Hermite-Hadamard inequality for preinvex functions

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Abstract. We derive integral inequalities of Hermite-Hadamard type for the functions that have preinvex absolute values of third order derivatives. Moreover, we also discuss applications to several special means.

Keywords: Hermite-Hadamard inequality, invex set, preinvex function, integral inequality.

1. Introduction

For convex functions, several inequalities have been studied by many authors, see [1], [2]-[9]. But the inequality obtained by Hadamard [8] is considered the most significant and rich in applications. Let $g : \mathcal{I} \subseteq R \to R$ be a convex

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function on the interval $\mathcal{I}$. The inequality in [8] is given by

$$g\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} g(u)du \leq \frac{g(\alpha)+g(\beta)}{2}, \quad \alpha,\beta \in \mathcal{I} \text{ and } \alpha<\beta. \tag{1}$$

As mentioned in [8]: "inequality (1) is known as the Hermite-Hadamard (H-H) inequality for convex functions". The inequalities will be reversed for a concave function. Hadamard inequality refines the concept of convexity and various classical inequalities can be derived from it.
Recently, several extensions, refinements and generalizations have been discussed by the many authors, see [2, 7, 9, 16, 18]. Dragomir et. al. [5] proved the following lemma for the class of convex functions.

Lemma 1.1 ([5]). *Suppose that $g : \mathcal{I}^o \subseteq R \to R$ be a differentiable mapping on $\mathcal{I}^o$, $\alpha, \beta \in \mathcal{I}^o$, such that $\alpha < \beta$. If $g' \in L[\alpha, \beta]$, then*

$$\frac{g(\alpha)+g(\beta)}{2} - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} g(u)du = \frac{\beta-\alpha}{2}\int_{0}^{1}(1-2t)g'(t\alpha+(1-t)\beta)dt. \tag{2}$$

Hanson [10] introduced the concept of invexity which is a significant generalization of covexity. The concept of preinvex functions was introduced by Weir and Mond [17], later Jeyakumar et. al. [13] investigated some properties of these functions. They [13] also studied the role of preinvex functions in optimization and mathematical programming. Yuan et. al. [18] investigated some new characterizations of preinvex and prequasi-invex function under some assumptions. Noor [14] derived H-H inequality for preinvex and log-preinvex functions, later Iqbal et. al. [11] investigated some refined integral inequalities and discussed its applications to special means.

The objective of this work is to formulate some new refined inequalities of H-H type for the functions that have preinvex absolute values of third derivatives. We have considered various special means to show its applications. Our findings extend the previously known results.

2. Preliminaries

The following definitions and known result will be used in the sequel.

Definition 2.1 ([10]). *A set $X \subseteq R^n$ is said to be invex with respect to $\eta : X \times X \to R^n$ if*

$$v + t\eta(u,v) \in X, \forall\ u,v \in X\ \&\ t \in [0,1]. \tag{3}$$

As discussed in [10], "the definition says that there is a path starting from v which is contained in X. It is not necessary that u should be one of the end points of the path. However, if we require that u be an end point of the path for every pair $u, v \in X$, then $\eta(u,v) = u - v$, reduces to convexity."

Define

$$P_{uy} := \{w : w = u + t\eta(v,u) : t \in [0,1]\}.$$

It represents the η-path joining the points u and $y := u + \eta(v,u)$ for every $u, v \in X$.

Definition 2.2 ([17]). *Let $X \subseteq R^n$ be an invex set with respect to $\eta : X \times X \to R^n$. Then, the function $g : X \to R$ is called preinvex with respect to η, if*

$$g(v + t\eta(u,v)) \leq tg(u) + (1-t)g(v), \ \forall\ u, v \in X \ \&\ t \in [0,1]. \tag{4}$$

Preinvex function is the generalized class of convex functions. The function $f(u) = - \mid u \mid$ is preinvex with respect to η, where

$$\eta(u,v) := \begin{cases} u - v, & \text{if } u \leq 0, v \leq 0 \text{ and } u \geq 0, v \geq 0, \\ v - u, & \text{otherwise.} \end{cases}$$

But it is not convex. Recently, Barani et. al. [1] extended the Lemma 1.1 for invex sets as follows:

Lemma 2.1 ([1]). *Suppose that $A \subseteq R$ be an open invex subset with respect to $\eta : A \times A \to R$ and $\alpha, \beta \in A$ with $\eta(\alpha,\beta) \neq 0$ and that $g : A \to R$ be differentiable function. If g' is integrable on the η path $P_{\beta\gamma}$, $\gamma = \beta + \eta(\alpha,\beta)$, then*

$$\begin{aligned} &- \frac{g(\beta) + g(\beta + \eta(\alpha,\beta))}{2} + \frac{1}{\eta(\alpha,\beta)} \int_{\beta}^{\beta+\eta(\alpha,\beta)} g(u)du \\ &= \frac{\eta(\alpha,\beta)}{2} \int_0^1 (1-2t) g'(\beta + t\eta(\alpha,\beta))dt. \end{aligned}$$

Using Lemma 2.1, Barani et. al. [1] established H-H type inequalities for preinvex functions.

3. Main results

We now extend the previous known results for the functions whose third derivatives absolute values are preinvex. Consider the function $\eta : A \times A \to R$ with $\eta(\alpha,\beta) \neq 0$, for $\alpha, \beta \in A$. Henceforth, we assume that $A \subseteq R$ is an open invex set with respect to η.

Lemma 3.1. *Let $g : A \to R$ be three times differentiable function and g''' is integrable on the η-path $P_{\beta\gamma}$, $\gamma = \beta + \eta(\alpha,\beta)$, then*

$$\frac{\eta(\alpha,\beta)}{12}[g'(\beta+\eta(\alpha,\beta)) - g'(\beta)] - \frac{1}{2}[g(\beta+\eta(\alpha,\beta)) + g(\beta)] + \frac{1}{\eta(\alpha,\beta)} \int_{\beta}^{\beta+\eta(\alpha,\beta)} g(u)du$$

$$= \frac{\eta(\alpha,\beta)^3}{12} \int_0^1 t(1-t)(2t-1) g'''(\beta + t\eta(\alpha,\beta))dt. \tag{5}$$

Proof. Let $\alpha, \beta \in A$. Since A is an invex set with respect to η, $\beta + t\eta(\alpha,\beta) \in A$ for every $t \in [0,1]$. Integrating by parts, we get

$$\begin{aligned}
&\int_0^1 t(1-t)(2t-1)g'''(b+t\eta(\alpha,\beta))dt \\
&= \left[\frac{t(1-t)(2t-1)g''(\beta+t\eta(\alpha,\beta))}{\eta(\alpha,\beta)}\right]_0^1 \\
&- \frac{1}{\eta(\alpha,\beta)}\int_0^1(-6t^2+6t-1)g''(\beta+t\eta(\alpha,\beta))dt \\
&= \frac{1}{\eta(\alpha,\beta)}\left[\frac{(6t^2-6t+1)g'(\beta+t\eta(\alpha,\beta))}{\eta(\alpha,\beta)}\right]_0^1 \\
&- \frac{1}{\eta(\alpha,\beta)^2}\int_0^1(12t-6)g'(\beta+t\eta(\alpha,\beta))dt \\
&= \frac{1}{\eta(\alpha,\beta)^2}[g'(\beta+\eta(\alpha,\beta))-g'(\beta)] - \frac{6}{\eta(\alpha,\beta)^3}[g(\beta+\eta(\alpha,\beta))+g(\beta)] \\
&+ \frac{12}{(\eta(\alpha,\beta)^4}\int_\beta^{\beta+\eta(\alpha,\beta)} g(u)du. \qquad \square
\end{aligned}$$

Using above lemma, we prove some interesting results for the preinvex functions.

Theorem 3.1. *Let $g : A \to R$ be three times differentiable function and g''' is integrable on the η-path $P_{\beta\gamma}$, $\gamma = \beta + \eta(\alpha,\beta)$. If $|g'''|$ is preinvex on A, then*

$$\begin{aligned}
&\left|\frac{\eta(\alpha,\beta)}{12}[g'(\beta+\eta(\alpha,\beta))-g'(\beta)]\right. \\
&\left.-\frac{1}{2}[g(\beta+\eta(\alpha,\beta))+g(\beta)] + \frac{1}{\eta(\alpha,\beta)}\int_\beta^{\beta+\eta(\alpha,\beta)} g(u)du\right| \\
&\leq \frac{|\eta(\alpha,\beta)|^3}{384}\left[\frac{25}{2}|g'''(\beta)| - |g'''(\alpha)|\right]
\end{aligned}$$

Proof. Applying Lemma 3.1 and using the preinvexity of $|g'''|$, we get

$$\begin{aligned}
&\left|\frac{\eta(\alpha,\beta)}{12}[g'(\beta+\eta(\alpha,\beta))-g'(\beta)] - \frac{1}{2}[g(\beta+\eta(\alpha,\beta))+g(\beta)]\right. \\
&\left.+\frac{1}{\eta(\alpha,\beta)}\int_\beta^{\beta+\eta(\alpha,\beta)} g(u)du\right| \\
&\leq \frac{|\eta(\alpha,\beta)|^3}{12}\int_0^1 t(1-t)|(2t-1)||g'''(\beta+t\eta(\alpha,\beta))|dt \\
&\leq \frac{|\eta(\alpha,\beta)|^3}{12}\,[\int_0^1 t(1-t)|(2t-1)|(t|g'''(\alpha)|+(1-t)|g'''(\beta)|)dt
\end{aligned}$$

$$= \frac{|\eta(\alpha,\beta)|^3}{12} \left[|g'''(\alpha)| \int_0^1 t^2(1-t)|(2t-1)|dt + |g'''(\beta)| \int_0^1 t(1-t)^2|(2t-1)|dt\right.$$

$$= \frac{|\eta(\alpha,\beta)|^3}{384} \left[\frac{25}{2}|g''(\beta)| - |g''(\alpha)|\right]. \qquad \square$$

Theorem 3.2. *Let $g : A \to R$ be three times differentiable function and g''' be integrable on the η-path $P_{\beta\gamma}$, $\gamma = \beta + \eta(\alpha,\beta)$. If $|g'''|^{p/p-1}$ is preinvex on A for $p > 1$, then*

$$\left|\frac{\eta(\alpha,\beta)}{12}[g'(\beta+\eta(\alpha,\beta)) - g'(\beta)] - \frac{1}{2}[g(\beta+\eta(\alpha,\beta)) + g(\beta)] + \frac{1}{\eta(\alpha,\beta)} \int_\beta^{\beta+\eta(\alpha,\beta)} g(u)du\right| \leq \frac{|\eta(\alpha,\beta)|^3}{96} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(|g'''(\alpha)|^q + |g'''(\beta)|^q\right)^{\frac{1}{q}}.$$

Proof. Using Lemma 3.1, preinvexity of $|g'''|^{p/p-1}$ and Holder's integral inequality, we obtain

$$\begin{aligned}
&\left|\frac{\eta(\alpha,\beta)}{12}[g'(\beta+\eta(\alpha,\beta)) - g'(\beta)] - \frac{1}{2}[f(\beta+\eta(\alpha,\beta)) + g(\beta)] + \frac{1}{\eta(\alpha,\beta)} \int_\beta^{\beta+\eta(\alpha,\beta)} g(u)du\right| \\
&\leq \frac{|\eta(\alpha,\beta)|^3}{12} \int_0^1 t(1-t)|(2t-1)||g'''(\beta+t\eta(\alpha,\beta))|dt \\
&\leq \frac{|\eta(\alpha,\beta)|^3}{12} \left(\int_0^1 t^p(1-t)^p|(2t-1)|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 |g'''(\beta+t\eta(\alpha,\beta))|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{|\eta(\alpha,\beta)|^3}{12} \left(\frac{1}{2^{2p+1}(p+1)}\right)^{\frac{1}{p}} \left(\int_0^1 t|g'''(\alpha)|^q + (1-t)|g'''(\beta)|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{|\eta(\alpha,\beta)|^3}{96} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(|g'''(\alpha)|^q + |g'''(\beta)|^q\right)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. $\square$

Theorem 3.3. *Let $g : A \to R$ be three times differentiable function and g''' be integrable on the η-path $P_{\beta\gamma}$, $\gamma = \beta + \eta(\alpha,\beta)$. If $|g'''|^q$ is preinvex on A for $q > 1$, then*

$$\left|\frac{\eta(\alpha,\beta)}{12}[g'(\beta+\eta(\alpha,\beta)) - g'(\beta)] - \frac{1}{2}[g(\beta+\eta(\alpha,\beta))\right.$$

$$+g(\beta)] + \frac{1}{\eta(\alpha,\beta)}\int_{\beta}^{\beta+\eta(\alpha,\beta)} g(u)du\Bigg|$$
$$\leq \frac{|\eta(\alpha,\beta)|^3}{192}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{25}{2}|g'''(\beta)|^q - |g'''(\alpha)|^q\right)^{\frac{1}{q}}.$$

Proof. Since $|g'''|^q$ is preinvex, using Lemma 3.1 and power-mean inequality, we obtain

$$\Bigg|\frac{\eta(\alpha,\beta)}{12}[g'(\beta+\eta(\alpha,\beta)) - g'(\beta)] - \frac{1}{2}[f(\beta+\eta(\alpha,\beta))$$
$$+g(\beta)] + \frac{1}{\eta(\alpha,\beta)}\int_{\beta}^{\beta+\eta(\alpha,\beta)} g(u)du\Bigg|$$
$$\leq \frac{|\eta(\alpha,\beta)|^3}{12}\int_0^1 t(1-t)|(2t-1)||g'''(\beta+t\eta(\alpha,\beta))|dt$$
$$\leq \frac{|\eta(\alpha,\beta)|^3}{12}\left(\int_0^1 t(1-t)|(2t-1)|dt\right)^{1-\frac{1}{q}}$$
$$\cdot\left(\int_0^1 t(1-t)|(2t-1)||g'''(\beta+t\eta(\alpha,\beta))|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{|\eta(\alpha,\beta)|^3}{12}\left(\frac{1}{16}\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)|(2t-1)|[t|g'''(\alpha)|^q + (1-t)|g'''(\beta)|^q]dt\right)^{\frac{1}{q}}$$
$$\leq \frac{|\eta(\alpha,\beta)|^3}{12}\left(\frac{1}{16}\right)^{1-\frac{1}{q}}\Bigg(|g''(\alpha)|^q\int_0^1 t^2(1-t)|(2t-1)|dt$$
$$+|g''(\beta)|^q\int_0^1 t(1-t)^2|(2t-1)|dt\Bigg)^{\frac{1}{q}}$$
$$\leq \frac{|\eta(\alpha,\beta)|^3}{12}\left(\frac{1}{16}\right)^{1-\frac{1}{q}}\left[|g'''(\alpha)|^q(-\frac{1}{32}) + |g'''(\beta)|^q(\frac{25}{64})\right]^{\frac{1}{q}}$$
$$= \frac{|\eta(\alpha,\beta)|^3}{192}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{25}{2}|g'''(\beta)|^q - |g'''(\alpha)|^q\right)^{\frac{1}{q}}. \qquad \square$$

4. Some applications

For distinct positive real numbers a_1 and a_2, we have:

Arithmetic mean: $A(a_1,a_2) = \dfrac{a_1+a_2}{2}$,

Logarithmic mean: $L_p(a_1,a_2) = \dfrac{a_1-a_2}{\ln a_1 - \ln a_2}$, and

generalized logarithmic mean: $L_p(a_1,a_2) = \left[\dfrac{a_1^{p+1}-a_2^{p+1}}{(p+1)(a_1-a_2)}\right]^{1/p}, p \neq -1, 0.$

Let us suppose that

$$g(u) = \frac{u^{n+3}}{(n+1)(n+2)(n+3)}$$

be a function and $a_3 = a_2 + \eta(a_1, a_2)$, then

$$\frac{g(a_3)+g(a_2)}{2} = \frac{1}{(n+1)(n+2)(n+3)} A(a_3^{n+3}, a_2^{n+3}),$$

$$\frac{1}{\eta(a_1,a_2)} \int_{a_2}^{a_3} g(u)du = \frac{1}{\eta(a_1,a_2)} \frac{1}{(n+1)(n+2)(n+3)} \left[\frac{a_3^{n+4} - a_2^{n+4}}{n+4}\right].$$

For $\eta(a_1, a_2) = a_1 - a_2$, it becomes

$$\frac{1}{a_1 - a_2} \int_{a_2}^{a_1} g(u)du = \frac{1}{(n+1)(n+2)(n+3)} L_{n+3}^{n+3}(a_1, a_2),$$

$$g'(a_2 + \eta(a_1,a_2) - g'(a_2) = \frac{(a_2 + \eta(a_1,a_2))^{n+2} - a_2^{n+2}}{(n+1)(n+2)}.$$

For $\eta(a_1, a_2) = a_1 - a_2$, it becomes

$$g'(a_1) - g'(a_2) = \frac{(a_1 - a_2)}{(n+1)} L_{n+1}^{n+1}(a_1^{n+2}, a_2^{n+2}).$$

Now, using the results of section 3, we discuss some applications to special means of real numbers.

Proposition 4.1. *For positive numbers a_1 and a_2 such that $a_1 > a_2$ and $0 < n \leq 1$, we have*

$$\begin{aligned}&\big|(a_1 - a_2)^2 (n+2)(n+3) L_{n+1}^{n+1}(a_1^{n+2}, a_2^{n+2}) \\ &-12A(a_1^{n+3}, a_2^{n+3}) + 12 L_{n+3}^{n+3}(a_1^{n+4}, a_2^{n+4})\big| \\ &\leq \frac{|(a_1 - a_2)|^3}{32} (n+1)(n+2)(n+3) \left[\frac{25}{2}|a_2^n| - |a_1^n|\right].\end{aligned}$$

Proof. The statement takes after from Theorem 3.3 connected to the function

$$g(u) = \frac{u^{n+3}}{(n+1)(n+2)(n+3)},$$

for $\eta(a_1, a_2) = a_1 - a_2$. □

Proposition 4.2. *For positive numbers a_1 and a_2 such that $a_1 > a_2$ and $0 < n \leq 1$, we have*

$$\begin{aligned}&\big|(a_1 - a_2)^2 (n+2)(n+3) L_{n+1}^{n+1}(a_1^{n+2}, a_2^{n+2}) - 12A(a_1^{n+3}, a_2^{n+3}) + 12 L_{n+3}^{n+3}(a_1^{n+4}, a_2^{n+4})\big| \\ &\leq \frac{|(a_1 - a_2)|^3}{8} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} (n+1)(n+2)(n+3)(|a_1^n|^q + |a_2^n|^q)^{\frac{1}{q}}.\end{aligned}$$

Proof. The statement takes after from Theorem 3.3 connected to the function

$$g(u) = \frac{u^{n+3}}{(n+1)(n+2)(n+3)},$$

for $\eta(a_1, a_2) = a_1 - a_2$. □

Proposition 4.3. *For positive numbers a_1 and a_2 such that $a_1 > a_2$, $0 < n \leq 1$ and $q > 1$, we have*

$$\left|(a_1-a_2)^2(n+2)(n+3)L_{n+1}^{n+1}(a_1^{n+2}, a_2^{n+2})-12A(a_1^{n+3}, a_2^{n+3})+12L_{n+3}^{n+3}(a_1^{n+4}, a_2^{n+4})\right|$$
$$\leq \frac{|(a_1-a_2)|^3}{16}\left(\frac{1}{2}\right)^q (n+1)(n+2)(n+3)\left[\frac{25}{2}|a_2^n|^q - |a_1^n|^q\right]^{\frac{1}{q}}.$$

Proof. The statement takes after from Theorem 3.3 connected to the function

$$g(u) = \frac{u^{n+3}}{(n+1)(n+2)(n+3)},$$

for $\eta(a_1, a_2) = a_1 - a_2$. □

5. Conclusion

In this paper, we have extended the estimates of right hand side of Hermite-Hadamard type inequality for the functions having pre-invex third derivative absolute values. To show its application, we have considered several special means for arbitrary real numbers. In the future, the results can be generalized for higher order derivatives. Moreover, it can be studied in the context of q-calculus, and various applications can be explored.

References

[1] A. Barani, A.G. Ghazanfari, S.S. Dragomir, *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex,* J. Inequal. Appl., 247 (2012).

[2] M. Bombardelli, S. Varosanec, *Properties of h-convex functions related to the Hermite-Hadamard-Fejer inequalities,* Comput. Math. Appl., 58 (2010), 1869-1877.

[3] S.S. Dragomir, *Two mappings in connection to Hadamard's inequalities,* J. Math. Anal. Appl., 167 (1992), 49-56.

[4] S.S. Dragomir, *On Hadamards inequalities for convex functions,* Math. Balk., 6 (1992), 215-222.

[5] S.S. Dragomir, R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula,* Appl. Math. Lett., 11 (1998), 91-95.

[6] S.S. Dragomir, C.E.M. Pearce, *Quasi-convex functions and Hadamard's inequality,* Bull. Austr. Math. Soc., 57 (1998), 377-385.

[7] S.S. Dragomir, Y.J. Cho, S.S. Kim, *Inequalities of Hadamard's type for Lipschitzian mappings and their applications,* J. Math. Anal. Appl., 245 (2000), 489-501.

[8] J. Hadamard, *Tude sur les proprietes des fonctions entieres en particulier dune fonction considre par Riemann,* J. Math. Pures Appl., 58 (1893), 171-215.

[9] N. Hadjisavvas, *Hadamard-type inequalities for quasi-convex functions,* J. Ineq. Pure Appl. Math., 4 (2003), article 13.

[10] M.A. Hanson, *On sufficiency of the Kuhn-Tucker conditions,* J. Math. Anal. Appl., 80 (1981), 545-550.

[11] A. Iqbal, V. Samhita, *Some integral inequalities for log-preinvex functions. Applied analysis in biological and physical sciences,* Springer Proceedings in Mathematics & Statistics, Springer, 2016, 373-384.

[12] D.A. Ion, *Some estimates on the Hermite-Hadamard inequality through quasi-convex functions,* Annals of University of Craiova. Math. Comp. Sci. Ser., 34 (2007), 19-22.

[13] V. Jeyakumar, *Strong and weak invexity in mathematical programming,* Meth. Oper. Res., 55 (1985), 109-125.

[14] M.A. Noor, *Hermite-Hadamard integral inequality for log-preinvex functions,* J. Math. Anal. Approx. Theory., 2 (2007), 126-131.

[15] P.M. Pardalos, P.G. Georgiev, H.M. Srivastava, (Eds), *Nonlinear analysis: stabiliy, approximation and inequalities,* Springer, 2012.

[16] S. Qaisar, S. Hussain, H. Chuanjiang, *On new inequalities of Hermite-Hadamard type for functions whose third derivatives absolute values are quasi-convex with applications,* J. Egyptian Math. Soc., 22 (2014), 137-146.

[17] T. Weir, B. Mond, *Preinvex functions in multiobjective optimization,* J. Math. Anal. Appl., 136 (1988), 29-38.

[18] G.S. Yang, D.Y. Hwang, K.L. Tseng, *Some inequalities for differentiable convex and concave mappings,* Comput. Math. Appl., 47 (2004), 207-216.

[19] D.H. Yuan, X. Liu, G. Lai, *Note on generalized invex functions,* Optim Lett., 7 (2013), 617-623.

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