Schatten class weighted composition operators on weighted Hilbert Bergman spaces of bounded strongly pseudoconvex domain

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Abstract. Let D be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$, $\delta(z) = d(z, \partial D)$ the Euclidean distance from the point z to the boundary ∂D and $H(D)$ the set of all holomorphic functions on D. For given $\beta \in \mathbb{R}$, the weighted Hilbert Bergman space on D, denoted by $A^2(D, \beta)$, consists of all $f \in H(D)$ such that

$$\|f\|_{2,\beta} = \Big[\int_D |f(z)|^2 \delta(z)^\beta dv(z)\Big]^{\frac{1}{2}} < +\infty,$$

where dv is the Lebesgue measure on D. The aim of the paper is to completely characterize the Schatten class of weighted composition operators on $A^2(D, \beta)$ when $\delta(z)$ satisfies certain integrable condition.

Keywords: weighted composition operator, strongly pseudoconvex domain, weighted Hilbert Bergman space, Schatten class.

1. Introduction

Let Ω be a domain in $\mathbb{C}^n$ and $H(\Omega)$ the set of all holomorphic functions on Ω. Let φ be a holomorphic self-map of Ω and $u \in H(\Omega)$. The well-known weighted composition operator on some subspaces of $H(\Omega)$ is defined by

$$W_{\varphi,u}f(z) = u(z)f(\varphi(z)), \;\; z \in \Omega.$$

When $u(z) \equiv 1$, it is reduced to the composition operator, usually denoted by C_φ. While $\varphi(z) = z$, it is reduced to the multiplication operator, usually

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denoted by M_u. Weighted composition operators have been widely studied (see, for example, [4, 5, 8, 9, 10, 15, 16, 17] and the related references therein).

Let $D \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, $\delta(z) = d(z, \partial D)$ the Euclidean distance from the point z to the boundary ∂D and dv the Lebesgue measure on D. The authors in [2] introduced the following weighted Bergman space by considering the distance function $\delta(z)$ as a weight on D. For given $\beta \in \mathbb{R}$ and $p \in [1, +\infty)$, the weighted Bergman space $A^p(D, \beta)$ consists of all $f \in H(D)$ such that

$$\|f\|_{p,\beta} = \Big[\int_D |f(z)|^p \delta(z)^\beta dv(z)\Big]^{\frac{1}{p}} < +\infty.$$

With the norm $\|\cdot\|_{p,\beta}$, $A^p(D, \beta)$ becomes a Banach space. If $\beta = 0$, then $A^p(D, \beta)$ is abbreviated to $A^p(D)$, usually called the Bergman space. In this paper, we consider the case of $p = 2$. For this case, it is a Hilbert space with the inner product

$$\langle f, g\rangle_\beta = \int_D f(z)\overline{g(z)}\delta(z)^\beta dv(z).$$

For a given separable Hilbert space H, the Schatten p-class of operators on H, $S_p(H)$, consists of those compact operators T on H with its sequence of singular numbers λ_n belonging to ℓ^p, the p-summable sequence space. When $p = 1$, it is usually called the trace class, and $p = 2$ is usually called the Hilbert-Schmidt class (see [22]). The theory of Schatten p-class of operators on the holomorphic function spaces has been widely studied (see, for example, [18, 7, 19, 14, 23, 12, 13, 6, 20] and the references therein). In particular, the authors in [20] characterized the Schatten p-class of weighted composition operators on $A^2(D)$.

Motivated by previous mentioned studies (in especial [20]), it is natural to consider how to characterize the Schatten p-class of weighted composition operators on $A^2(D, \beta)$. After a long time of careful consideration, we find that if the parameter β satisfies the condition

$$\int_D K(z, z)\delta(z)^\beta dv(z) = +\infty,$$

then it is a difficult problem. However, if β satisfies the condition

$$\int_D K(z, z)\delta(z)^\beta dv(z) < +\infty,$$

we can completely characterize the Schatten p-class of weighted composition operators on $A^2(D, \beta)$ by borrowing the methods obtained in [2] and [21]. We hope that this paper can attract people's more attention to such problems.

Let $K(z, w) : D \times D \to \mathbb{C}$ be the Bergman kernel of D. For every $w \in D$, the normalized Bergman kernel of D, denoted by $k_w(z)$, is defined by

$$k_w(z) = \frac{K(z, w)}{\sqrt{K(w, w)}} = \frac{K(z, w)}{\|K(\cdot, w)\|_{2,\beta}}.$$

For μ a finite complex Borel measure on D, the Berezin transform $\tilde{\mu}(z)$ is defined by

$$\tilde{\mu}(z)=\int_D |k_z(w)|^2 d\mu(w).$$

Let $\beta(z,w)$ be the Kobayashi distance function on D. For $z\in D$ and $r\in(0,1)$, let

$$B(z,r)=\{w\in D:\beta(z,w)<r\}$$

denote the Kobayashi ball with center z and radius $\frac{1}{2}\ln\frac{1+r}{1-r}$. We define $v_\beta(B(z,r))$ by

$$v_\beta(B(z,r))=\int_{B(z,r)}\delta(w)^\beta dv(w).$$

The function $\hat{\mu}^r(z)$ on D is defined by

$$\hat{\mu}^r(z)=\frac{\mu(B(z,r))}{v_\beta(B(z,r))}.$$

For φ the holomorphic self-map of D and $u\in H(D)$, we define $dv_{2,\beta}(z)=|u(z)|^2\delta(z)^\beta dv(z)$ and $\mu_{2,\beta}=v_{2,\beta}\circ\varphi^{-1}$, respectively. In this paper, we will use the Berezin transform $\tilde{\mu}_{2,\beta}$ and the function $\hat{\mu}^r_{2,\beta}$ to characterize the Schatten p-class of weighted composition operators on $A^2(D,\beta)$.

In this paper, the positive constants are denoted by C which may differ from one occurrence to the next.

2. Preliminary results

In this section, we present some results from [1] on the Kobayashi geometry of bounded strongly pseudoconvex domain.

Lemma 2.1. *Let $D\subseteq\mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then, for $z_0\in D$ and $r\in(0,1)$, there exists a positive constant C independent of $z\in B(z_0,r)$ such that*

$$\frac{1-r}{C}\delta(z_0)\le\delta(z)\le\frac{C}{1-r}\delta(z_0).$$

Lemma 2.2. *Let $D\subseteq\mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then, for $\beta\in\mathbb{R}$ and $r\in(0,1)$, there exist two positive constants C_1 and C_2 such that*

$$C_1\delta(\cdot)^{n+1+\beta}\le v_\beta(B(\cdot,r))\le C_2\delta(\cdot)^{n+1+\beta}.$$

By using Lemma 2.1 and Lemma 2.2, we have the following result.

Corollary 2.1. *For $r,s,R\in(0,1)$, there exists a positive constant C independent of z_1,z_2 with $\beta(z_1,z_2)\le R$ such that*

$$C^{-1}\le\frac{v_\beta(B(z_1,r))}{v_\beta(B(z_2,s))}\le C.$$

We also need the following result on the Bergman kernel obtained in [1] and [11].

Lemma 2.3. *Let $D \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then, for $r \in (0,1)$, there exist positive constants C and δ such that, if $z_0 \in D$ satisfies $\delta(z_0) < \delta$, then*

$$\frac{C}{\delta(z_0)^{n+1}} \leq |K(z, z_0)| \leq \frac{1}{C\delta(z_0)^{n+1}}$$

and

$$\frac{C}{\delta(z_0)^{n+1}} \leq |k_{z_0}(z)|^2 \leq \frac{1}{C\delta(z_0)^{n+1}},$$

for all $z \in B(z_0, r)$.

From Lemmas 2.2 and 2.3, the following result follows.

Corollary 2.2. *Let $D \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then, for $r \in (0,1)$, there exist positive constants C and δ such that, if $z_0 \in D$ satisfies $\delta(z_0) < \delta$, then*

$$\frac{C}{v_\beta(B(z_0, r))} \leq |K(z, z_0)| \leq \frac{1}{Cv_\beta(B(z_0, r))}$$

and

$$\frac{C}{v_\beta(B(z_0, r))} \leq |k_{z_0}(z)|^2 \leq \frac{1}{Cv_\beta(B(z_0, r))},$$

for all $z \in B(z_0, r)$.

We also need the following cover of D (see [1]).

Lemma 2.4. *Let $D \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then, for $r \in (0,1)$, there exist an $m \in \mathbb{N}$ and a sequence $\{z_i\} \subseteq D$ such that $D = \bigcup_{i=1}^{\infty} B(z_i, r)$ and any point in D belongs to at most m balls of the form $B(z_i, R)$ where $R = \frac{1}{2}(1+r)$.*

3. Main results and proofs

First, we have the following result.

Lemma 3.1. *If $T \in S_1(A^2(D, \beta))$, then*

$$tr(T) = \int_D \langle TK(\cdot, z), K(\cdot, z)\rangle_\beta \delta(z)^\beta dv(z).$$

Proof. Let $\{e_j(z)\}$ be an orthonormal basis for $A^2(D,\beta)$. We have

$$K(z,w)=\sum_{j=1}^{\infty} e_j(z)\overline{e_j(w)}.$$

Then, from this it follows that

$$\begin{aligned}
\operatorname{tr}(T) &= \sum_{j=1}^{\infty} \langle Te_j, e_j\rangle_\beta = \sum_{j=1}^{\infty} \int_D Te_j(z)\overline{e_j(z)}\delta(z)^\beta dv(z)\\
&= \sum_{j=1}^{\infty} \int_D \langle Te_j, K(\cdot,z)\rangle_\beta \overline{e_j(z)}\delta(z)^\beta dv(z)\\
&= \sum_{j=1}^{\infty} \int_D \langle e_j, T^*K(\cdot,z)\rangle_\beta \overline{e_j(z)}\delta(z)^\beta dv(z)\\
&= \int_D \delta(z)^\beta \int_D \Big(\sum_{j=1}^{\infty} e_j(w)\overline{e_j(z)}\Big)\overline{T^*K(\cdot,z)(w)}\delta(w)^\beta dv(w)dv(z)\\
&= \int_D \delta(z)^\beta \int_D K(w,z)\overline{T^*K(\cdot,z)(w)}\delta(w)^\beta dv(w)dv(z)\\
&= \int_D \langle K(\cdot,z), T^*K(\cdot,z)\rangle_\beta \delta(z)^\beta dv(z) = \int_D \langle TK(\cdot,z), K(\cdot,z)\rangle_\beta \delta(z)^\beta dv(z).
\end{aligned}$$

From this, the desired result follows. This completes the proof. □

In the following result, we give an estimate for the finite positive Borel measure on D.

Lemma 3.2. *Let $D\subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, μ a finite positive Borel measure on D and $r\in(0,1)$. Then, there exists a positive constant C depending on r such that*

$$\mu(B(a,r)) \le \frac{C}{v_\beta(B(a,r))}\int_{B(a,r)} \mu(B(z,r))\delta(z)^\beta dv(z).$$

Proof. For any $a\in D$, we have

$$\begin{aligned}
&\int_{B(a,r)} \mu(B(z,r))\delta(z)^\beta dv(z) = \int_{B(a,r)} \delta(z)^\beta dv(z)\int_{B(z,r)} d\mu(w)\\
&= \int_{B(a,r)} \delta(z)^\beta dv(z)\int_D \chi_{B(z,r)}(w)d\mu(w) = \int_D d\mu(w)\int_{B(a,r)} \chi_{B(z,r)}(w)\delta(z)^\beta dv(z).
\end{aligned}$$

Noting that $\chi_{B(w,r)}(z)=\chi_{B(z,r)}(w)$, for all w and z in D, we have

$$\begin{aligned}
\int_{B(a,r)} \mu(B(z,r))\delta(z)^\beta dv(z) &= \int_D d\mu(w)\int_{B(a,r)} \chi_{B(w,r)}(z)\delta(z)^\beta dv(z)\\
&= \int_D v_\beta(B(a,r)\cap B(w,r))d\mu(w) \ge \int_{B(a,r)} v_\beta(B(a,r)\cap B(w,r))d\mu(w),
\end{aligned}$$

where $\chi_{B(w,r)}(z)$ is the characteristic function of the set $B(w,r)$. Let $\alpha(t)$ $(0 \leq t < 1)$ be the geodesic (in the Bergman metric) from a to w and $m_{(a,w)} = \alpha(\frac{1}{2})$. By using Lemma 3 in [21], we obtain

$$\int_{B(a,r)} \mu(B(z,r))\delta(z)^{\beta} dv(z) \geq \int_{B(a,r)} v_{\beta}\left(B(m_{(a,w)}, \frac{r}{2})\right) d\mu(w).$$

From Corollary 2.1, it follows that there exists a positive constant C depending only on r such that

$$C v_{\beta}\left(B\left(m_{(a,w)}, \frac{r}{2}\right)\right) \geq v_{\beta}(B(a,r)),$$

for all $w \in B(a,r)$. Therefore, we have

$$C \int_{B(a,r)} \mu(B(z,r))\delta(z)^{\beta} dv(z) \geq \int_{B(a,r)} v_{\beta}(B(a,r)) d\mu(w),$$

that is,

$$\mu(B(a,r)) \leq \frac{C}{v_{\beta}(B(a,r))} \int_{B(a,r)} \mu(B(z,r))\delta(z)^{\beta} dv(z).$$

This completes the proof. □

Corollary 3.1. *Let $D \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, μ a finite positive Borel measure on D and $r \in (0,1)$. Then, there exists a positive constant C depending on r such that*

$$[\mu_{2,\beta}(B(z_j,r))]^{\frac{p}{2}} \leq \frac{C}{v_{\beta}(B(z_j,r))} \int_{B(z_j,r)} [\mu_{2,\beta}(B(z_j,r))]^{\frac{p}{2}} \delta(z)^{\beta} dv(z).$$

Corollary 3.2. *Let $D \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, μ a finite positive Borel measure on D and $r \in (0,1)$. Then, for every $r, R \in (0,1)$, there exists a positive constant C depending on r and R such that*

$$\left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_{\beta}(B(z_j,r))}\right]^{\frac{p}{2}} \leq \frac{C}{v_{\beta}(B(z_j,r))} \int_{B(z_j,r)} \left[\frac{\mu_{2,\beta}(B(z,r))}{v_{\beta}(B(z,r))}\right]^{\frac{p}{2}} \delta(z)^{\beta} dv(z),$$

for all z_j, z with $\beta(z_j,z) \leq R$.

As an application of Corollary 3.2, we can introduce the following complex measure. For $p \in [2,+\infty)$, the complex measure $\mu_{2,\beta,\zeta}$ is defined by

$$\mu_{2,\beta,\zeta}(z) = \sum_{j=1}^{\infty} \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_{\beta}(B(z_j,r))}\right]^{\frac{p}{2}\zeta - 1} \chi_{B(z_j,r)}(z)\mu_{2,\beta}(z),$$

where ζ is a complex number with $0 \leq \mathrm{Re}\zeta \leq 1$ and $\chi_{B(z_j,r)}(z)$ is the characteristic function of the set $B(z_j,r)$.

Lemma 3.3. *Let $\zeta = \frac{2}{p}$. Then, it follows that*

$$T_{\mu_{2,\beta}} \leq T_{\mu_{2,\beta,\frac{2}{p}}} \leq m T_{\mu_{2,\beta}}.$$

Proof. Obviously, it follows that

$$\mu_{2,\beta,\frac{2}{p}}(z)=\sum_{j=1}^{\infty}\chi_{B(z_j,r)}(z)\mu_{2,\beta}(z)\geq\mu_{2,\beta}(z).$$

Then, we have

$$T_{\mu_{2,\beta,\frac{2}{p}}}f(z)=\int_D f(w)K(w,z)d\mu_{2,\beta,\frac{2}{p}}(w)\geq\int_D f(w)K(w,z)d\mu_{2,\beta}(w)=T_{\mu_{2,\beta}}f(z),$$

which shows $T_{\mu_{2,\beta,\frac{2}{p}}}\geq T_{\mu_{2,\beta}}$.

Conversely, it follows from Lemma 2.4 that $\mu_{2,\beta,\frac{2}{p}}(z)\leq m\mu_{2,\beta}(z)$. Similarly, we can get $T_{\mu_{2,\beta,\frac{2}{p}}}\leq mT_{\mu_{2,\beta}}$. This completes the proof. □

Lemma 3.4. *Let T_1, T_2 be two compact operators on Hilbert space H and $0\leq T_1\leq T_2$. Then*

$$\|T_1\|_{S_p(H)}\leq\|T_2\|_{S_p(H)}.$$

Proof. By Lemma 14 in [21], we have $s_j(T_1)\leq s_j(T_2)$ for $j\in\mathbb{N}$. Since

$$\|T\|_{S_p}=\left[\sum_{j=1}^{\infty}(s_j(T))^p\right]^{\frac{1}{p}},$$

we have

$$\|T_1\|_{S_p(H)}=\left[\sum_{j=1}^{\infty}(s_j(T_1))^p\right]^{\frac{1}{p}}\leq\left[\sum_{j=1}^{\infty}(s_j(T_2))^p\right]^{\frac{1}{p}}=\|T_2\|_{S_p(H)}.$$

This completes the proof. □

Now, we prove the main result of this paper. We assume that β satisfies the condition

$$\int_D K(z,z)\delta(z)^{\beta}dv(z)<+\infty. \tag{1}$$

Remark 3.1. We consider the condition (1) for the special case $D=\{z\in\mathbb{C}: |z|<1\}$, the open unit disk. For this case, we have (see, for example, [22])

$$K(z,w)=\frac{1}{(1-z\overline{w})^2}.$$

For the case, it is easy to see that $\delta(z)=1-|z|^2$. Then, we have

$$\int_{\mathbb{D}}K(z,z)\delta(z)^{\beta}dv(z)=\int_{\mathbb{D}}(1-|z|^2)^{\beta-2}dv(z)=2\pi\int_0^1(1-r^2)^{\beta-2}rdr. \tag{2}$$

From a direct calculation, it follows that (2) is finite if and only if $\beta \in (1, +\infty)$. This shows that Theorem 3.1 excludes the result of the Bergman space (that is, corresponding to $\beta = 0$). Maybe it is caused by the different definitions of the weights. For example, in [21] the author defined the weighted Bergman space on bounded symmetric domains by the weight $K(z,z)^{\lambda}$.

Theorem 3.1. *Let $D \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, $p \in [2, +\infty)$, φ a holomorphic self-map of D and $u \in H(D)$. Then, the following statements are equivalent:*

(*i*) $W_{\varphi,u} \in S_p(A^2(D,\beta))$;

(*ii*) $\tilde{\mu}_{2,\beta} \in L^{\frac{p}{2}}(D, K(z,z)\delta(z)^{\beta}dv(z))$;

(*iii*) $\hat{\mu}^r_{2,\beta} \in L^{\frac{p}{2}}(D, K(z,z)\delta(z)^{\beta}dv(z))$;

(*iv*) $\sum_{j=1}^{\infty} \left(\hat{\mu}^r_{2,\beta}(z_j)\right)^{\frac{p}{2}} < +\infty$, *where $\{z_j\}$ is the sequence in Lemma* 2.4.

Proof. For $f, g \in A^2(D,\beta)$, we have

$$\begin{aligned}\langle (W_{\varphi,u})^*(W_{\varphi,u})f, g\rangle_{\beta} &= \langle (W_{\varphi,u})f, (W_{\varphi,u})g\rangle_{\beta} = \int_D |u(z)|^2 f(\varphi(z))\overline{g(\varphi(z))}\delta(z)^{\beta}dv(z)\\ &= \int_D f(\varphi(z))\overline{g(\varphi(z))}dv_{2,\beta}(z) = \int_D f(w)\overline{g(w)}d\mu_{2,\beta}(w).\end{aligned}$$

Considering the Toeplitz operator on $A^2(D,\beta)$

$$T_{\mu_{2,\beta}}f(z) = \int_D f(w)K(w,z)d\mu_{2,\beta}(w),$$

we have

$$\begin{aligned}\langle T_{\mu_{2,\beta}}f, g\rangle_{\beta} &= \int_D \int_D f(w)K(w,z)d\mu_{2,\beta}(w)\overline{g(z)}\delta(z)^{\beta}dv(z)\\ &= \int_D f(w)\overline{\int_D K(z,w)g(z)\delta(z)^{\beta}dv(z)}d\mu_{2,\beta}(w)\\ &= \int_D f(w)\overline{g(w)}d\mu_{2,\beta}(w),\end{aligned}$$

which shows that

$$T_{\mu_{2,\beta}} = (W_{\varphi,u})^*(W_{\varphi,u}).$$

This implies that $T_{\mu_{2,\beta}}$ is a positive operator on $A^2(D,\beta)$.

$(i) \Rightarrow (ii)$. From Theorem 1.4.6 in [22], we know that $W_{\varphi,u} \in S_p(A^2(D,\beta))$ if and only if $T_{\mu_{2,\beta}} \in S_{\frac{p}{2}}(A^2(D,\beta))$. Since $T_{\mu_{2,\beta}}$ is positive, by using Lemma 3.1, we have

$$\|T_{\mu_{2,\beta}}\|_{S_{\frac{p}{2}}}^{\frac{p}{2}} = \operatorname{tr}(T_{\mu_{2,\beta}}^{\frac{p}{2}}) = \int_D \big\langle T_{\mu_{2,\beta}}^{\frac{p}{2}} K(\cdot,z), K(\cdot,z)\big\rangle_\beta \delta(z)^\beta dv(z)$$
$$= \int_D K(z,z)\big\langle T_{\mu_{2,\beta}}^{\frac{p}{2}} k(\cdot,z), k(\cdot,z)\big\rangle_\beta \delta(z)^\beta dv(z).$$

Since $\frac{p}{2} \geq 1$ and each k_z is a unit vector in $A^2(D,\beta)$, by Proposition 6.4 in [3] we get

$$\|T_{\mu_{2,\beta}}\|_{S_{\frac{p}{2}}(A^2(D,\underline{\ }))}^{\frac{p}{2}} \geq \int_D K(z,z)\big[\,\big\langle T_{\mu_{2,\beta}} k(\cdot,z), k(\cdot,z)\big\rangle_\beta\,\big]^{\frac{p}{2}} \delta(z)^\beta dv(z)$$
$$= \int_D K(z,z)(\tilde{\mu}_{2,\beta}(z))^{\frac{p}{2}} \delta(z)^\beta dv(z),$$

which shows that $\tilde{\mu}_{2,\beta} \in L^{\frac{p}{2}}(D, K(z,z)\delta(z)^\beta dv(z))$.

$(ii) \Rightarrow (iii)$. Form Corollary 2.2, there exists a positive constant C such that

$$C\tilde{\mu}_{2,\beta}(z_0) = C\int_D |k_{z_0}(z)|^2 d\mu_{2,\beta}(z) \geq C\int_{B(z_0,r)} |k_{z_0}(z)|^2 d\mu_{2,\beta}(z)$$
$$\geq \frac{1}{v_\beta(B(z_0,r))}\int_{B(z_0,r)} d\mu_{2,\beta}(z) = \hat{\mu}_{2,\beta}^r(z_0).$$

Thus

$$\int_D (\hat{\mu}_{2,\beta}^r(z))^{\frac{p}{2}} K(z,z)\delta(z)^\beta dv(z) \leq C\int_D (\tilde{\mu}_{2,\beta}(z))^{\frac{p}{2}} K(z,z)\delta(z)^\beta dv(z) < +\infty.$$

$(iii) \Rightarrow (iv)$. Let $\{z_j\}$ be the sequence in Lemma 2.4. By Corollary 3.2, we have

$$\left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}} \leq \frac{C}{v_\beta(B(z_j,r))}\int_{B(z_j,r)} \left[\frac{\mu_{2,\beta}(B(z,r))}{v_\beta(B(z,r))}\right]^{\frac{p}{2}} \delta(z)^\beta dv(z).$$

From Corollary 2.2, letting $z_0 = z$, there exists a positive constant C such that

$$\left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}} \leq C\int_{B(z_j,r)} \left[\frac{\mu_{2,\beta}(B(z,r))}{v_\beta(B(z,r))}\right]^{\frac{p}{2}} K(z,z)\delta(z)^\beta dv(z).$$

By Lemma 2.4, there exists an $m \in \mathbb{N}$ such that

$$\sum_{j=1}^\infty \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}} \leq Cm\int_D \left[\frac{\mu_{2,\beta}(B(z,r))}{v_\beta(B(z,r))}\right]^{\frac{p}{2}} K(z,z)\delta(z)^\beta dv(z),$$

that is,

$$\sum_{j=1}^{\infty}\left(\hat{\mu}^r_{2,\beta}(z_j)\right)^{\frac{p}{2}} \leq Cm\int_D \left(\hat{\mu}^r_{2,\beta}(z)\right)^{\frac{p}{2}} K(z,z)\delta(z)^\beta dv(z).$$

$(iv) \Rightarrow (i)$. We use the complex interpolation method in [21] to prove this statement. We want to show that $T_{\mu_{2,\beta}} \in S_{\frac{p}{2}}(A^2(D,\beta))$ and

$$\|T_{\mu_{2,\beta}}\|^{\frac{p}{2}}_{S_{\frac{p}{2}}(A^2(D,))} \leq C\sum_{j=1}^{\infty}\left(\hat{\mu}^r_{2,\beta}(z_j)\right)^{\frac{p}{2}}.$$

For $p=2$, by Corollary 2.2, there exists a positive constant C such that

$$\begin{aligned}
&\|T_{\mu_{2,\beta}}\|_{S_1(A^2(D,\beta))} = \int_D \left\langle T_{\mu_{2,\beta}}K(\cdot,z), K(\cdot,z)\right\rangle_\beta \delta(z)^\beta dv(z)\\
&= \int_D K(z,z)\left\langle T_{\mu_{2,\beta}}k_z(\cdot), k_z(\cdot)\right\rangle_\beta \delta(z)^\beta dv(z) = \int_D K(z,z)(\tilde{\mu}_{2,\beta}(z))\delta(z)^\beta dv(z)\\
&= \int_D K(z,z)\int_D |k_z(w)|^2 d\mu_{2,\beta}(w)\delta(z)^\beta dv(z) = \int_D\int_D |K(w,z)|^2 d\mu_{2,\beta}(w)\delta(z)^\beta dv(z)\\
&= \int_D\int_D |K(w,z)|^2\delta(z)^\beta dv(z)d\mu_{2,\beta}(w) = \int_D K(w,w)d\mu_{2,\beta}(w)\\
&= \int_D K(z,z)d\mu_{2,\beta}(z) \leq \sum_{j=1}^{\infty}\int_{B(z_j,r)} |K(z,z)|d\mu_{2,\beta}(z) \leq C\sum_{j=1}^{\infty}\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))},
\end{aligned}$$

for all $z_j \in B(z,r)$ and $j\in\mathbb{N}$. For $1<\frac{p}{2}<+\infty$, since $\sum_{j=1}^{\infty}\left(\hat{\mu}^r_{2,\beta}(z_j)\right)^{\frac{p}{2}} < +\infty$, we can assume that

$$\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))} < 1,$$

for all $j\in\mathbb{N}$. By Corollary 2.2 and Lemma 2.4, we have

$$\begin{aligned}
&|\mu_{2,\beta,\zeta}|(D) \leq \sum_{j=1}^{\infty}\left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}\mathrm{Re}\zeta-1}\mu_{2,\beta}(B(z_j,r))\\
&\leq \sum_{j=1}^{\infty}\left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{-1}\mu_{2,\beta}(B(z_j,r)) = \sum_{j=1}^{\infty} v_\beta(B(z_j,r))\\
&\leq C\sum_{j=1}^{\infty}\int_{B(z_j,r))} K(z,z)\delta(z)^\beta dv(z) \leq Cm\int_D K(z,z)\delta(z)^\beta dv(z) < +\infty.
\end{aligned}$$

For every ζ with $0\leq \mathrm{Re}\zeta \leq 1$, we consider the Toeplitz operator $T_{\mu_{2,\beta,\zeta}}$ on $A^2(D,\beta)$ defined by

$$T_{\mu_{2,\beta,\zeta}}f(z) = \int_D K(z,w)f(w)d\mu_{2,\beta,\zeta}(w).$$

By Lemma 3.3 and Lemma 3.4, we have

$$\|T_{\mu_{2,\beta}}\|_{S_p(A^2(D,\beta))} \leq \|T_{\mu_{2,\beta,\frac{2}{p}}}\|_{S_p(A^2(D,\beta))} \leq m\|T_{\mu_{2,\beta}}\|_{S_p(A^2(D,\beta))}.$$

Thus, $T_{\mu_{2,\beta}} \in S_{\frac{p}{2}}(A^2(D,\beta))$ is equivalent to $T_{\mu_{2,\beta,\frac{2}{p}}} \in S_{\frac{p}{2}}(A^2(D,\beta))$. By complex interpolation (see [21]), we have

$$\|T_{\mu_{2,\beta,\frac{2}{p}}}\|_{S_{\frac{p}{2}}(A^2(D,\beta))} \leq M_0^{1-\frac{2}{p}} M_1^{\frac{2}{p}},$$

where

$$M_0 = \sup\left\{\|T_{\mu_{2,\beta,\zeta}}\| : \text{Re}\zeta = 0\right\} \text{ and } M_1 = \sup\left\{\|T_{\mu_{2,\beta,\zeta}}\|_{S_1} : \text{Re}\zeta = 1\right\}.$$

Now, we show that M_0 and M_1 are bounded. For $\text{Re}\zeta = 0$,

$$\begin{aligned}|\mu_{2,\beta,\zeta}|(B(z_k,r)) &\leq \sum_{j=1}^{\infty}\left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{-1}\int_{B(z_k,r)}\chi_{B(z_j,r)}(z)d\mu_{2,\beta}(z)\\ &= \sum_{j=1}^{\infty}\left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{-1}\mu_{2,\beta}(B(z_k,r)\cap B(z_j,r)).\end{aligned}$$

Since $B(z_k,r)\cap B(z_j,r) \neq 0$, by Lemma 2.4, for any fixed positive integer k, there exists $N_k \leq N$ such that

$$\begin{aligned}|\mu_{2,\beta,\zeta}|(B(z_k,r)) &\leq \sum_{i=1}^{N_k}\left[\frac{\mu_{2,\beta}(B(z_{j_i},r))}{v_\beta(B(z_{j_i},r))}\right]^{-1}\mu_{2,\beta}(B(z_k,r)\cap B(z_{j_i},r))\\ &\leq \sum_{i=1}^{N_k}\left[\frac{\mu_{2,\beta}(B(z_{j_i},r))}{v_\beta(B(z_{j_i},r))}\right]^{-1}\mu_{2,\beta}(B(z_{j_i},r))\\ &= \sum_{i=1}^{N_k} v_\beta(B(z_{j_i},r)).\end{aligned}$$

Since $B(z_{j_i},r)\cap B(z_k,r) \neq 0$, by Corollary 2.1 there exists a positive constant C such that

$$v_\beta(B(z_{j_i},r)) \leq C v_\beta(B(z_k,r)).$$

Thus, for all $k \in \mathbb{N}$, we have

$$|\mu_{2,\beta,\zeta}|(B(z_k,r)) \leq CN_k v_\beta(B(z_k,r)) \leq CN v_\beta(B(z_k,r)).$$

From Theorem 3.4 in [1], we know that $|\mu_{2,\beta,\zeta}|$ is a Carleson measure of $A^2(D,\beta)$. By Corollary and Theorem 7 in [21], there exists a positive constant C such that

$$\int_D |f(z)|^2 d|\mu_{2,\beta,\zeta}|(z) \leq C\int_D |f(z)|^2\delta(z)^\beta dv(z),$$

for all f in $A^2(D,\beta)$. Therefore,

$$\begin{aligned}\left|\langle T_{\mu_{2,\beta,\zeta}} f, g\rangle_\beta\right| &= \left|\int_D f(z)\overline{g(z)} d|\mu_{2,\beta,\zeta}|(z)\right| \\ &\le \left[\int_D |f(z)|^2 d|\mu_{2,\beta,\zeta}|(z)\right]^{\frac{1}{2}} \left[\int_D |g(z)|^2 d|\mu_{2,\beta,\zeta}|(z)\right]^{\frac{1}{2}} \\ &\le C\left[\int_D |f(z)|^2 \delta(z)^\beta dv(z)\right]^{\frac{1}{2}} \left[\int_D |g(z)|^2 \delta(z)^\beta dv(z)\right]^{\frac{1}{2}},\end{aligned}$$

which implies that $\|T_{\mu_{2,\beta,\zeta}}\| \le C$, for all ζ with $\mathrm{Re}\zeta = 0$, that is, M_0 is bounded.

For $\mathrm{Re}\zeta = 1$, by Corollary 2.2, there exists a positive constant C such that

$$\begin{aligned}\int_D K(z,z) d|\mu_{2,\beta,\zeta}|(z) &\le \sum_{j=1}^{\infty} \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}-1} \int_{B(z_j,r)} K(z,z) d\mu_{2,\beta}(z) \\ &\le C\sum_{j=1}^{\infty} \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}-1} \frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))} \\ &= C\sum_{j=1}^{\infty} \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}}.\end{aligned}$$

For any orthonormal bases $\{f_j\}$ and $\{g_j\}$ of $A^2(D,\beta)$ and $\mathrm{Re}\zeta = 1$, we have

$$\begin{aligned}\sum_{j=1}^{\infty} \left|\langle T_{\mu_{2,\beta,\zeta}} f_j(z), g_j(z)\rangle_\beta\right| &\le \int_D \sum_{j=1}^{\infty} |f_j(z)||g_j(z)| d|\mu_{2,\beta,\zeta}|(z) \\ &\le \int_D \left[\sum_{j=1}^{\infty} |f_j(z)|^2\right]^{\frac{1}{2}} \left[\sum_{j=1}^{\infty} |g_j(z)|^2\right]^{\frac{1}{2}} d|\mu_{2,\beta,\zeta}|(z) \\ &= \int_D K(z,z)|d|\mu_{2,\beta,\zeta}|(z) \\ &\le C\sum_{j=1}^{\infty} \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}}.\end{aligned}$$

Therefore, for all $\mathrm{Re}\zeta = 1$, we have

$$\|T_{\mu_{2,\beta,\zeta}}\|_{S_1(A^2(D,\beta))} \le C\sum_{j=1}^{\infty} \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}},$$

that is,

$$M_1 \le C\sum_{j=1}^{\infty} \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}} = C\sum_{j=1}^{\infty} \left(\hat{\mu}^r_{2,\beta}(z_j)\right)^{\frac{p}{2}}.$$

Hence,

$$\|T_{\mu_{2,\beta}}\|_{S_p(A^2(D,\beta))} \leq M_0^{1-\frac{2}{p}} M_1^{\frac{2}{p}} \leq C \left(\sum_{j=1}^{\infty} \left(\hat{\mu}^r_{2,\beta}(z_j) \right)^{\frac{p}{2}} \right)^{\frac{2}{p}}.$$

This completes the proof. □

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References

[1] M. Abate, A. Saracco, *Carleson measures and uniformly discrete sequences in strongly pseudoconvex domains,* J. London. Math. Soc., 83 (2011), 587-605.

[2] M. Abate, J. Raissy, A. Saracco, *Toeplitz operators and Carleson measures in strongly pseudoconvex domains,* J. Funct. Anal., 263 (2012), 3449-3491.

[3] J. Arazy, S. D. Fisher, J. Peetre, *Hankel operators on weighted Bergman spaces,* Amer. J. Math., 110 (1988), 989-1053.

[4] F. Colonna, S. Li, *Weighted composition operators from the minimal Möbius invariant space into the Bloch space,* Mediter. J. Math., 10 (2013), 395-409.

[5] K. Esmaeili, M. Lindström, *Weighted composition operators between Zygmund type spaces and their essential norms,* Integral Equ. Oper. Theory, 75 (2013), 473-490.

[6] E. A. Gallardo-Gutiérrez, R. Kumar, J. R. Partington, *Boundedness, compactness and Schatten-class membership of weighted composition operators*, Integr. Equ. Oper. Theory., 67 (2010), 467-479.

[7] J. Isralowitz, J. Virtanen, L. Wolf, *Schatten class Toeplitz operators on generalized Fock spaces,* J. Math. Anal. Appl., 421 (2015), 329-337.

[8] Z. J. Jiang, *Weighted composition operators from weighted Bergman spaces to some spaces of analytic functions on the upper half plane,* Util. Math., 93 (2014), 205-212.

[9] Z. J. Jiang, Z. A. Li, *Weighted composition operators on Bers-type spaces of Loo-keng Hua domains,* Bull. Korean Math. Soc., 57 (2020), 583-595.

[10] A. S. Kucik, *Weighted composition operators on spaces of analytic functions on the complex half-plane*, Complex Anal. Oper. Theory., 12 (2018), 1817-1833.

[11] H. Li, *BMO, VMO and Hankel operators on the Bergman space of strictly pseudoconvex domains,* J. Funct. Anal., 106 (1992), 375-408.

[12] S. Y. Li, B. Russo, *Schatten class composition operators on weighted Bergman spaces of bounded symmetric domains,* Ann. Mat. Pur Appl., 172 (1997), 379-394.

[13] T. Mengestie, *Schatten class weighted composition operators on weighted Fock spaces,* Arch. Math., 101 (2013), 349-360.

[14] B. F. Sehba, *Schatten class Toeplitz operators on weighted Bergman spaces of tube domains over symmetric cones,* Sehba Complex Anal. Synerg, 4 (2018), 3-27.

[15] S. Stević, *Weighted composition operators from Bergman-Privalov-type spaces to weighted-type spaces on the unit ball,* Appl. Math. Comput., 217 (2010), 1939-1943.

[16] S. Stević, Z. J. Jiang, *Differences of weighted composition operators on the unit polydisk*, Mat. Sb., 52 (2011), 358-371.

[17] S. Stević, R. Chen, Z. Zhou, *Weighted composition operators between Bloch type spaces in the polydisc,* Mat. Sb., 201 (2010), 289-319.

[18] X. F. Wang, J. Xia, G. F. Cao, *Schatten p-class Toeplitz operators with unbounded symbols on pluriharmonic Bergman space*, Acta Mathematica Sinica, English Series., 29 (2013), 2355-2366.

[19] L. H. Xiao, X. F. Wang, J. Xia, *Schatten p-class* $(0 < p \leq \infty)$ *Toeplitz operators on generalized Fock spaces,* Acta Mathematica Sinica, 31 (2015), 703-714.

[20] X. D. Yang, *Schatten class weighted composition operators on Bergman spaces of bounded strongly pseudoconvex domains,* Cent. Eur. J. Math., 11 (2013), 74-84.

[21] K. Zhu, *Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains,* J. Operator Theory., 20 (1988), 329-357.

[22] K. Zhu, *Operator theory in function spaces*, Marecl Dekker, New York, 1990.

[23] K. Zhu, *Schatten class composition operators on weighted Bergman spaces of the disk,* J. Operator Theory., 46 (2001), 173-181.

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