\mathbf{Z})$ denotes the $s \times s$ full matrix ring over the integer ring $\mathbb{Z}$.

Proof. Here $s = \mid \Delta_0 \mid$. Since A is of finite global dimension, for any $i \in \{1, \dots, s\}$ and the corresponding simple A-module S_i there is a projective resolution

$$0 \to Q_{m_i} \to \dots \to Q_1 \to Q_0 \to S_i \to 0$$

in mod$A$ for a positive integer m_i.

From Proposition 2.1, it follows that $\mathbf{dim}S_i = \sum_{l=1}^{m_i} (-1)^l \mathbf{dim}Q_l$. Because $P_1, \dots, P_s$ are the complete set of non-isomorphic indecomposable projective A-modules, each Q_l is a direct sum of finitely many copies of $P_1, \dots, P_s$. Thus, for each i, $\mathbf{dim}S_i$ is a linear combination of the vectors $\mathbf{dim}P_1, \dots, \mathbf{dim}P_s$ with integral coefficients. Thus, there exists $B \in \mathcal{M}_s(\mathbb{Z})$ such that

$$\begin{pmatrix} n_1^{-1} & & \\ & \ddots & \\ & & n_s^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{dim}S_1 & \dots & \mathbf{dim}S_s \end{pmatrix} = \begin{pmatrix} \mathbf{dim}P_1 & \dots & \mathbf{dim}P_s \end{pmatrix} B.$$

But, $P(i) = n_i P_i$, so $\mathbf{dim}P(i) = n_i \mathbf{dim}P_i$ for $i = 1, \ldots, s$. Hence,

$$\begin{pmatrix} n_1^{-1} & & \\ & \ddots & \\ & & n_s^{-1} \end{pmatrix} = \begin{pmatrix} n_1^{-1}\mathbf{dim}P(1) & \ldots & n_s^{-1}\mathbf{dim}P(s) \end{pmatrix} B$$

$$= \begin{pmatrix} C_A\mathbf{dim}S_1 & \ldots & C_A\mathbf{dim}S_s \end{pmatrix} \begin{pmatrix} \frac{1}{n_1^2 dim_k D_1} & & \\ & \ddots & \\ & & \frac{1}{n_s^2 dim_k D_s} \end{pmatrix} B$$

$$= C_A \begin{pmatrix} \frac{1}{n_1^3 dim_k D_1} & & \\ & \ddots & \\ & & \frac{1}{n_s^3 dim_k D_s} \end{pmatrix} B.$$

Thus,

$$C_A^{-1} = \begin{pmatrix} \frac{1}{n_1^3 dim_k D_1} & & \\ & \ddots & \\ & & \frac{1}{n_s^3 dim_k D_s} \end{pmatrix} B \begin{pmatrix} n_1 & & \\ & \ddots & \\ & & n_s \end{pmatrix}. \quad \square$$

Note that, when $n_i = 1$ and $dim_k D_i = 1$ for all i, A is a basic algebra and $C_A^{-1} = B$ is an integer matrix.

We use the Cartan matrix C_A to define a nonsymmetric $\mathbb{Z}$-bilinear form on the $\mathbb{Z}^s$.

Definition 2.4. *Let A be an artinian algebra with radical r of finite global dimension such that $A/r = A_1 \oplus \ldots \oplus A_s$ where each A_i is simple ideals of A/r which is isomorphic to the matrix algebra of order n_i over a division k-algebra D_i. Let C_A be the Cartan matrix of A.*

(i) The Euler characteristic of A is the $\mathbb{Z}$-bilinear form $\langle -, - \rangle_A : \mathbb{Z}^s \times \mathbb{Z}^s \to \mathbb{Z}$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle_A = \mathbf{x}^t \begin{pmatrix} n_1^{-1} & & \\ & \ddots & \\ & & n_s^{-1} \end{pmatrix} (C_A^{-1})^t \begin{pmatrix} n_1^3 dim_k D_1 & & \\ & \ddots & \\ & & n_s^3 dim_k D_s \end{pmatrix} \mathbf{y}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^s$;

(ii) The Euler quadratic form of A is the quadratic form $q_A : \mathbb{Z}^s \to \mathbb{Z}$ defined by $q_A(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle_A$ for $\mathbf{x} \in \mathbb{Z}^s$.

This definition makes sense due to Proposition 2.4.

2.2 Dimension vectors of representations of a pre-modulation

Given a valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$ and a vertex $i \in \mathcal{G}$, define an operation, denoted by δ_i, on the orientation Ω to get the orientation $\delta_i\Omega$ as follows: we reverse all arrows along edges containing i and leave all others unchanged in Ω.

With respect to the orientation Ω, call *admissible sequence of sinks* an ordering

$$(k_1, k_2, \ldots, k_n)$$

of all the vertices of $\mathcal{G}$ such that k_1 is a sink with respect to Ω, k_2 a sink with respect to $\delta_{k_1}\Omega$, and so on, that is, k_t is a sink with respect to $\delta_{k_{t-1}} \ldots \delta_{k_1}\Omega$ for $2 \le t \le n$. Similarly, *admissible sequence of sources* can be defined. We shall call an orientation admitting an admissible sequence of sinks *admissible.* As known in [10], the orientation Ω is admissible if and only if the valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$ has no oriented cycle. In general, there are many different admissible sequences with respect to a given orientation.

Suppose that $\mathcal{M} = (A_i, {}_iM_j)$ is a k-pre-modulation of a valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$ whose orientation Ω is admissible. Let $k(Q, \mathcal{A}) = T(M, A_0)$ be the constructed corresponding normal generalized path algebra in [16], where $M = \bigoplus_{i,j} A_i\Omega(i,j)A_j$ for $A_i\Omega(i,j)A_j \cong {}_iM_j$ and $A_0 = \oplus_{i\in Q_0} A_i$. Then, $Q_0 = \mathcal{G} = \{1, 2, \ldots, s\}$ and the arrow set $Q_1 = \bigcup_{i,j} \Omega(i,j)$ is decided by the number t_{ij} of generators in the A_i-A_j-basis of ${}_iM_j$ as free A_i-A_j-bimodule.

Denote $A = k(Q, \mathcal{A})$. Q is a finite acyclic quiver since the orientation Ω is admissible. Then, A is artinian. Due to [15], $k(Q, \mathcal{A})$ is just the corresponding generalized path algebra $k(\Delta_A, \mathcal{A})$, that is, the ideal I is zero in Theorem 2.1.

Let $\mathcal{V} = (V_i, {}_j\varphi_i)$ be a representation of $\mathcal{M}$. Then, $V = \oplus_{i\in Q_0} V_i$ is a right module over $A = k(Q, \mathcal{A})$ with right A_i-module V_i such that $VA_i = V_i$ but $V_iA_j = 0$ for $i, j \in Q_0, i \neq j$. However, $A/r \cong A_0 = \oplus_{i\in Q_0} A_i$ for the radical of A. So, let $Q_0 = \mathcal{G} = \{1, 2, \ldots, s\}$, the dimension vector of V

$$\mathbf{dim}V = \begin{pmatrix} \frac{dim_k VA_1}{dim_k A_1} \\ \vdots \\ \frac{dim_k VA_s}{dim_k A_s} \end{pmatrix} = \begin{pmatrix} \frac{dim_k V_1}{dim_k A_1} \\ \vdots \\ \frac{dim_k V_s}{dim_k A_s} \end{pmatrix}$$

in $\mathbb{Q}^s$. We call $\mathbf{dim}V$ the dimension vector of the representation $\mathcal{V} = (V_i, {}_j\varphi_i)$ of $\mathcal{M}$, denoted as $\mathbf{dim}\mathcal{V} = \begin{pmatrix} \frac{dim_k V_1}{dim_k A_1} \\ \vdots \\ \frac{dim_k V_s}{dim_k A_s} \end{pmatrix}$.

For a k-pre-modulation $\mathcal{M} = (A_i, {}_iM_j)$ of a valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$, we define the bilinear forms $B(\mathbf{x}, \mathbf{y})$ and $(\mathbf{x}, \mathbf{y})$ by

$$B(\mathbf{x}, \mathbf{y}) = \sum_{i\in\mathcal{G}} x_i y_i dim_k A_i - \sum_{i\to j} d_{ij} x_i y_j dim_k A_j,$$

$$(\mathbf{x}, \mathbf{y}) = B(\mathbf{x}, \mathbf{y}) + B(\mathbf{y}, \mathbf{x}),$$

where $\mathbf{x} = (x_i)_{i\in\mathcal{G}}$ and $\mathbf{y} = (y_i)_{i\in\mathcal{G}}$ in $\mathbb{Q}^s$. We call $B(-,-)$ the *Euler form* and $(-,-)$ the *symmetric Euler form* respectively. Moreover, we can define the quadratic form $q_{\mathcal{M}} : \mathbb{Q}^s \to \mathbb{Q}^s$ by $q_{\mathcal{M}}(x) = B(x,x)$ for $x \in \mathbb{Q}^s$, which is called *the quadratic form of the pre-modulation* $\mathcal{M}$.

In the trivial case that $A_i = k$ for all $i \in \mathcal{G}$, we can get a quiver Q with $Q_0 = \mathcal{G}$ and Q_1 consisting of t_{ij} arrows from i to j by $t_{ij} = d_{ij}/\varepsilon_i = d_{ji}/\varepsilon_j$. Then, the quadratic form $q_{\mathcal{M}}$ is just that of the quiver Q defined in [2]. In this trivial case, it was shown in Lemma VII4.1 of [2] that this quadratic form $q_{\mathcal{M}}$ and the Euler quadratic form q_A coincide for $A = kQ$. The proof of this result in [2] was dependent on the homological interpretation of the Euler characteristic.

However, for a general $A = k(Q,\mathcal{A})$, it is difficult for us to try to get the similar relation between the quadratic form $q_{\mathcal{M}}$ and the Euler quadratic form q_A in the reason that the inverse matrix of the Cartan matrix C_A is so complicated for computing that we cannot give the homological interpretation of the Euler characteristic $\langle -,-\rangle_A$. Hence, on the other hand, we will give the homological interpretation of the Euler form $B(-,-)$ as follows.

Theorem 2.2. *Assume that $\mathcal{M} = (A_i,\ {}_iM_j)$ is a pre-modulation over a field k of a valued quiver $(\mathcal{G},\mathcal{D},\Omega)$. For two representations $\mathcal{X} = (X_i,\ {}_i\varphi_j)$ and $\mathcal{Y} = (Y_i,\ {}_i\psi_j)$ in rep$(\mathcal{M})$,*

$$B(\mathbf{dim}\mathcal{X},\mathbf{dim}\mathcal{Y}) = dim_k Hom(\mathcal{X},\mathcal{Y}) - dim_k Ext^1(\mathcal{X},\mathcal{Y}).$$

Proof. Firstly, define a map:

$$\Delta_{\mathcal{X},\mathcal{Y}} : \bigoplus_{i\in\mathcal{G}} Hom_{A_i}(X_i,\ Y_i) \longrightarrow \bigoplus_{j\to i} Hom_{A_i}(X_j \otimes_{A_j} {}_jM_i, Y_i)$$

with $\Delta_{\mathcal{X},\mathcal{Y}}((\alpha_i)_{i\in\mathcal{G}}) = ({}_i\psi_j(\alpha_j\otimes 1) - \alpha_i\, {}_i\varphi_j)_{j\to i}$, for any

$$(\alpha_i)_{i\in\mathcal{G}} \in \bigoplus_{i\in\mathcal{G}} Hom_{A_i}(X_i,\ Y_i).$$

Due to the definition of morphisms between representations, it is easy to see that $Ker\Delta_{\mathcal{X},\mathcal{Y}} = Hom(\mathcal{X},\mathcal{Y})$.

Secondly, we can show that $Coker\Delta_{\mathcal{X},\mathcal{Y}} = Ext^1(\mathcal{X},\mathcal{Y})$ as follows.

Let $\Sigma = ({}_i\sigma_j)$ belong to $\bigoplus_{j\to i} Hom_{A_i}(X_j \otimes {}_jM_i, Y_i)$. Then we can get an extension $E(\Sigma) = (Y_j \oplus X_j, \begin{pmatrix} {}_i\psi_j & {}_i\sigma_j \\ 0 & {}_i\varphi_j \end{pmatrix})$ of representations $\mathcal{X}$ and $\mathcal{Y}$. Conversely, any extension of $\mathcal{X}$ and $\mathcal{Y}$ can be denoted as this form. So, there exists the one-one correspondence between all elements of $\bigoplus_{j\to i} Hom_{A_i}(X_j \otimes {}_jM_i, Y_i)$ and all of extensions of representations $\mathcal{X}$ and $\mathcal{Y}$.

Let $\Sigma' = ({}_i\sigma'_j)$ be another element in $\bigoplus_{j\to i} Hom_{A_i}(X_j \otimes {}_jM_i, Y_i)$ with its corresponding extension $E(\Sigma') = (Y_j \oplus X_j, \begin{pmatrix} {}_i\psi_j & {}_i\sigma'_j \\ 0 & {}_i\varphi_j \end{pmatrix})$.

Then $E(\Sigma)$ and $E(\Sigma')$ are equivalent if and only if there exists an invertible morphism τ of $rep(\mathcal{M})$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{Y} & \xrightarrow{i} & E(\Sigma) & \xrightarrow{p} & \mathcal{X} & \longrightarrow & 0 \\ & & \downarrow id & & \downarrow \tau & & \downarrow id & & \\ 0 & \longrightarrow & \mathcal{Y} & \xrightarrow{i'} & E(\Sigma') & \xrightarrow{p'} & \mathcal{X} & \longrightarrow & 0 \end{array}$$

commutes where i and i' are both embedding maps, p and p' are both projectors. It can be easily checked that τ must be the form of $\tau = \{\begin{pmatrix} 1 & \tau_i \\ 0 & 1 \end{pmatrix} : i \in \mathcal{G}\}$ where τ_i is an A_i-homomorphism from X_i to Y_i, $i \in \mathcal{G}$. And, obviously, for any such τ_i, the given τ always makes this diagram to be commutative. Hence, $E(\Sigma)$ and $E(\Sigma')$ are equivalent if and only if there exists a morphism $\tau = \{\begin{pmatrix} 1 & \tau_i \\ 0 & 1 \end{pmatrix} : i \in \mathcal{G}\}$ of $rep(\mathcal{M})$ for an A_i-homomorphism τ_i from X_i to Y_i, $i \in \mathcal{G}$.

Since the τ is admitted to be a morphism in $rep(\mathcal{M})$, the following square commutes:

$$\begin{array}{ccc} (Y_j \oplus X_j) \otimes {}_jM_i & \xrightarrow{\begin{pmatrix} {}_i\psi_j & {}_i\sigma_j \\ 0 & {}_i\varphi_j \end{pmatrix}} & Y_i \oplus X_i \\ {\scriptstyle \begin{pmatrix} 1 & \tau_j \\ 0 & 1 \end{pmatrix} \otimes 1} \big\downarrow & & \big\downarrow {\scriptstyle \begin{pmatrix} 1 & \tau_i \\ 0 & 1 \end{pmatrix}} \\ (Y_j \oplus X_j) \otimes {}_jM_i & \xrightarrow[\begin{pmatrix} {}_i\psi_j & {}_i\sigma'_j \\ 0 & {}_i\varphi_j \end{pmatrix}]{} & Y_i \oplus X_i \end{array}$$

Then

$$\begin{pmatrix} {}_i\psi_j & {}_i\sigma'_j \\ 0 & {}_i\varphi_j \end{pmatrix} (\begin{pmatrix} 1 & \tau_j \\ 0 & 1 \end{pmatrix} \otimes 1) = \begin{pmatrix} 1 & \tau_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}_i\psi_j & {}_i\sigma_j \\ 0 & {}_i\varphi_j \end{pmatrix}.$$

It follows that ${}_i\sigma_j + \tau_i\, {}_i\varphi_j = {}_i\psi_j(\tau_j \otimes 1) + {}_i\sigma'_j$, hence

$${}_i\sigma_j - {}_i\sigma'_j = {}_i\psi_j(\tau_j \otimes 1) - \tau_i\, {}_i\varphi_j.$$

It means that $\Sigma - \Sigma' \in Im(\Delta_{\mathcal{X},\mathcal{Y}})$ due to the definition of $\Delta_{\mathcal{X},\mathcal{Y}}$.

Hence, we get that $E(\Sigma)$ and $E(\Sigma')$ are equivalent if and only if $\Sigma - \Sigma' \in Im(\Delta_{\mathcal{X},\mathcal{Y}})$, which implies that $Cok(\Delta_{\mathcal{X},\mathcal{Y}}) \cong Ext^1(\mathcal{X}, \mathcal{Y})$.

Next, we need the following lemma:

Lemma 2.3. *Suppose A and B are simple algebras over a field k and X, Y are both right A-modules, Z is a right B-module and M is a free B-A-bimodule. Then,*

(5) $\qquad dim_k Hom_A(X, Y) = (dim_k X\ dim_k Y)/dim_k A,$

(6) $\qquad dim_k Hom_A(Z \otimes_B M, Y) = (rank_A M dim_k Z dim_k Y)/dim_k B.$

Proof. Since A is a simple algebra, we have $A \cong M_n(D)$ for some positive integer n and D a divisible k-algebra.

It is easy to see that for any simple A-modules X and Y, we have $X \cong Y$, then $Hom_A(X,Y) \cong D$ and $dim_k Hom_A(X,Y) = dim_k D$; simultaneously, $dim_k X = ndim_k D$, $dim_k Y = ndim_k D$ and $dim_k A = n^2 dim_k D$. Therefore,

$$dim_k Hom_A(X,Y) = (dim_k X\ dim_k Y)/dim_k A = dim_k D.$$

In general, let A-modules X and Y be any A-modules which are not necessarily simple. Since A is simple, X and Y are semisimple A-modules. Let $X = X_1 \oplus \cdots \oplus X_s$ and $Y = Y_1 \oplus \cdots \oplus Y_t$.

Then, $dim_k Hom_A(X,Y) = dim_k Hom_A(X_1 \oplus \cdots \oplus X_s, Y_1 \oplus \cdots \oplus Y_t) = dim_k \oplus_{i,j} Hom_A(X_i, Y_j) = \oplus_{i,j} dim_k Hom_A(X_i,Y_j) = (st) dim_k D$.

On the other hand,

$$\begin{aligned}(dim_k X\ dim_k Y)/dim_k A &= (dim_k(X_1 \oplus \cdots \oplus X_s)\ dim_k(Y_1 \oplus \cdots \oplus Y_t))/dim_k A \\ &= (\oplus_{i=1}^s dim_k X_i)(\oplus_{i=1}^t dim_k Y_i)/dim_k A \\ &= ((sn)dim_k D(tn)dim_k D)/(n^2 dim_k D) = (st)dim_k D.\end{aligned}$$

Therefore, we get $dim_k Hom_A(X,Y) = (dim_k X\ dim_k Y)/dim_k A$.

According to the adjoint-isomorphism theorem,

$$Hom_A(Z \otimes_B M, Y) \cong Hom_B(Z, Hom_A(M,Y)).$$

Hence, due to (5), we have

$$\begin{aligned}dim_k Hom_A(Z \otimes_B M, Y) &= dim_k Hom_B(Z, Hom_A(M,Y)) \\ &= dim_k Z dim_k Hom_A(M,Y)/dim_k B \\ &= dim_k Z(dim_k M dim_k Y/dim_k A)/dim_k B \\ &= (rank_A M dim_k Z dim_k Y)/dim_k B. \qquad \square\end{aligned}$$

Now, return to the proof of the proposition:

By the definition of B, we have

$$\begin{aligned}B(\mathbf{dim}\mathcal{X}, \mathbf{dim}\mathcal{Y}) &= \sum_{i \in \mathcal{G}} dim_k A_i \frac{dim_k X_i}{dim_k A_i} \frac{dim_k Y_i}{dim_k A_i} \\ &- \sum_{j \to i} d_{ji} dim_k A_i \frac{dim_k X_j}{dim_k A_j} \frac{dim_k Y_i}{dim_k A_i} \\ &= \sum_{i \in \mathcal{G}} (dim_k X_i dim_k Y_i)/dim_k A_i \\ &- \sum_{j \to i} (rank_{A_i}({}_iM_j) dim_k X_j dim_k Y_i)/dim_k A_j \\ &= \sum_{i \in \mathcal{G}} dim_k Hom_{A_i}(X_i, Y_i) - \sum_{j \to i} dim_k Hom_{A_i}(X_j \otimes_{A_j} {}_jM_i, Y_i)\end{aligned}$$

$$
\begin{aligned}
&= dim_k \bigoplus_{i\in\mathcal{G}} Hom_{A_i}(X_i, Y_i) - dim_k \bigoplus_{j\to i} Hom_{A_i}(X_j \otimes_{A_j} {}_jM_i, Y_i) \\
&= dim_k Ker\Delta_{\mathcal{X},\mathcal{Y}} - dim_k Coker\Delta_{\mathcal{X},\mathcal{Y}} \\
&= dim_k Hom(\mathcal{X},\mathcal{Y}) - dim_k Ext^1(\mathcal{X},\mathcal{Y}). \qquad \square
\end{aligned}
$$

According to the discussion above before this theorem, we leave as a question as follows.

Problem 2.1. *Characterize the relationship between the Euler characteristic and the Euler form and that between their corresponding quadratic forms.*

3. Berstein-Gelfand-Ponomarev theory in category of pre-modulations

3.1 Reflection functors of a pre-modulation

A k-pre-modulation $\mathcal{M} = (A_i, {}_iM_j)$ of a valued graph $(\mathcal{G},\mathcal{D})$ is defined in [16] as a set of artinian k-algebras $\{A_i\}_{i\in\mathcal{G}}$, together with a set $\{{}_iM_j\}_{(i,j)\in\mathcal{G}\times\mathcal{G}}$ of finitely generated free unital A_i-A_j-bimodules ${}_iM_j$ such that $rank({}_iM_j)_{A_j} = d_{ij}$ and $rank_{A_i}({}_iM_j) = d_{ji}$.

Assume that $\mathcal{M} = (A_i, {}_iM_j)$ is a k-pre-modulation over a connected valued quiver $(\mathcal{G},\mathcal{D},\Omega)$ with the admissible sequence of sinks $\{1, 2, \ldots, n\}$, that is, $(\mathcal{G},\mathcal{D},\Omega)$ has no oriented cycles. Let $dim_kA_i = f_i$ which is finite by the definition for any $i \in \mathcal{G}$, and let $rank({}_iM_j)_{A_j} = d_{ij}$ and $rank_{A_i}({}_iM_j) = d_{ji}$. Then $d_{ji}f_i = dim_k {}_iM_j = d_{ij}f_j$.

Denote by $\underline{A}_l$ the representation of $rep(\mathcal{M})$ corresponding to the vertex $l \in \mathcal{G}$ defined by $\underline{A}_l = (X_i, {}_i\varphi_j)$ where $X_i = \begin{cases} A_l, & \text{if } i = l \\ 0, & \text{if } i \neq l \end{cases}$ and ${}_i\varphi_j = 0$ for all $i \to j$. All $\underline{A}_l$ $(l \in \mathcal{G})$ are called the *elementary representations* of $rep(\mathcal{M})$.

Since A_l $(l \in \mathcal{G})$ is a simple algebra, let $dim_kA_l = s_l^2$ for a positive integer s_l. As A_l-module, A_l can be decomposed into a direct sum of s_l simple A_l-modules which are isomorphic each other, that is, every A_l has a unique simple A_l-submodule under isomorphism. Equivalently, every $\underline{A}_l$ can be decomposed into a direct sum of some simple representations which are isomorphic each other, that is, we have:

Fact 3.1. *For any vertex $l \in \mathcal{G}$, $\underline{A}_l$ in the category $rep(\mathcal{M})$ has a unique simple direct summand under isomorphism.*

Lemma 3.1. *$\underline{A}_1$ is projective and $\underline{A}_n$ is injective in $rep(\mathcal{M})$.*

Proof. Since $\underline{A}_1$ is non-zero only in the first coordinate, suppose there is the diagram:
where $\beta_i = 0$ for any $i \neq 1$ and the row sequence is exact. Thus, it follows that:

$$\begin{tikzcd} & \underline{A}_1 \arrow[d, "\underline{\beta}=(\beta_i)"] & \\ \mathcal{X} \arrow[r, "\pi"] & \mathcal{Y} \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} & A_1 \arrow[d, "\beta_1"] & \\ X_1 \arrow[r, "\pi_1"] & Y_1 \arrow[r] & 0 \end{tikzcd}$$

But, since A_1 is a simple algebra, A_1 is projective as A_1-module. So, there is γ_1 such that the following diagram commutes:

$$\begin{tikzcd} & A_1 \arrow[dl, "\gamma_1"'] \arrow[d, "\beta_1"] & \\ X_1 \arrow[r, "\pi_1"] & Y_1 \arrow[r] & 0 \end{tikzcd}$$

Hence, the first diagram can be completed by $\underline{\gamma} = (\gamma_i)$ with $\gamma_i = 0$ for $i \neq 1$, that is, the following diagram commutes:

Moreover, it is necessary to explain that $\underline{\gamma}$ is a morphism in $rep(\mathcal{M})$. Indeed, since 1 is a sink, there exists no arrow $1 \to i$ for any i. If there is an arrow $j \to 1$ for some j, the following diagram is always commutative:

$$\begin{tikzcd} 0 \otimes_{A_j} {}_jM_1 \arrow[r, "0"] \arrow[d, "0\otimes id"] & A_1 \arrow[d, "\gamma_1"] \\ X_j \otimes_{A_j} {}_jM_1 \arrow[r, "{}_j\psi_1"] & X_1 \end{tikzcd}$$

From this diagram and $A_j = 0$, $\gamma_j = 0$ for any $j \neq 1$, it follows that $\underline{\gamma}$ is a morphism in $rep(\mathcal{M})$.

Therefore $\underline{A}_1$ is projective in $rep(\mathcal{M})$.

Dually, it can be proved similarly that $\underline{A}_n$ is injective in $rep(\mathcal{M})$ since A_n is injective as A_n-module. $\square$

Corollary 3.1. *In the category $rep(\mathcal{M})$, the unique simple direct summand S_1 under isomorphism of $\underline{A}_1$ is projective and that of $\underline{A}_n$ is injective.*

$$\begin{tikzcd}
& \underline{A}_1 \arrow[dl, "\underline{\gamma}"'] \arrow[d, "\underline{\beta} = (\beta_i)"] & \\
\mathcal{X} \arrow[r, "\pi"] & \mathcal{Y} \arrow[r] & 0
\end{tikzcd}$$

Corollary 3.2. *For* $i,\, j \in \mathcal{G}$, $dim_k Ext^1(\underline{A}_i,\, \underline{A}_j) = d_{ij} f_j$, $dim_k Ext^1(\underline{A}_i,\, \underline{A}_j) = d_{ji} f_i$.

Proof. If $i \to j$, by the definition, we have

$$B(\mathbf{dim}\, \underline{A}_i,\, \mathbf{dim}\, \underline{A}_j) = -d_{ij} f_j.$$

Since $Hom(\underline{A}_i,\, \underline{A}_j) = 0$, by the Theorem 2.2 we deduce that

$$dim_k Ext^1(\underline{A}_i,\, \underline{A}_j) = -B(\mathbf{dim}\, \underline{A}_i,\, \mathbf{dim}\, \underline{A}_j) = d_{ij} f_j,$$

and hence the first equality follows. On the other hand, if there is no arrow $i \to j$, the first equality are trivial as $0 = 0$.

The second equality is an immediate consequence of the fact that $d_{ij} f_j = d_{ji} f_i$. □

Given any vertex $k \in \mathcal{G}$ of a valued quiver $(\mathcal{G},\, \mathcal{D},\, \Omega)$, we define a reflection $\delta_k : \mathbb{Q}^{\mathcal{G}} \to \mathbb{Q}^{\mathcal{G}}$ satisfying that if $\underline{x} = (x_i)_{i \in \mathcal{G}}$, then $\delta_k \underline{x} = \underline{y} = (y_i)_{i \in \mathcal{G}}$ is given by:

$$\begin{aligned} y_i &= x_i, \forall i \neq k, \\ y_k &= -x_k + \sum_{i \in \mathcal{G}} d_{ik} x_i. \end{aligned}$$

Corollary 3.3. (i) *Let* $\mathcal{X}$ *be a representation with no direct summand isomorphic to the unique simple direct summand of* $\underline{A}_1$, *then*

$$(\delta_1(\mathbf{dim}\mathcal{X}))_1 = \frac{dim_k Ext^1(\mathcal{X},\, \underline{A}_1)}{dim_k A_1}.$$

(ii) *Let* $\mathcal{X}$ *be a representation with no direct summand isomorphic to* $\underline{A}_n$, *then*

$$(\delta_n(\mathbf{dim}\mathcal{X}))_n = \frac{dim_k Ext^1(\underline{A}_n,\, \mathcal{X})}{dim_k A_n}.$$

Proof. (i) If $\mathcal{X}$ has no direct summand isomorphic to the unique simple direct summand of $\underline{A}_1$, then $Hom(\mathcal{X},\, \underline{A}_1) = 0$. Hence

$$B(\mathbf{dim}\mathcal{X},\, \mathbf{dim}\, \underline{A}_1) = -dim_k Ext^1(\mathcal{X},\, \underline{A}_1).$$

On the other hand,

$$\begin{aligned} B(\mathbf{dim}\mathcal{X}, \mathbf{dim}\ \underline{A}_1) &= f_1 \frac{dim_k X_1}{f_1} - \sum_{i \to 1} d_{i1} f_1 \frac{dim_k X_i}{f_i} \\ &= -f_1(-\frac{dim_k X_1}{f_1} + \sum_{i \in \mathcal{G}} d_{i1} \frac{dim_k X_i}{f_i}) \\ &= -f_1(\delta_1(\mathbf{dim}\mathcal{X}))_1, \end{aligned}$$

where the second equality uses the fact that the vertex 1 is a sink.

(ii) can be proved dually. □

Now, to any sink (respectively, source) k of the graph $\mathcal{G}$, we shall associate a functor Δ_k^+ (respectively, Δ_k^-) of $rep(\mathcal{M}, \Omega)$ into $rep(\mathcal{M}, \delta_k\Omega)$, which are called *the reflection functors of the pre-modulation* $\mathcal{M} = (A_i, {}_iM_j)$.

In accordance with our convention, 1 is a sink, and n a source of Ω, thus we shall content ourselves with defining Δ_1^+ and Δ_n^-.

Let $\mathcal{X} = (X_i, {}_i\varphi_j)$ be an object of $rep(\mathcal{M}, \Omega)$, we recall that ${}_i\varphi_j : X_j \otimes_{A_j} {}_jM_i \to X_i$ is an A_i-map. We can attach to it an A_j-map $\overline{{}_i\varphi_j} : X_j \to X_i \otimes_{A_i} {}_iM_j$ in the following way.

By the adjoint isomorphism theorem, we have

$$Hom_{A_i}(X_j \otimes_{A_j} {}_jM_i, X_i) \cong Hom_{A_j}(X_j, Hom_{A_i}({}_jM_i, X_i)).$$

Lemma 3.2. *Let A be a semisimple algebra and B another finite-dimensional algebra over a field k, X be right an A-module and M a left-right free B-A-bimodule with basis of a finite number of generators. Then, as right B-modules,*

$$Hom_A(M, X) \cong X \otimes_A Hom_A(M, A).$$

Proof. Define $\pi :\ X \otimes_A Hom_A(M, A) \to Hom_A(M, X)$ satisfying

$$\pi(\sum_i x_i \otimes f_i)(m) = x_i f_i(m),$$

for all $x_i \in X, f_i \in Hom_A(M, A)$ and $m \in M$. Then, π is a right B-module homomorphism. In fact, $\pi((\sum_i x_i \otimes f_i)b)(m) = \pi(\sum_i x_i \otimes f_i b)(m) = \sum_i x_i(f_i b)(m) = \sum_i x_i f_i(bm) = \pi(\sum_i x_i \otimes f_i)(bm) = (\pi(\sum_i x_i \otimes f_i)b)(m)$, it follows that $\pi((\sum_i x_i \otimes f_i)b) = \pi(\sum_i x_i \otimes f_i)b$.

Let $\{\varepsilon_1, \ldots, \varepsilon_s\}$ be the basis of M as right A-module. Define f_i be from M to A satisfying $f_i(\varepsilon_j) = \begin{cases} 1, & \text{if } i \neq j \\ 0, & \text{otherwise} \end{cases}$. Then f_i can be expended into a right A-homomorphism and $\{f_1, \ldots, f_s\}$ is the basis of $Hom_A(M, A)$ as left free A-module. For any $g \in Hom_A(M, X)$, let $\chi = \sum_{i=1}^s g(\varepsilon_i) \otimes f_i$, then $\chi \in X \otimes_A Hom_A(M, A)$ satisfying $\pi(\chi) = g$. Therefore, π is surjective.

Write $M_A \cong \oplus_\lambda A_A$, thus, we get the following right A-isomorphisms:

$Hom_A(M,X) \cong Hom_A(\oplus_\lambda A_A, X) \cong \oplus_\lambda Hom_A(A_A, X) \cong \oplus_\lambda X_A \cong \oplus_\lambda X_A \otimes A \cong X_A \otimes (\oplus_\lambda A_A) \cong X_A \otimes Hom(\oplus_\lambda A_A, A) \cong X_A \otimes Hom(M, A)$.
Then, $dim_k(Hom_A(M,X)) = dim_k(X \otimes_A Hom_A(M,A))$.

Hence from the fact that the surjective right B-module homomorphism π is also a surjective k-linear map of spaces, we know that π is an isomorphism. $\square$

Dealing with finite-dimensional modules, by Lemma 3.2, we get that

$$Hom_{A_i}({}_jM_i,\, X_i) \cong X_i \otimes_{A_i} Hom_{A_i}({}_jM_i,\, A_i) \cong X_i \otimes_{A_i} {}_iM_j,$$

giving a canonical isomorphism

$$Hom_{A_i}(X_j \otimes_{A_j} {}_jM_i,\, X_i) \cong Hom_{A_j}(X_j,\, X_i \otimes_{A_i} {}_iM_j). \tag{7}$$

Thus to ${}_i\varphi_j$ there corresponds $\overline{{}_i\varphi_j} : X_j \to X_i \otimes_{A_i} {}_iM_j$ which will be referred to as the *adjoint* of ${}_i\varphi_j$. Now we can define $\Delta_1^+\mathcal{X} = \mathcal{Y} = (Y_i,\, {}_i\psi_j)$ as follows:

If $j \neq 1$, take $Y_j = X_j$, and ${}_i\psi_j = {}_i\varphi_j$.

If $j = 1$,for every $i \in \mathcal{G}$ such $\exists\, i \to 1$, we have a mapping ${}_1\varphi_i : X_i \otimes_{A_i} {}_iM_1 \to X_1$. Let $\varphi_1 = \bigoplus_{j\to1} {}_1\varphi_j :\ \bigoplus_{j\to1} X_j \otimes_{A_j} {}_jM_1 \to X_1$. Let $Y_1 = Ker\varphi_1$, κ_1 the embedding map from $Ker\varphi_1$ to $\bigoplus_{j\to1} X_j \otimes_{A_j} {}_jM_1$ and ${}_i\kappa_1 = \pi_i\kappa_1 : Y_1 \to X_i \otimes_{A_i} {}_iM_1 = Y_i \otimes_{A_i} {}_iM_1$ (where π_i is the canonical projection if there exists an arrow $i \to 1$):

$$\begin{tikzcd}
 & & X_i \otimes_{A_i} {}_iM_1 \arrow[dr, "{}_1\varphi_i"] & \\
0 \arrow[r] & Y_1 \arrow[ur, "{}_i\kappa_1"] \arrow[r, "\kappa_1"] & \bigoplus_{j\to1}(X_j \otimes_{A_j} {}_jM_1) \arrow[u, "\pi_i"] \arrow[r, "\varphi_1"] & X_1
\end{tikzcd}$$

According to (7), we put ${}_i\psi_1 = \overline{{}_i\kappa_1} : Y_1 \otimes_{A_1} {}_1M_i \to X_i = Y_i$. Thus we have defined $\Delta_1^+\mathcal{X} = \mathcal{Y}$ in $rep(\mathcal{M},\, \delta_1\Omega)$.

If $\alpha : \mathcal{X} \to \mathcal{X}'$ is a morphism of $rep(\mathcal{M},\, \Omega)$, $\beta = \Delta_1^+\alpha$ is defined as follows: if $j \neq 1$, take $\beta_j = \alpha_j$ and $\beta_1 : Y_1 \to Y_1'$ is the restriction to Y_1 of the mapping

$$\bigoplus_{i\to1}(\alpha_i \otimes 1) :\ \bigoplus_{i\to1} X_i \otimes_{A_i} {}_iM_1 \to \bigoplus_{i\to1} X_i' \otimes_{A_i} {}_iM_1.$$

If $\exists$ arrow $i \to 1$ in Ω, then

$$\begin{tikzcd}
0 \arrow[r] & Y_1 \arrow[r, "\kappa_1"] \arrow[d, "\beta_1"] & \bigoplus_{j\to1} X_j \otimes_{A_j} {}_jM_1 \arrow[r, "\varphi_1"] \arrow[d, "\bigoplus_{j\to1}(\alpha_j\otimes1)"] & X_1 \arrow[d, "\alpha_1"] \\
0 \arrow[r] & Y_1' \arrow[r, "\kappa_1'"] & \bigoplus_{j\to1} X_j' \otimes_{A_j} {}_jM_1 \arrow[r, "\varphi_1'"] & X_1'
\end{tikzcd}$$

Thus,

$$\begin{tikzcd} Y_1 \arrow[r, "{}_i\kappa_1"] \arrow[d, "\beta_1"] & Y_i \otimes_{A_i} {}_iM_1 \arrow[d, "\beta_i\otimes 1"] \\ Y_1' \arrow[r, "{}_i\kappa_1'"] & Y_i' \otimes_{A_i} {}_iM_1 \end{tikzcd}$$

It follows that

$$\begin{tikzcd} Y_1 \otimes_{A_1} {}_1M_i \arrow[r, "{}_i\psi_1"] \arrow[d, "\beta_1\otimes 1"] & Y_i \arrow[d, "\beta_i"] \\ Y_1' \otimes_{A_1} {}_1M_i \arrow[r, "{}_i\psi_1'"] & Y_i' \end{tikzcd}$$

And, if $i \neq 1$, $\beta_i = \alpha_i$ which are morphisms in $rep(\mathcal{M}, \Omega)$.

Hence, all β_i are morphisms in $rep(\mathcal{M}, \delta_1\Omega)$. Thus, β is a morphism of $rep(\mathcal{M}, \delta_1\Omega)$.

In summary, Δ_1^+ is a functor from $rep(\mathcal{M}, \Omega)$ to $rep(\mathcal{M}, \delta_1\Omega)$.

Dually, $\Delta_n^-\mathcal{X} = \mathcal{Y} = (Y_i, {}_i\psi_j)$ is the object of $rep(\mathcal{M}, \delta_n\Omega)$ defined as follows:
(i) If $i \neq n$, take $Y_i = X_i$, and ${}_i\psi_j = {}_i\varphi_j$; (ii) If $i = n$, let Y_n be the cokernel in the diagram:

$$\begin{tikzcd} & X_j \otimes_{A_j} {}_jM_n \arrow[d, shift left, "\iota_j"] \arrow[dr, "{}_n\eta_i"] & & \\ X_n \arrow[ur, "{}_j\bar{\varphi}_n"] \arrow[r, "({}_j\bar{\varphi}_n)"'] & \oplus_{n\to j}(X_j \otimes_{A_j} {}_jM_n) \arrow[u, shift left, "\pi_j"] \arrow[r, "\eta_n"'] & Y_n \arrow[r] & 0 \end{tikzcd}$$

and ${}_n\psi_j = {}_n\eta_j$.

For a morphism $\alpha : \mathcal{X} \to \mathcal{X}'$, we define $\beta = \Delta_n^-\alpha$ by letting $\beta_i = \alpha_i$ for $i \neq n$, while $\beta_n : Y_n \to Y_n'$ is the mapping induced on the cokernels by

$$\bigoplus_{n\to j}(\alpha_j \otimes 1) : \quad \bigoplus_{n\to j} X_j \otimes_{A_j} {}_jM_n \to \bigoplus_{n\to j} X_j' \otimes_{A_j} {}_jM_n.$$

In summary, Δ_n^- is a functor from $rep(\mathcal{M}, \Omega)$ to $rep(\mathcal{M}, \delta_n\Omega)$.

As a direct consequence of the definition, Δ_1^+ preserves monomorphisms, while Δ_n^- preserves epimorphisms, and both preserve finite direct sums.

3.2 Construction of indecomposable projectives/injective representations

In this part, we use reflection functors to construct indecomposable projective/injective representations of a hereditary algebra.

Lemma 3.3. *Let $(\mathcal{G}, \mathcal{D}, \Omega)$ be a connected valued quiver with admissible orientation Ω and $\mathcal{M}$ be a k-pre-modulation. Then for every representation $\mathcal{X}$ of $\mathcal{M}$:*

(i) $\mathcal{X} \cong \Delta_1^-\Delta_1^+\mathcal{X} \oplus \mathcal{P}$, where $\mathcal{P} = (P_i, {}_i\pi_j)$ with $P_i = 0$ if $i \neq 1$ and P_1 is a (semisimple) A_1-module. Thus if $\mathcal{X}$ is indecomposable, either (a) $\mathcal{X} \cong \mathcal{P}$ (equivalently, $\Delta_1^+\mathcal{X} = 0$) in which case $\mathcal{P}$ is the unique simple direct summand of $\underline{A}_1$ under isomorphism or (b) $\mathcal{X} \cong \Delta_1^-\Delta_1^+\mathcal{X}$ (equivalently, $\Delta_1^+\mathcal{X} \neq 0$) in which case $End(\Delta_1^+\mathcal{X}) \cong End(\mathcal{X})$ and thus $\Delta_1^+\mathcal{X}$ is indecomposable and $\mathbf{dim}(\Delta_1^+\mathcal{X}) = \delta_1(\mathbf{dim}\mathcal{X})$;

(ii) $\mathcal{X} \cong \Delta_n^+\Delta_n^-\mathcal{X} \oplus \mathcal{I}$, where $\mathcal{I} = (I_i, {}_i\tau_j)$ with $I_i = 0$, if $i \neq n$ and I_n is a (semisimple) A_n-module. Thus, if $\mathcal{X}$ is indecomposable, either (a) $\mathcal{X} \cong \mathcal{I}$ (equivalently, $\Delta_n^-\mathcal{X} = 0$) in which case $\mathcal{I}$ is the unique simple direct summand of $\underline{A}_n$ under isomorphism or (b) $\mathcal{X} \cong \Delta_n^+\Delta_n^-\mathcal{X}$ (equivalently, $\Delta_n^-\mathcal{X} \neq 0$) in which case $End(\Delta_n^-\mathcal{X}) \cong End(\mathcal{X})$ and thus $\Delta_n^-\mathcal{X}$ is indecomposable and $\mathbf{dim}(\Delta_n^-\mathcal{X}) = \delta_n(\mathbf{dim}\mathcal{X})$.

Proof. Firstly, We give the prove of (i).

Since $\mathcal{X} \in rep(\mathcal{M}, \Omega)$ and 1 is a sink in Ω, $\mathcal{Y} = \Delta_1^+\mathcal{X} \in rep(\mathcal{M}, \delta_1\Omega)$ and 1 is a source in $\delta_1\Omega$. Then, by the definition of Δ_1^-, we have

$$(\Delta_1^-\Delta_1^+\mathcal{X})_1 = cokY_1 = cok(ker\varphi_1) = Im\varphi_1 \overset{\mu_1}{\hookrightarrow} X_1.$$

Thus, we obtain the following diagram in the first coordinate from the construction of $\Delta_1^-\Delta_1^+\mathcal{X}$:

$$\begin{tikzcd}
0 \arrow[r] & (\Delta_1^+\mathcal{X})_1 \arrow[r, "\kappa_1"] & \oplus_{j\to 1}(X_j \otimes_{A_j} {}_jM_1) \arrow[r, "\varphi_1"] \arrow[d] & X_1 \\
 & & (\Delta_1^-\Delta_1^+\mathcal{X})_1 \arrow[ur, "\mu_1"'] \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

Due to the above mention, $(\Delta_1^-\Delta_1^+\mathcal{X})_1$ can be seen as an A_1-submodule of X_1. But, since A_1 is a simple algebra, all its modules are projective and then $(\Delta_1^-\Delta_1^+\mathcal{X})_1$ is a direct summand of X_1 as an A_1-module. Let $X_1 \cong (\Delta_1^-\Delta_1^+\mathcal{X})_1 \oplus P_1$ where P_1 is a semisimple A_1-module. Thus, by the definition of $\Delta_1^-\Delta_1^+\mathcal{X}$, $\mathcal{X} \cong \Delta_1^-\Delta_1^+\mathcal{X} \oplus \mathcal{P}$, where $\mathcal{P} = (P_i, {}_i\pi_j)$ with $P_i = 0$ if $i \neq 1$.

Hence, if $\mathcal{X}$ is indecomposable, we have either (a) $\mathcal{X} \cong \mathcal{P}$, equivalently, $\Delta_1^-\Delta_1^+\mathcal{X} = 0$ or (b) $\mathcal{X} \cong \Delta_1^-\Delta_1^+\mathcal{X}$, equivalently, $\mathcal{P} = 0$.

In the case (a), if $\Delta_1^+\mathcal{X} = 0$, clearly $\Delta_1^-\Delta_1^+\mathcal{X} = 0$; conversely, if $\Delta_1^-\Delta_1^+\mathcal{X} = 0$, then in the above diagram all $X_j = 0$ $(j \neq 1)$ which means $(\Delta_1^+\mathcal{X})_1 = 0$ and it follows that $\Delta_1^+\mathcal{X} = 0$. Therefore, $\mathcal{X} \cong \mathcal{P}$ is equivalent to $\Delta_1^+\mathcal{X} = 0$.

Moreover, in the case (b), $\mathcal{X} \cong \Delta_1^-\Delta_1^+\mathcal{X}$ is equivalent to $\Delta_1^+\mathcal{X} \neq 0$. Then, $\Delta_1^+\Delta_1^-\Delta_1^+\mathcal{X} \cong \Delta_1^+\mathcal{X}$ and φ_1 is surjective.

From X_1 to get $(\Delta_1^+\mathcal{X})_1$, we have the following:

$$\begin{tikzcd}
0 \arrow[r] & (\Delta_1^+\mathcal{X})_1 \arrow[r, "\kappa_1"] \arrow[d, "\widetilde{f}_1"] & \bigoplus\limits_{j\to 1} X_j \otimes_{A_j} {}_jM_1 \arrow[r, "\varphi_1"] \arrow[d, "\bigoplus_{j\to 1}(f_j\otimes 1)"] & X_1 \arrow[d, "f_1"] \\
0 \arrow[r] & (\Delta_1^+\mathcal{X})_1 \arrow[r, "\kappa_1"] & \bigoplus\limits_{j\to 1} X_j \otimes_{A_j} {}_jM_1 \arrow[r, "\varphi_1"] & X_1
\end{tikzcd}$$

From $(\Delta_1^+\mathcal{X})_1$ to get $(\Delta_1^-\Delta_1^+\mathcal{X})_1$, we have the following:

$$\begin{tikzcd}
0 \arrow[r] & (\Delta_1^+\mathcal{X})_1 \arrow[r, "\kappa_1"] \arrow[d, "\widetilde{f}_1"] & \bigoplus\limits_{j\to 1} X_j \otimes_{A_j} {}_jM_1 \arrow[r, "\varphi_1"] \arrow[d, "\bigoplus_{j\to 1}(f_j\otimes 1)"] & (\Delta_1^-\Delta_1^+\mathcal{X})_1 \arrow[d, "\widetilde{\widetilde{f}}_1"] \\
0 \arrow[r] & (\Delta_1^+\mathcal{X})_1 \arrow[r, "\kappa_1"] & \bigoplus\limits_{j\to 1} X_j \otimes_{A_j} {}_jM_1 \arrow[r, "\varphi_1"] & (\Delta_1^-\Delta_1^+\mathcal{X})_1
\end{tikzcd}$$

where $(\Delta_1^-\Delta_1^+\mathcal{X})_1 = Im\varphi_1$ is embedded into X_1 by μ_1. But, φ_1 is surjective in the case (b), $(\Delta_1^-\Delta_1^+\mathcal{X})_1 = Im\varphi_1$ is isomorphic to X_1. So, f_1 and $\widetilde{\widetilde{f}}_1$ are one-one correspondence via $\Delta_1^-\Delta_1^+$. Therefore, in the series of maps:

$$End(\mathcal{X}) \xrightarrow{\Delta_1^+} End(\Delta_1^+\mathcal{X}) \xrightarrow{\Delta_1^-} End(\Delta_1^-\Delta_1^+\mathcal{X}) \xrightarrow{\Delta_1^+} End(\Delta_1^+\Delta_1^-\Delta_1^+\mathcal{X}),$$

we get $End(\mathcal{X}) \overset{\Delta_1^-\Delta_1^+}{\cong} End(\Delta_1^-\Delta_1^+\mathcal{X})$ and similarly,

$$End(\Delta_1^+\mathcal{X}) \overset{\Delta_1^+\Delta_1^-}{\cong} End(\Delta_1^+\Delta_1^-\Delta_1^+\mathcal{X}).$$

From them, it follows that $End(\Delta_1^+\mathcal{X}) \overset{\Delta_1^-}{\cong} End(\Delta_1^-\Delta_1^+\mathcal{X})$, and then $End(\mathcal{X}) \overset{\Delta_1^+}{\cong} End(\Delta_1^+\mathcal{X})$. Naturally, the above isomorphisms still hold under the meaning of the endomorphism algebras of these representations.

Now the indecomposability of $\mathcal{X}$ implies that $End(\mathcal{X})$ is local, hence so is $\mathrm{End}(\Delta_1^+\mathcal{X})$ through the isomorphism and then $\Delta_1^+\mathcal{X}$ is indecomposable.

Lastly, we verify that $\mathbf{dim}(\Delta_1^+\mathcal{X}) = \delta_1(\mathbf{dim}\mathcal{X})$ in the case (b). By the definitions of $\Delta_1^+\mathcal{X}$ and δ_1, it is enough to show that $\frac{dim_k(\Delta_1^+\mathcal{X})_1}{dim_k A_1} = (\delta_1(\mathbf{dim}\mathcal{X}))_1$.

On the one hand,

$$(\delta_1(\mathbf{dim}\mathcal{X}))_1 = -\frac{dim_k X_1}{dim_k A_1} + \sum_{i\to 1} d_{i1}\frac{dim_k X_i}{dim_k A_i}.$$

On the other hand, in this case, φ_1 is surjective, then we have the short exact sequence

$$0 \longrightarrow (\Delta_1^+\mathcal{X})_1 \longrightarrow \bigoplus_{i\to 1}(X_i \otimes_{A_i} {}_iM_1) \longrightarrow X_1 \longrightarrow 0$$

which gives $dim_k(\Delta_1^+\mathcal{X})_1 = \sum_{i\to 1} dim_k(X_i \otimes_{A_i} {}_iM_1) - dim_k X_1$. Thus,

$$\frac{dim_k(\Delta_1^+\mathcal{X})_1}{dim_k A_1} = -\frac{dim_k X_1}{dim_k A_1} + \sum_{i\to 1}\frac{dim_k(X_i \otimes_{A_i} {}_iM_1)}{dim_k A_1}.$$

Hence, it is enough for us to prove that for any arrow $i \to 1$, $\frac{dim_k(X_i\otimes_i M_1)}{dim_k A_1} = d_{i1}\frac{dim_k X_i}{dim_k A_i}$.

In fact, $dim_k({}_iM_1) = d_{i1} dim_k A_1$, so $d_{i1}\frac{dim_k X_i}{dim_k A_i} = \frac{dim_k({}_iM_1)}{dim_k A_1}\frac{dim_k X_i}{dim_k A_i}$. Since A_i is a simple algebra over k and X_i is its right module, there is $dim_k A_i = s_i^2$ for some positive integer s_i, $X_i = W_1 \oplus \ldots \oplus W_t$ for some right A_i-simple submodules $W_1, \ldots, W_t$ and $dim_k W_i = s_i$ for all i. And, ${}_iM_1$ is a left free A_i-module with d_{1i} the rank of a basis which we write $d_{1i} = rank_{A_i}({}_iM_1)$. Then, ${}_iM_1 = \oplus\, d_{1i}A_i$ and

$$X_i \otimes_{A_i} {}_iM_1 = (\oplus_{j=1}^t W_j)\otimes_{A_i}(\oplus\, d_{1i}A_i) = \oplus\, d_{1i}(\oplus_{j=1}^t W_j \otimes_{A_i} A_i) = \oplus\, d_{1i}(\oplus_{j=1}^t W_j).$$

Thus, $dim_k(X_i \otimes_{A_i} {}_iM_1) = d_{1i} t s_i$.

On the other hand, $(dim_k({}_iM_1) dim_k X_i)/dim_k A_i = rank_{A_i}({}_iM_1) dim_k X_i = d_{1i} t s_i$.

Hence, $\frac{dim_k(X_i\otimes_i M_1)}{dim_k A_1} = d_{i1}\frac{dim_k X_i}{dim_k A_i}$, then $\frac{dim_k(\Delta_1^+\mathcal{X})_1}{dim_k A_1} = (\delta_1(\mathbf{dim}\mathcal{X}))_1$. It means that $\mathbf{dim}(\Delta_1^+\mathcal{X}) = \delta_1(\mathbf{dim}\mathcal{X})$.

The proof of (ii) can be given dually by considering the following diagram:

$$\begin{tikzcd}
X_n \arrow[r, "({}_i\bar{\varphi}_n)"] & \oplus_{n\to i}(X_i \otimes_{A_i} {}_iM_1) \arrow[r, "\eta_n"] & (\Delta_n^-\mathcal{X})_n \arrow[r] & 0 \\
& (\Delta_n^+\Delta_n^-\mathcal{X})_n = ker({}_i\bar{\varphi}_n) \arrow[u] & & \\
& 0 \arrow[u] & &
\end{tikzcd}$$

The direct sum $\mathcal{X} \cong \Delta_n^+\Delta_n^-\mathcal{X} \oplus \mathcal{I}$ is from the fact A_n is a simple algebra and then X_n is projective as A_n-module. The further discussion is similar in dual. $\square$

Theorem 3.2. *(i) The full subcategory $rep^{(1)}(\mathcal{M}, \Omega)$ of all representations in $rep(\mathcal{M}, \Omega)$ with no direct summand isomorphic to the unique projective simple direct summand (under isomorphism) of $\underline{A}_1$ is equivalent to the full subcategory $rep_{(1)}(\mathcal{M}, \Omega)$ of all representations in $rep(\mathcal{M}, \delta_1\Omega)$ with no direct summand isomorphic to the unique injective simple direct summand (under isomorphism) of $\underline{A}_1$.*

(ii) The full subcategory $rep_{(n)}(\mathcal{M}, \Omega)$ of all representations in $rep(\mathcal{M}, \Omega)$ with no direct summand isomorphic to the unique injective simple direct summand (under isomorphism) of $\underline{A}_n$ is equivalent to the full subcategory $rep^{(n)}(\mathcal{M}, \Omega)$ of all representations in $rep(\mathcal{M}, \delta_n\Omega)$ with no direct summand isomorphic to the unique projective simple direct summand (under isomorphism) of $\underline{A}_n$.

Proof. By Lemma 3.1, $\underline{A}_1$ is projective and $\underline{A}_n$ is injective in $rep(\mathcal{M}, \Omega)$. Then by the definitions of δ_1 and δ_n, in $rep(\mathcal{M}, \delta_1\Omega)$ and $rep(\mathcal{M}, \delta_n\Omega)$ respectively, $\underline{A}_1$ is injective and $\underline{A}_n$ is projective. Then, so are their direct summands respectively.

Any $\mathcal{X} \in rep(M, \Omega)$ can be written as $\mathcal{X} = \mathcal{P}^{(1)}+\ldots+\mathcal{P}^{(s)}+\mathcal{X}^{(1)}+\ldots+\mathcal{X}^{(t)}$ where all $\mathcal{P}^{(i)}$are indecomposable and $\Delta_1^+\mathcal{P}^{(i)} = 0$, all $\mathcal{X}^{(j)}$are indecomposable and $\Delta_1^+\mathcal{X}^{(j)} \neq 0$. Then by Lemma 3.3, $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(s)}$ are all the (possible) direct summands of X isomorphic to the unique simple direct summand of $\underline{A}_1$, and $\Delta_1^-\Delta_1^+\mathcal{X} = \Delta_1^-\Delta_1^+\mathcal{X}^{(1)} + \ldots + \Delta_1^-\Delta_1^+\mathcal{X}^{(t)} = \mathcal{X}^{(1)} + \ldots + \mathcal{X}^{(t)}$. Therefore, $\mathcal{X} = \Delta_1^-\Delta_1^+\mathcal{X}$ if and only if X has no direct summands isomorphic to the unique simple direct summand of $\underline{A}_1$. It means $\mathcal{X} \in rep^{(1)}(M, \Omega)$ if and only if $\mathcal{X} = \Delta_1^-\Delta_1^+\mathcal{X}$. Moreover, through the functors Δ_1^-, Δ_1^+ in $rep^{(1)}(M, \Omega)$, for any morphism $\alpha : \mathcal{X} \to \mathcal{X}'$, we get also $\alpha = \Delta_1^-\Delta_1^+\alpha$.

Similarly, $\mathcal{Y} = \Delta_1^-\Delta_1^+\mathcal{Y}$ for any object $\mathcal{Y}$ in $rep_{(1)}(\mathcal{M}, \Omega)$ and $\beta = \Delta_1^+\Delta_1^-\beta$ for a morphism β in $rep_{(1)}(\mathcal{M}, \Omega)$. Thus, $\mathcal{X} \in rep^{(1)}(M, \Omega)$ means $\Delta_1^+\mathcal{X} = \Delta_1^+\Delta_1^-(\Delta_1^+\mathcal{X})$. So, $\Delta_1^+\mathcal{X}$ is in $rep_{(1)}(M, \Omega)$. Similarly, for any morphism α in $rep^{(1)}(M, \Omega)$, $\Delta_1^+\alpha$ is in $rep_{(1)}(M, \Omega)$. That is, Δ_1^+ is a functor from $rep^{(1)}(M, \Omega)$ to $rep_{(1)}(M, \Omega)$.

Similarly, Δ_1^- is a functor from $rep_{(1)}(M, \Omega)$ to $rep^{(1)}(M, \Omega)$.

Trivially, Δ_1^- and Δ_1^+ are mutual invertible. Hence, Δ_1^+ and Δ_1^- implement the desired equivalence.

The part (ii) can be discussed similarly. $\square$

The following corollary can be got easily from the relations $\mathcal{X} = \Delta_1^-\Delta_1^+\mathcal{X}$ and $\alpha = \Delta_1^-\Delta_1^+\alpha$:

Corollary 3.4. *(i) For two objects $\mathcal{X}$, $\mathcal{X}'$ in $rep^{(1)}(\mathcal{M},\Omega)$,*

$$Ext^1(\mathcal{X},\mathcal{X}') \cong Ext^1(\Delta_1^+\mathcal{X},\Delta_1^+\mathcal{X}');$$

(ii) For two objects $\mathcal{Y}$, $\mathcal{Y}'$ in $Rep_{(1)}(\mathcal{M},\Omega)$,

$$Ext^1(\mathcal{Y},\mathcal{Y}') \cong Ext^1(\Delta_1^-\mathcal{Y},\Delta_1^-\mathcal{Y}').$$

Now, define the functors:

$$\Delta^+ = \Delta_n^+\Delta_{n-1}^+ \dots \Delta_2^+\Delta_1^+ : \ rep(\mathcal{M},\Omega) \to rep(\mathcal{M},\Omega)$$

and

$$\Delta^- = \Delta_1^-\Delta_2^- \dots \Delta_{n-1}^-\Delta_n^- : \ rep(\mathcal{M},\Omega) \to rep(\mathcal{M},\Omega).$$

These endofunctors are called the *Coxter functors*. For each $u \in \mathcal{G}$, define the representations $\underline{P}_u = \Delta_1^-\Delta_2^- \dots \Delta_{u-1}^-\underline{A}_u$ with $\underline{A}_u \in rep(\mathcal{M}, \delta_u\delta_{u+1}\dots\delta_n\Omega)$, $\underline{Q}_u = \Delta_n^+\Delta_{n-1}^+ \dots \Delta_{u+1}^+\underline{A}_u$ with $\underline{A}_u \in rep(\mathcal{M}, \delta_u\delta_{u-1}\dots\delta_1\Omega)$.

Since A_u is a simple algebra over k, let $dimA_u = s_u^2$ for a positive integer s_u, then $A_u = W_u^{(1)} + \dots + W_u^{(s_u)}$ with the mutual-isomorphic simple A_u-modules $W_u^{(1)}, \dots, W_u^{(s_u)}$, and $\underline{A}_u = \underline{W}_u^{(1)} + \dots + \underline{W}_u^{(s_u)}$ where all mutual-isomorphic simple representations $\underline{W}_u^{(i)}$ are defined by $\underline{W}_u^{(i)} = (X_j, \ {}_j\varphi_l)$ for $X_j = \begin{cases} W_u^{(i)}, & \text{if } j = u \\ 0, & \text{if } j \neq u \end{cases}$ and ${}_j\varphi_l = 0$ for all $j \to l$.

It is clear to understand that the set $\{\underline{W}_u^{(1)}\}_{1\leq u\leq n}$ consists of the set of all mutual non-isomorphic simple representations in $rep(\mathcal{M},\Omega)$. Then, $\underline{P}_u = \mathcal{P}_u^{(1)} \oplus \dots \oplus \mathcal{P}_u^{(s_u)}$ and $\underline{Q}_u = \mathcal{Q}_u^{(1)} \oplus \dots \oplus \mathcal{Q}_u^{(s_u)}$ with mutual-isomorphic indecomposable representations $\mathcal{P}_u^{(i)} = \Delta_1^-\Delta_2^- \dots \Delta_{u-1}^-\underline{W}_u^{(i)}$ for $i = 1, \dots, s_u$ and $\mathcal{Q}_u^{(i)} = \Delta_n^+\Delta_{n-1}^+ \dots \Delta_{u+1}^+\underline{W}_u^{(i)}$ for $i = 1, \dots, s_u$ by Lemma 3.3.

For any distinct u, v, $\mathcal{P}_u^{(i)}$ and $\mathcal{P}_v^{(j)}$ are non-isomorphic each other for all i, j, since $\underline{W}_u^{(i)}$ and $\underline{W}_v^{(j)}$ are so. Now, we can obtain:

Theorem 3.3. *The set* $\{\mathcal{P}_u^{(1)}\}_{1\leq u\leq n}$ *(respectively,* $\{\mathcal{Q}_u^{(1)}\}_{1\leq u\leq n}$*) consists of the set of all non-isomorphic indecomposable projective (respectively, injective) representations in* $rep(\mathcal{M},\Omega)$ *for a connected valued quiver* $(\mathcal{G},\mathcal{D},\Omega)$ *with the admissible orientation* Ω *and the admissible sequence of sinks* $\{1, 2, \dots, n\}$.

Proof. According to the one-one correspondence between simple representations and indecomposable projective representations via modulo the latter radical in $rep(\mathcal{M},\Omega)$ and the above fact all $\mathcal{P}_u^{(i)}$ are indecomposable representations, it suffices to prove all $\mathcal{P}_u^{(1)}$ are projective, for this implies these indecomposable representations are, indeed, all non-isomorphic indecomposable projective ones.

We use induction u. First, for $u = 1$, $\mathcal{P}_1^{(1)}$ is just the unique simple direct summand under isomorphism of $\underline{A}_1$ which is projective by Corollary 3.1. Next, assume that for all $l < u$, $\mathcal{P}_l^{(1)}$ is projective for its corresponding admissible orientation of the graph. Then, in particular, $\widetilde{\mathcal{P}}_u^{(1)} = \Delta_2^- \dots \Delta_{u-1}^-\underline{W}_u^{(1)}$ is projective. We have $\mathcal{P}_u^{(1)} = \Delta_1^-\widetilde{\mathcal{P}}_u^{(1)}$.

Firstly, since $\widetilde{\mathcal{P}}_u^{(1)}$ is indecomposable, we have

$$\Delta_1^-\Delta_1^+\mathcal{P}_u^{(1)} = \Delta_1^-(\Delta_1^+\Delta_1^-\widetilde{\mathcal{P}}_u^{(1)}) = \Delta_1^-\widetilde{\mathcal{P}}_u^{(1)} = \mathcal{P}_u^{(1)} \tag{8}$$

$$\begin{tikzcd} & \mathcal{P}_u^{(1)} \arrow[d, "\beta"] & \\ \mathcal{X} \arrow[r, "\alpha"] & \mathcal{Y} \arrow[r] & 0 \end{tikzcd}$$

by Lemma 3.3, which means $\mathcal{P}_u^{(1)}$ is indecomposable. In order to prove the projectivity of $\mathcal{P}_u^{(1)}$, consider the diagram
whose row is exact. We show that it may be assumed that such a diagram is in the category $rep^{(1)}(\mathcal{M}, \Omega)$ in Theorem 3.2.

Indeed, by Lemma 3.3, we have $\mathcal{X} \cong \Delta_1^- \Delta_1^+ \mathcal{X} \oplus \mathcal{P}$, where $\mathcal{P} = (P_i, {}_i\pi_j)$ with $P_i = 0$ if $i \neq 1$ and P_1 is a (semisimple) A_1-module. We claim that

$$\alpha(\Delta_1^- \Delta_1^+ \mathcal{X}) = \Delta_1^- \Delta_1^+ \mathcal{Y}.$$

In fact, clearly $\Delta_1^- \Delta_1^+ \mathcal{X} \in rep^{(1)}(\mathcal{M}, \Omega)$, then $\alpha(\Delta_1^- \Delta_1^+ \mathcal{X}) \subseteq \Delta_1^- \Delta_1^+ \mathcal{Y}$. If this inclusion is proper, the fact that α is an epimorphism implies that some copy of the unique simple direct summand S_1 of $\underline{A}_1$ lies in $\Delta_1^- \Delta_1^+ \mathcal{Y}$. It is a contradiction.

Also, $\beta(\mathcal{P}_u^{(1)}) \subseteq \Delta_1^- \Delta_1^+ \mathcal{Y}$. Otherwise, there would exist a non-zero map $\mathcal{P}_u^{(1)} \to S_1$. This map must be an epimorphism since S_1 is simple and thus S_1 is a direct summand of $\mathcal{P}_u^{(1)}$ since S_1 is projective by Corollary 3.1. But, due to (8), $\mathcal{P}_u^{(1)}$ is indecomposable and non-isomorphic to S_1. This is a contradiction.

Thus, without loss of generality, assume that the above diagram lies in the category $rep^{(1)}(\mathcal{M}, \Omega)$ in Theorem 3.2. Then, applying Δ_1^+, we have $\Delta_1^+ \mathcal{P}_u^{(1)} = \Delta_1^+ \Delta_1^- \widetilde{\mathcal{P}}_u^{(1)} \cong \widetilde{\mathcal{P}}_u^{(1)}$ and get the following diagram:

$$\begin{tikzcd} & \widetilde{\mathcal{P}}_u^{(1)} \arrow[dl, "\gamma^+"'] \arrow[d, "\Delta_1^+\beta"] & \\ \Delta_1^+ \mathcal{X} \arrow[r, "\Delta_1^+\alpha"] & \Delta_1^+ \mathcal{Y} \arrow[r] & 0 \end{tikzcd}$$

where γ^+ exists by the projectivity of $\widetilde{\mathcal{P}}_u^{(1)}$ which makes this diagram to be commutative.

By Theorem 3.2, Δ_1^- and Δ_1^+ are mutual invertible between $rep_{(1)}(M, \Omega)$ and $rep^{(1)}(M, \Omega)$. So, $\Delta_1^- \Delta_1^+ \mathcal{X} \cong \mathcal{X}$, $\Delta_1^- \Delta_1^+ \mathcal{Y} \cong \mathcal{Y}$, $\Delta_1^- \Delta_1^+ \alpha \cong \alpha$, $\Delta_1^- \Delta_1^+ \beta \cong \beta$. But, $\mathcal{P}_u^{(1)} = \Delta_1^- \widetilde{\mathcal{P}}_u^{(1)}$. Thus, we get the following commutative diagram:
which means the projectivity of $\mathcal{P}_u^{(1)}$.

The statement on $\{\mathcal{Q}_u^{(i)}\}_{1 \leq i \leq s_u;\, 1 \leq u \leq n}$ can be shown in dual, according to the one-one correspondence between simple representations and indecomposable

$$\begin{tikzcd} & \mathcal{P}_u^{(1)} \arrow[dl, "\Delta_1^-\gamma^+"'] \arrow[d, "\beta"] & \\ \mathcal{X} \arrow[r, "\alpha"] & \mathcal{Y} \arrow[r] & 0 \end{tikzcd}$$

injective representations via the frontal, as the socles, are embedded into the latter in $rep(\mathcal{M}, \Omega)$. $\square$

According to Theorem 3.3 and the mutual constructions between a normal generalized path algebra and the corresponding pre-modulation in Section 2, we can give all indecomposable projective or injective representations of a normal generalized path algebra as follows:

Corollary 3.5. *Let $k(Q, \mathcal{A})$ be a normal $\mathcal{A}$-path algebra over a field k with connected acyclic quiver Q and the corresponding k-pre-modulation $\mathcal{M} = (A_i,\ {}_iM_j)$. Denote by $\{1, 2, \ldots, n\}$ the admissible sequence of sinks in Q and $\{\mathcal{P}_u^{(1)}\}_{1\leq u\leq n}$ (respectively, $\{\mathcal{Q}_u^{(1)}\}_{1\leq u\leq n}$) the set of all mutual non-isomorphic indecomposable projective (respectively, injective) representations in $rep(\mathcal{M}, \Omega)$ as in Theorem 3.3. Write $\mathcal{P}_u^{(1)} = (X_j^{(u)},\ {}_j\varphi_i)_{i,j\in Q_0}$ and $\mathcal{Q}_u^{(1)} = (Y_j^{(u)},\ {}_j\psi_i)_{i,j\in Q_0}$, let $\mathbf{P}_u = \sum_{j\in Q_0} X_j^{(u)}$ and $\mathbf{Q}_u = \sum_{j\in Q_0} Y_j^{(u)}$ for $u = 1, \ldots, n$. Then, in the category $modk(Q, \mathcal{A})$, under isomorphism, $\{\mathbf{P}_u\}_{1\leq u\leq n}$ (respectively, $\{\mathbf{Q}_u\}_{1\leq u\leq n}$) is the set of all indecomposable projective (respectively, injective) modules.*

We have known in [18] that if an artinian algebra A of Gabriel-type with admissible ideal is hereditary, then A is isomorphic to its related generalized path algebra $k(\Delta_A, \mathcal{A})$. Therefore, we can construct all indecomposable projective and injective modules over this kind of artinian hereditary algebras using of the method given in Corollary 3.5.

Remark 3.4. In [11], V.Dlab and C.M.Ringel generalize the Bernstein-Gelfand-Ponomarev theory in two directions. On one hand, they use valued graphs instesd of graphs, and show the relationship between the dimension vectors of indecomposable representations of elementary artinian algebras over skew-fields and the positive roots of the quadratic forms which is a bijection. On the other hand, they discuss the extended Dynkin diagrams and describe all there indecomposable representations. Note when the skew-fields are fields then the elementary artinian algebras are basic.

In our work, we use the natural quiver of a (non-basic) hereditary artinian algebra and the reformed modulations via generalized path algebras isomorphic to the hereditary algebras to construct all non-isomorphic indecomposable projective and injective representations of the generalized path algebras with acyclic quivers.

4. Representation-type of a generalized path algebra and its natural quiver

As one knows, according to Gabriel theory, representation type of a classical path algebra over an algebraically closed field or the modulation of a valued quiver is decided by the type of the quiver. Naturally, it is motivated to consider representation type of a generalized path algebra, equivalently, of a generalized modulation through the type of the corresponding natural quiver. First let us review the discussion given in [18].

We say a quiver to be of *almost Dynkin-affine type* provide that when one looks upon all arrows with same direction between an ordered pair of vertices as an arrow then the quiver becomes a quiver of either Dynkin or affine type; moreover, if it is of neither Dynkin nor affine type, we call this *proper almost Dynkin-affine type*. Respectively, we can give the definitions of (*proper*) *almost Dynkin type* and (*proper*) *almost affine type*.

By the classical Gabriel theory, if A is a hereditary k-spitting artinian algebra, A is of finite type if and only if Γ_A is of Dynkin type, A is of tame type if and only if Γ_A is of affine type. About the natural quiver Δ_A, it firstly was given that:

Proposition 4.1 ([18]). *For a hereditary k-splitting artinian algebra A, let m_{ij} be the number of arrows from a vertex i to another vertex j in the Ext-quiver Γ_A of A. Then, the natural quiver $\Delta_A = \Gamma_A$ if $m_{ij} \leq 1$ for any $i, j \in \Gamma_A$. Moreover, if A is of either finite type or tame type, then its natural quiver Δ_A is of either Dynkin type or affine type respectively.*

By Drozd's tame-and-wild Theorem, a finite-dimensional algebra A over an algebraically closed field k, which is not of finite type, is of either tame type or wild type. Then, the following holds:

Corollary 4.1 ([18]). *A finite-dimensional hereditary algebra A over an algebraically closed field k is of wild type if its natural quiver Δ_A is of neither Dynkin type nor affine type.*

The converse result is not true, that is, when A is of wild type, Δ_A is also possible to be of either Dynkin type or affine type.

Motivated by this discussion, it is asked how to characterize the kind of finite-dimensional (more generally, artinian) hereditary algebras of wild type whose natural quivers are of either Dynkin type or affine type?

As a part of this question, a class of wild algebras whose natural quivers are of either Dynkin type or affine type was constructed as in the following:

Proposition 4.2 ([18]). *For a normal generalized path algebra $k(Q, \mathcal{A})$ over an algebraically closed field k with Q a finite acyclic quiver, let $\mathcal{A} = \{A_i : i \in Q_0\}$ and $n_i = \sqrt{dim_k A_i}$ for any $i \in Q_0$.*

(*i*) *If there is an arrow from i to j in Q with $n_i n_j > 1$, then $k(Q, \mathcal{A})$ is of wild type;*

(ii) If the quiver Q is of either Dynkin or affine type and there is an arrow from i to j in Q with $n_in_j > 1$, then the Ext-quiver of $k(Q,\mathcal{A})$ is of proper almost Dynkin-affine type.

Theorem 4.1. *For a normal generalized path algebra $k(Q,\mathcal{A})$ over an algebraically closed field k with Q a finite connected acyclic quiver, let $\mathcal{A} = \{A_i : i \in Q_0\}$ and $n_i = \sqrt{dim_k A_i}$ for any $i \in Q_0$. If Q is of Dynkin type (resp. affine type), then*

(i) $k(Q,\mathcal{A})$ is of finite type (resp. tame type) if and only if $A_i \cong k$ for each vertex $i \in Q_0$, or equivalently say, $k(Q,\mathcal{A}) \cong kQ$;

(ii) in the otherwise case, $k(Q,\mathcal{A})$ is of wild type.

Proof. (i) "if": It is trivial according to the classical Gabriel theory.

"only if": As we have known in [17, 18], Q is just the natural quiver of $k(Q,\mathcal{A})$. Let Γ denote the Ext-quiver of $k(Q,\mathcal{A})$. Then, the relation is given in [21, 18] that $g_{ij} = n_in_jt_{ij}$ for the numbers g_{ij} and t_{ij} arrows from i to j in Γ and Q respectively.

Suppose there is one $p \in Q_0$ such that $A_p \ncong k$, that is, $n_p > 1$. Since Q is connected, p is either a head or a tail of some arrow in Q. No loss of generality, let p be the head of an arrow $\alpha : p \to q$ in Q. Then, $g_{pq} = n_pn_qt_{pq} > 1$ due to $n_p > 1$. Thus, Γ is neither of Dynkin type nor of affine type. By Gabriel theory, $k(Q,\mathcal{A})$ is neither of finite type nor of tame type.

(ii): It follows from the proof of "only if" above and Drozd's tame-and-wild Theorem. □

In the case of basic hereditary algebras, Gabriel's theorem tell us the hereditary algebra KQ is representation-finite if and only if the underlying graph of Q is one of the Dynkin diagrams. Theorem 4.1 discusses the representation type of normal generalized path algebra $k(Q,A)$, where Q is Dynkin quiver. It shows that a normal generalized path algebra $k(Q,A)$ to be representation-finite type in the case the quiver is of Dynkin type if and only if all algebras at the vertices are isomorphic to fields. As analogue for affine type, we also discuss the condition for a generalized path algebra to be of tame type in the case the quiver is of affine type.

It is easy to see that in the case of Theorem 4.1 (ii), the Ext-quiver of $k(Q,\mathcal{A})$ is certainly of proper almost Dynkin-affine type.

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