On open question of prominent interior GE-filters in GE-algebras

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Abstract. In an interior GE-algebra, the concept of prominent interior GE-filter of type 1 was introduced to serve as a generalization of prominent interior GE-filters. However, there are some work need to be done for this goal. For example, the extension property for prominent interior GE-filter of type 1 still remains unproved so there is an open question on the extension property of such GE-filters need to be proved that 'Let (X, f) be an interior GE-algebra. Let F and G be interior GE-filters in (X, f). If $F \subseteq G$ and F is a prominent interior GE-filter of type 1 in (X, f), then is G also a prominent interior GE-filter of type 1 in (X, f)?' In this paper, we propose the condition for an interior GE-filter to be a prominent interior GE-filter of type 1 , then we prove the extension property for prominent interior GE-filter of type 1 in an interior GE-algebra, and thus the open question is solved.
Keywords: GE-algebra, GE-filter, prominent interior GE-filter of type 1, extension property, open question.

1. Introduction

Henkin and Scolem introduced Hilbert algebra in the implication investigation inintuitionistic logics and other non classical logics [1, 2, 3, 4, 5, 6, 7, 8]. Bandaru et al. introduced GE-algebra as a generalization of Hilbert algebra, and studied its properties [9]. Later some scholars studied interior operators on different

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algebraic structures, such as bounded residuated lattices, GMV-algebras and GE-algebras, and thus different kinds of interior GE-algebras were introduced [10, 11, 12, 13].

Filters theory plays a vital role not only in studying of algebraic structure, but also in non classical logic computer science and logical semantics From the aspect of logical point, filters correspond to various provable formulae sets [14, 15]. Song et al. introduced the notions of an interior GE-filter, a weak interior GE-filter and a belligerent interior GE-filter, and investigate their relations and properties [16]. They provided relations between belligerent interior GE-filter and interior GE-filter and conditions for an interior GE-filter to be a belligerent interior GE-filter is considered. Given a subset and an element, they established an interior GE-filter, and they considered conditions for a subset to be a belligerent interior GE-filter. They studied the extensibility of the belligerent interior GE-filter and established relationships between a weak interior GE-filter and a belligerent interior GE-filter of type 1, type 2 and type 3. Rezaei et al. [12]studied prominent GE-filters in GE algebras.

Afterwards, Song et al. introduced the concept of a prominent interior GE-filter (of type 1 and type 2), and investigated their properties. The relationship between a prominent GE-filter and a prominent interior GE-filter and the relationship between an interior GE-filter and a prominent interior GE-filter are discussed. Also conditions for an interior GE-filter to be a prominent interior GE-filter are given and conditions under which an internal GE-filter larger than a given internal GE-filter can become a prominent internal GE-filter are considered. The relationship between a prominent interior GE-filter and a prominent interior GE-filter of type 1 is discussed [17].

After that, because of the lack of some properties for prominent interior GE-filters of type 1 and of type 2 to serve as a generalization of prominent interior GE-filters, [17] proposed an open question of prominent interior GE-filters of type 1 and of type 2 in GE-algebras that "Let (X, f) be an interior GE-algebra. Let F and G be interior GE-filters in (X, f). If $F \subseteq G$ and F is a prominent interior GE-filter of type 1 in (X, f), then is G also a prominent interior GE-filter of type 1 in (X, f)?"

The motivation of this paper is to further study the prominent interior GE-filter and solve the open question. We prove that in an interior GE-algebra, every prominent interior GE-filter of type 1 is a GE-filters as a complement of [17]. We propose the condition for an interior GE-filter to be a prominent interior GE-filter of type 1. Based on this, we prove the extension property for prominent interior GE-filter of type 1, and thus an open question on such GE-filters of type 1 is solved. As an application, the proof method of the extension property for prominent interior GE-filter of two types can improve the extension theory for other filters in GE-algebras and enrich the generalization theory for filter generation in other logic algebras.

2. Preliminaries

Definition 2.1 ([9]). *A GE-algebra is a non-empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms for all $u, v, w \in X$:*
$(GE1)$ $u * u = 1$;
$(GE2)$ $1 * u = u$;
$(GE3)$ $u * (v * w) = u * (v * (u * w))$.

In a GE-algebra X, a binary relation $\leq$ is defined by $(\forall x, y \subseteq X)(x \leq y \Leftrightarrow x * y = 1)$.

Definition 2.2 ([9]). *A GE-algebra X is said to be transitive if it satisfies:*

$$(\forall x, y, z \in X)(x * y \leq (z * x) * (z * y)).$$

Proposition 2.1 ([9]). *Every GE-algebra X satisfies the following items for* $\forall u, v, w \in X$:
(1) $u * 1 = 1$;
(2) $u * (u * v) = u * v$;
(3) $u \leq v * u$;
(4) $u * (v * w) \leq v * (u * w)$;
(5) $1 \leq u \Rightarrow u = 1$;
(6) $u \leq (v * u) * u$;
(7) $u \leq (u * v) * v$;
(8) $u \leq v * w \Leftrightarrow v \leq u * w$.
If X is transitive, then:
(9) $u \leq v \Rightarrow w * u \leq w * v, v * w \leq u * w$;
(10) $u * v \leq (v * w) * (u * w)$.

Lemma 2.1 ([9]). *In a GE-algebra X, the following facts are equivalent for* $\forall x, y, z \in X$:
(1) $x * y \leq (z * x) * (z * y)$;
(2) $x * y \leq (y * z) * (x * z)$.

Definition 2.3 ([9]). *A subset F of a GE-algebra X is called a GE-filter of X if it satisfies for* $\forall x, y \in X$:
(1) $1 \in F$;
(2) $x * y \in F, x \in F \Rightarrow y \in F$.

Lemma 2.2 ([9]). *In a GE-algebra X , every non-empty subset F of X is a filter if and only if it satisfies:*
(1) $1 \in F$ *and* $(\forall x, y \in X)(x \leq y, x \in F \Rightarrow y \in F)$;
(2) $(\forall x, y \in F, z \in X)(x \leq y * z \Rightarrow z \in F)$.

Definition 2.4 ([10]). *A subset F of a GE-algebra X is called a prominent GE-filter of X if it satisfies* $1 \in F$ *and* $(\forall x, y, z \in X)(x * (y * z) \in F, x \in F \Rightarrow ((z * y) * y) * z \in F)$.

Note that every prominent GE-filter is a GE-filter in a GE-algebra (see [10]).

Definition 2.5 ([4]). *By an interior GE-algebra we mean a pair (X, f) in which X is a GE-algebra and $f : X \to X$ is a mapping such that for $\forall x, y \in X$:*
(1) $x \leq f(x)$;
(2) $(f \circ f)(x) = f(x)$;
(3) $(x \leq y \Rightarrow f(x) \leq f(y))$.

Definition 2.6 ([11]). *Let (X, f) be an interior GE-algebra. A GE-filter F of X is said to be interior if it satisfies:*

$$(*) \qquad (\forall x \in X)(f(x) \in F \Rightarrow x \in F).$$

Definition 2.7 ([17]). *Let (X, f) be an interior GE-algebra. Then a subset F of X is called a prominent interior GE-filter in (X, f) if F is a prominent GE-filter of X which satisfies the condition $(*)$.*

Theorem 2.8 ([17]). *In an interior GE-algebra, every prominent interior GE-filter is an interior GE-filter.*

Theorem 2.9 ([17]). *Every interior GE-filter F in an interior GE-algebra (X, f) is a prominent interior GE-filter if and only if it satisfies:*
$(\forall x, y \in X)(x * y \in F \Rightarrow ((y * x) * x) * y \in F)$.

3. Prominent interior GE-filters of type 1

Definition 3.1 ([17]). *Let (X, f) be an interior GE-algebra and let F be a subset of X which satisfies $1 \in F$, then F is called a prominent interior GE-filter of type 1 in (X, f) if it satisfies:*

$$(\forall x, y, z \in X)(x * (y * f(z)) \in F, f(x) \in F \Rightarrow ((f(z) * y) * y) * f(z) \in F).$$

By example [17] shows that interior GE-filter and prominent interior GE-filter of type 1 are independent of each other.

Theorem 3.2 ([17]). *In an interior GE-algebra, every prominent interior GE-filter is of type 1, but the converse may not be true.*

Proposition 3.1. *In an interior GE-algebra, every prominent interior GE-filter of type 1 is a GE-filters.*

Proof. Let F be prominent interior GE-filter of type 1 of X and for any $x, y \in X$. Let $x * y \in F$, $x \in F$, if we let f be the identity mapping and $z = 1$, then we get $f(z) = 1$ and $f(x) \in F$, by definition , we have $(\forall x, y, z = 1 \in X) x * (z * f(y)) = x * (1 * y) = x * y \in F, f(x) = x \in F \Rightarrow (f(y) * z) * z) * f(y) = (y * 1) * 1) * y = y \in F$. It follows from F is a GE-filters. □

Theorem 3.3. *An interior GE-filter F of X is a prominent interior GE-filter of type 1 if and only if it satisfies:*

$$y * f(z) \in F \ \ \textit{implies} \ \ ((f(z) * y) * y) * f(z) \in F \ \ \textit{for all}\, y, z \in X.$$

Proof. Assume that F is a prominent interior GE-filter of type 1 of X and let $y, z \in X$ be such that $y * f(z) \in F$. Then $1 * (y * f(z)) = y * f(z) \in F$ and $f(1) = 1 \in F$. It follows from that $((f(z) * y) * y) * f(z) \in F$.

Conversely, let F be an interior GE-filter of X satisfying the above condition and let $x, y, z \in X$ be such that $x * (y * f(z)) \in F$ and $f(x) \in F$. Then $x \in F$, $y * f(z) \in F$ and hence $((f(z) * y) * y) * f(z) \in F$. Therefore, F is a prominent interior GE-filter of type 1 of X. □

[17] proposed an open question of prominent interior GE-filters of type 1 and of type 2 in GE-algebras: Let (X, f) be an interior GE-algebra. Let F and G be interior GE-filters in (X, f). If $F \subseteq G$ and F is a prominent interior GE-filter of type 1 in (X, f), then is G also a prominent interior GE-filter of type 1 in (X, f)?

For this open question for type 1, based on the previous work, we can solve it in the following theorem.

Theorem 3.4 (Extension property for prominent interior GE-filter of type 1.)**.** *Let F and G be prominent interior GE-filters of type 1 of X such that $F \subseteq G$. If F is a prominent interior GE-filter of type 1, then so is G.*

Proof. Let $y, z \in X$ be such that $y * f(z) \in G$. Then $y * ((y * f(z)) * f(z)) \leq (y * f(z)) * (y * f(z)) = 1 \in F$. Since F is prominent interior of type 1, it follows that $((((y * f(z)) * f(z)) * y) * y) * ((y * f(z)) * f(z)) \in F$ so, that $(y * f(z)) * (((((y * f(z)) * f(z)) * y) * y) * f(z)) \in F \subseteq G$.

Since $y * f(z) \in G$, therefore $((((y * f(z)) * f(z)) * y) * y) * f(z) \subseteq G$. But $1 = (y * f(z)) * 1 = (y * f(z)) * (f(z) * f(z)), \leq f(z) * ((y * f(z)) * f(z)), \leq (((y * f(z)) * f(z)) * y) * (f(z) * y), \leq ((f(z) * y) * y) * ((((y * f(z)) * f(z)) * y) * y), \leq (((((y * f(z)) * f(z)) * y) * y) * f(z)) * (((f(z) * y) * y) * f(z))$. Using Lemma 2.6. (2), we get $(((((y * f(z)) * f(z)) * y) * y) * f(z)) * (((f(z) * y) * y) * f(z)) = 1$ and $((f(z) * y) * y) * f(z) \in G$.

Hence, by Theorem 3.4, G is a prominent interior GE-filter of type 1 of X. □

4. Conclusion

Filter theory is of great significance in the study of algebraic domain. Many scholars introduced the concepts and relationships among a varieties of filters from different aspects. Several open questions on this topic thus appeared. In GE-algebras, an open question of prominent interior GE-filters of type 1 and of type 2 is proposed.

The purpose of the study is to solve the open question. On the basis of previous work, in this paper, we prove that in an interior GE-algebra, every prominent interior GE-filter of type 1 is a GE-filters as a complement. We also propose the condition for an interior GE-filter to be a prominent interior GE-filter of type 1. Based on this, we prove the extension property for prominent interior GE-filter of type 1 is proved, and thus an open question on such GE-filters of type 1 is solved. We hope that will bring us enlightment in the study of this field.

For the future work, we will further study the prominent interior GE-filter of type 2 and solve the open question of it completely. If the extension property for prominent interior GE-filter of type 2 also holds, we will try to find a generalization of the two types in an interior GE-algebra.

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