The ω-continuity of group operation in the first (second) variable

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Abstract. The present paper aims to introduce and study the ω-continuity of the group operation in the first (resp., second) variable and some basic properties and relationships concerning left and right translation functions are obtained. Also, we have shown that the group operation is ω-continuous at the first (resp., second) variable if and only if it is ω-irresolute at the first (resp., second) variable.
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1. Introduction

Topology is a special type of geometry and includes several fields of study and it has many interesting applications in graph theory. Hdeib H. Z. [7] defined and studied ω-closed sets and ω-open sets. He used ω-closed sets to define a new type of mappings called ω-closed functions. He obtained many properties and relationships concerning these concepts. Also, he used ω-open sets to define $\omega - continuous$ mappings [8] and he studied this new type of continuous

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mapping and obtained certain properties and relationships concerning this type of continuous mappings.

The notion of a topological group goes back to the second half of the nineteenth century. Topological groups are objects that combine two separate algebraic structures with the topology structure and the requirement links them that multiplication and inversion are continuous functions.

In this article, we study the ω-continuity of a group operation at the first (resp., second) variable respectively and obtain some basic properties of this kind of ω-continuity of groups.

2. Preliminaries

Let A be a subset of a topological space (X, τ), the interior and closure of A are denoted by $Int(A)$ and $Cl(A)$, respectively. A point x of X is called a condensation point of A [9] if $G \cap A$ is uncountable for each open set G containing x. A is called ω-closed [7] if it contains all it is condensation points. The complement of an ω-closed set is called an ω-open set. The intersection of all ω-closed subsets of X which contain A is called ω-closure of A and is denoted by ωClA [4] and [7]. A point $x \in A$ is said to be an ω-interior point of A [8], if there exists an ω-open set U containing x such that $U \subseteq A$. The set of all ω-interior points of A is denoted by $\omega IntA$.

The discrete topology is denoted by τ_{dis}, and the family of all ω-open subsets of a space (X, τ), denoted by τ^ω from a topology on X finer than τ ([4]). A compact space is a topological space for which every covering of that space by a collection of open sets has a finite subcover.

Definition 2.1 ([5]). *A space (X, τ) is said to be $\omega - compact$ provided that every ω-open cover of X has a finite subcover.*

Definition 2.2 ([5]). *A space (X, τ) is said to be $\omega - lindelof$ provided that every ω-open cover of X has a countable subcover.*

Definition 2.3 ([4]). *A space (X, τ) is said to be locally countable if each point of X has a countable open neighbourhood.*

Theorem 2.1 ([3]). *Let (X, τ) be a topological space, then $\tau^\omega = \tau_{dis}$ if and only if the space (X, τ) is locally-countable.*

Theorem 2.2 ([4]). *For any topological space (X, τ) and any subset A of X, $(\tau_A)^\omega = \tau^\omega_A$.*

The proof of the following lemma can be found in [15]. Also, we can find a similar proof in [14], Lemma 3 and [16], Lemma 3.3].

Definition 2.4 ([8]). *Let $f : (X, \tau) \to (Y, \rho)$ be a mapping, f is said to be ω-continuous at a point $x \in X$, if for each open subset V in Y containing $f(x)$ there exists an ω-open subset U of X contains x such that $f(U) \subseteq V$, and f is called ω-continuous if it is ω-continuous at each point x of X.*

Definition 2.5 ([1]). *Let $f : (X, \tau) \to (Y, \rho)$ be a mapping. Then f is said to be ω-irresolute, if $f^{-1}(F)$ is an ω-closed in X for each ω-closed set F in Y.*

Definition 2.6 ([10]). *Let $f : (X, \tau) \to (Y, \rho)$ be a mapping, then f is an ω-homeomorphism if and only if f is bijective and f, f^{-1} are ω-irresolute.*

Definition 2.7 ([11]). *A space (X, τ) is said to be lindelof provided that every open cover of X has a countable subcover.*

Lemma 2.1 ([4]). *Let (X, τ) be a topological space. Then X is ω-lindelof if and only if it is lindelof.*

Definition 2.8 ([12]). *A topological space (X, τ) is called a normal space if given any disjoint closed sets E and F, there are neighbourhoods U of E and V of F with $U \cap V = \phi$.*

Definition 2.9 ([13]). *Let X be a nonempty set and $\mu : X \to X$ be a binary operation defined by $\mu(g_1, g_2) = g_1 * g_2$. The pair $(X, *)$ is a group if the following three properties hold:*

1. *For all $a, b, c \in X$ we have $(a * b) * c = a * (b * c)$ (associative law);*
2. *There exists an $e \in X$ such that for all $a \in X$ we have $a * e = e * a = a$ (existence of identity element);*
3. *For all $a \in X$ there exists $a^{-1} \in X$ such that $a * a^{-1} = a^{-1} * a = e$ (each element has inverse).*

Definition 2.10 ([13]). *Let $(X, *)$ be a group. If X has the property that $a * b = b * a$ for all $a, b \in X$, then we call X abelian.*

Definition 2.11 ([17]). *Let $(X, *)$ be a group and H be a subset of X. We call H a subgroup of X when the following hold:*

1. *$H \neq \phi$;*
2. *If $x, y \in H$, then $x * y \in H$;*
3. *If $x \in H$, then $x^{-1} \in H$.*

Definition 2.12 ([17]). *Let X be a group, H a subgroup of H and $g \in X$. The sets $gH = \{g * h, h \in H\}$ and $Hg = \{h * g, h \in H\}$ are called the left and right cosets of H in X, respectivly.*

3. The results

We introduce the following definition

Definition 3.1. *Let $(X, *)$ be a group and τ be a topology on X. The multiplication map $\mu : X * X \to X$ is said to be ω-continuous at the first (second) variable if, for any fixed point $a \in X$, any point $b \in X$ and any open set G in X which contains $\mu(b, a) = b * a$, $(\mu(a, b) = a * b)$, there exists an ω-open set V in X such that $b \in V$ and $V * a \subseteq G$, $(a * V \subseteq G)$.*

In our first result, we prove that for abelian groups, the ω-continuity of multiplication maps at the first and second variable are equivalent.

Theorem 3.1. *If $(X, *)$ is an abelian group and τ a topology on X. Then, the multiplication map μ is ω-continuous at the first variable if and only if it is ω-continuous at the second variable.*

Proof. Let μ be ω-continuous at the first variable. Suppose a is any fixed point of X and b is an arbitrary point of X. To show μ is ω-continuous at the second variable. Let O be any open subset of X which contains $a * b$. But, since $a * b = b * a$, so, $b * a \in O$, Since μ is ω-continuous at the first variable, then by Definition 3.1, there is an ω-open subset V of X which contains b and $V * a \subseteq O$. But, $V * a = a * V$, so $a * V \subseteq O$. Hence, μ is ω-continuous at the second variable. The converse part is followed similarly. □

Theorem 3.2. *If $(X, *)$ is any group and τ a topology on X such that (X, τ) is locally countable, then the multiplication map of X is $\omega - continuous$ at the first variable as well as at the second variable.*

Proof. Since (X, τ) is locally countable, so, by Theorem 2.1, $\tau^\omega = \tau_{dis}$. For any $a, b \in X$ and any open subset G of X such that $a * b \in G$, we have $\{a\}, \{b\} \in \tau^\omega$, $a * \{b\} = \{a * b\} \subseteq G$ and $\{a\} * b = \{a * b\} \subseteq G$. Thus, μ is ω-continuous at the first and second variables. □

Remark 3.1. The following example shows that the ω-continuity of the multiplication map in the first and second variable does not imply that the group is abelian and also does not imply that the group is semi-topological.

Example 3.1. Consider the symmetric group S_3 of the set $A = \{1, 2, 3\}$. The elements of this group are $f_1 = 1, f_2 = (1, 2), f_3 = (2, 3), f_4 = (1, 3), f_5 = (1, 2, 3), f_6 = (1, 3, 2)$, so, that S_3={1,(1,2),(1,3),(2,3), (1,2,3), (132)} with the usual composition of maps $(S_3, \circ)$ forms a non-commutative group, let $\tau = \{\phi, S_3, \{f_1\}, \{f_2, f_3, f_4\}, \{f_1, f_2, f_3, f_4\}, \{f_1, f_5, f_6\}\}$ be a topology on S_3 then, the multiplication map is not continuous neither in the first nor in the second variable because for $i = 2, 3, 4$ we have $f_i \circ f_i = f_1$ and $\{f_2, f_3, f_4\} \circ f_i \nsubseteq \{f_1\}$. Also, $f_1 \circ f_2, f_3, f_4 \subseteq f_1$ since $S_3 \times S_3$ is finite, so, by $(\tau \times \tau)_\omega = \tau_{dis}$, S_3 is finite $\tau^\omega = \tau_{dis}$, and $\tau^\omega \times \tau^\omega = \tau_{dis} \times \tau_{dis} = (\tau \times \tau)^\omega = \tau^\omega \times \tau^\omega$.

Theorem 3.3. *Let $(X, *)$ be a group and τ be a topology on X and the multiplication map μ is ω-continuous at the second (first) variable. For any $A, B \subseteq X$ and $a \in X$, the following statements are true:*

1. $a * \omega\ ClB \subseteq Cl(a * B)$and $((\omega ClB) * a \subseteq Cl(B * a))$.

2. $\omega Cl(a * B) \subseteq a * ClB$ and $(\omega Cl(B * a) \subseteq (ClB) * a)$.

3. $A * \omega ClB \subseteq Cl(A * B)$ and $((\omega ClB) * A \subseteq Cl(B * A))$.

Proof. 1. Let $y \in a * \omega ClB$, and let G be an open subset of X, such that $y \in G$. Then, there is $x \in \omega ClB$ such that y= $a * x$. Since μ is $\omega - continuous$ at the second variable, there exists an ω-open set V in X such that $x \in V$ and $a * V \subseteq G$. Since $x \in V$ and $x \in \omega ClB$, then $V \cap B \neq \phi$, so, there is, $s \in V \cap B$. Then, $a * s \in a * V$ and $a * s \in a * B$, so, $(a * V) \cap (a * B) \neq \phi$. Hence, $G \cap (a * B) \neq \phi$. This means that, $y \in Cl(a * B)$. Thus, $a * \omega ClB \subseteq Cl(a * B)$.

2. By (1) we have $a^{-1}(\omega(Cla*B)) \subseteq Cl(a^{-1}*(a*B)) = Cl(a^{-1}*a)*B = ClB$. Therefore, $a * (a^{-1} * (\omega Cl(a * B)) \subseteq a * ClB$. That is, $\omega Cl(a * B) \subseteq a * ClB$.

3. By (1) $A*\omega ClB = \bigcup_{a \in A}(a*\omega ClB) \subseteq \bigcup_{a \in A} Cl(a*B) \subseteq Cl \bigcup_{a \in A}(a*B) = Cl(A * B)$. □

Theorem 3.4. *Let $(X, *)$ be a group and τ be a topology on X, in which the multiplication map μ is ω-continuous at the second (first) variable. Then, for each $A, B \subseteq X$ and $a \in X$, the following statements hold:*

1. $Int(a * B) \subseteq a * \omega IntB$ and $(Int(B * a) \subseteq (\omega IntB) * a)$;

2. $a * IntB \subseteq \omega Int(a * B)$ and $((IntB) * a \subseteq \omega Int(B * a))$;

3. $A * IntB \subseteq \omega Int(A * B)$ and $((IntB) * A \subseteq \omega Int(B * A))$.

Proof. 1. Let $y \in Int(a * B)$. Then, there is an open set O in X such that $y \in O \subseteq a * B$, then there is $b \in B$ such that $y = a * b$. By ω-continuity of μ at the second variable, there exists an ω-open subset V of X such that $b \in V$ and $a * V \subseteq O$, that is, $a * V \subseteq a * B$, so, $a^{-1} * (a * V) \subseteq a^{-1} * (a * B)$, hence $V \subseteq B$. This means that, $b \in \omega IntB$, so, $y = a * b \in a * \omega IntB$. Hence, $Int(a * B) \subseteq a * \omega IntB$.

2. $a * IntB = a * Int(e * B) = a * Int(a^{-1} * (a * B)) \subseteq a * (a^{-1}\omega Int(a * B)) = (a * a^{-1}) * \omega Int(a * B) = e * \omega Int(a * B) = \omega Int(a * B)$.

3. $A * IntB = \bigcup_{a \in A}(a * IntB) \subseteq \bigcup_{a \in A} \omega Int(a * B) \subseteq \omega Int \bigcup_{a \in A}(a * B) = \omega Int(A * B)$. □

Theorem 3.5. *Let $(X, *)$ be a group and τ be a topology on X, then:*

1. *the multiplication map μ is ω-continuous at the second variable if and only if the left translation function $\iota_a : X \to X$ is ω-continuous, for each $a \in X$;*

2. *the multiplication map μ is ω-continuous at the first variable if and only if the right translation function $r_a : X \to X$ is ω-continuous, for each $a \in X$.*

Proof. We prove (1) and the proof of (2) is completely similar.

Let the multiplication map μ is $\omega - continuous$ at the second variable. To show that ι_a is $\omega - continuous$, for each $a \in X$.

Let $x \in X$ and O be any open subset of X such that $\iota_a(x) \in O$ (That is, $a * x \in O$.) So, there is an $\omega - open$ set V in X such that $x \in V$ and $a * V \subseteq O$, that is $\iota_a(V) \subseteq O$, this means that, ι_a is $\omega - continuous$ at x. But, since a and x are arbitrary points of X, therefore, ι_a is ω-continuous for each $a \in X$.

Suppose that ι_a is ω-continuous, for each $a \in X$. Now, let a be a fixed point of X, $x \in X$ and O be an arbitrary open subset of X such that $a * x \in O$. That is, $\iota_a \in O$. By ω-continuity of ι_a, there is an ω-open set V in X such that $x \in V$ and $\iota_a(V) \subseteq G$. Hence, $a * V \subseteq O$, so, that μ is $\omega - continuous$ at the second variable. □

Corollary 3.1. *Let τ be any topology on a group $(X, *)$, then:*

1. *the multiplication map μ is ω-continuous at the second variable if and only if the left translation function ι_a is ω-irresolute, for each $a \in X$;*
2. *the multiplication map μ is ω-continuous at the first variable if and only if the right translation function r_a is ω-irresolute, for each $a \in X$.*

Proof. 1. Since μ is $\omega - continuous$ at the second variable, so, by Theorem 3.5, the left translation function ι_a is $\omega - continuous$, for each $a \in X$. Since ι_a is bijective, ι_a is ω-irresolute, for each $a \in X$.

Conversely, let ι_a be $\omega - irresolute$ for each $a \in X$. Then, it is ω-continuous, for each $a \in X$. By Theorem 3.5, μ is ω-continuous at the second variable.

2. The proof is similar to the proof of (1). □

Proposition 3.1. *Let τ be a topology on a group $(X, *)$. Then, the left (right) translation function $\iota_a(\ r_a)$ is ω-continuous if and only if it is ω-homeomorphism, for each $a \in X$.*

Proof. Let ι_a (r_a) be an ω-continuous function, for each $a \in X$. Then, ι_a (r_a) respectively, is ω-irresolute for each $a \in X$. Since ι_a (r_a) is a bijective function with $(\iota a)^{-1}(V) = \iota a^{-1}(V) = V * a^{-1}$, and $a^{-1} \in X$, then $\iota_a^{-1}(\ r_a^{-1})resp.$, is an $\omega - irresolute$ function. Hence, ι_a (r_a) is ω-homeomorphism, for each $a \in X$. □

Proposition 3.2. *Let τ be a topology on a group $(X, *)$. Then:*

1. the multiplication map μ is ω-irresolute at the second variable if and only if the left translation function ι_a is ω-irresolute, for each $a \in X$.

2. the multiplication map μ is ω-irresolute at the first variable if and only if the right translation function r_a is ω-irresolute, for each $a \in X$.

Proof. The proof is completely similar to the proof of the Theorem 3.5. □

Proposition 3.3. *Let τ be a topology on a group $(X, *)$. The multiplication map μ is ω-irresolute at the second (resp., first) variable if and only if it is $\omega - continuous$ at the second (first) variable.*

Proof. Let μ is $\omega-irresolute$ at the second (first) variable if and only if $\iota_a(resp.,\ r_a)$ is $\omega-irresolute$, for each $a\in X$ by Proposition 3.3 if and only if μ is $\omega-continuous$ at the second (resp., second) variable by Corollary 3.1. □

Proposition 3.4. *Let τ be a topology on a group $(X,*)$. Then, the multiplication map μ is ω-irresolute at the second (first) variable if and only if it is ω-continuous at the second (first) variable.*

Proof. we can show that μ is $\omega-irresolute$ at the second (first) variable by the same way as we have proved Theorem 3.5 and Corollary 3.1, we will get the left translation ι_a (right translation r_a) function is $\omega-irresolute$, for each $a\in X$. If and only if μ is ω-continuous at the second (first) variable. □

Theorem 3.6. *If τ is a topology on a group $(X,*)$ such that the multiplication map μ is ω-continuous at the second variable, then for each $A,B\subseteq X$ and $a\in X$, we have:*

1. $a*\omega ClB=\omega Cl(a*B)$.
2. $a*\omega IntB=\omega Int(a*B)$.
3. *B is ω-open if and only if* $a*B\,is\,\omega-open$.
4. *B is ω-closed if and only if* $a*B\,is\,\omega-closed$.
5. $A*\omega ClB\subseteq\omega Cl(A*B)$.
6. $A*\omega IntB\subseteq\omega Int(A*B)$.
7. $\omega IntA*\omega IntB\subseteq\omega Int(A*B)$.
8. $\omega ClA*\omega ClB\subseteq\omega Cl(A*B)$.
9. *If B is $\omega-open$, then $A*B$ is $\omega-open$.*
10. *If B is ω-closed and A is finite, then $A*B$ is ω-closed.*

Proof. 1. Let $y\in a*\omega ClB$. Then, $y=a*b$ for some $b\in\omega ClB$. Let G be any $\omega-open$ subset of X such that $y=a*b\in G$. By Proposition 3.2 there exists an ω-open subset V of X such that $b\in V$ and $a*V\subseteq G$. Since $b\in\omega ClB$, so, $V\cap B\neq\phi$. Therefore, $a*V\cap a*B\neq\phi$. Since $a*V\subseteq G$, so, $G\cap(a*B)\neq\phi$. This means that, $y\in\omega Cl(a*B)$. That is, $a*\omega ClB\subseteq\omega Cl(a*B)$. Also, $a^{-1}*(\omega Cl(a*B)\subseteq\omega Cl(a^{-1}*(a*B))=\omega Cl((a*a^{-1})*B)=\omega Cl(e*B)=\omega ClB$. Then, $a*(a^{-1}*\omega Cl(a*B))\subseteq a*\omega ClB$, so, that $\omega Cl(a*B)\subseteq a*\omega ClB$. Hence, $a*\omega ClB=\omega Cl(a*B)$.

2. Let $y\in\omega Int(a*B)$. Then, there exists $x\in B$ and an $\omega-open$ set V in X such that $y=a*x\in V\subseteq a*B$. By Proposition 3.2, there exists an $\omega-open$ set U in X such that $x\in U$ and $a*U\subseteq V$. Thus, $a*U\subseteq a*B$,

so, $U \subseteq B$. This means that, $x \in \omega IntB$. Then, $y = a * x \in a * \omega IntB$. So, $\omega Int\, a * B \subseteq a\,\omega IntB$. Now, Since $a^{-1} \in X$ and $a * B \subseteq X$, we get $\omega IntB = \omega Int(e * B) = \omega Int(a^{-1} * (a * B)) \subseteq a^{-1} * \omega Int(a * B)$. Therefore, $a*\omega IntB \subseteq (a*a^{-1}*\omega Int(a*B) = \omega Int(a*B)$. Hence, $a*\omega IntB = \omega Int(a*B)$.

3. Let B be $\omega - open$ in X. From Corollary 3.1, we have ι_a^{-1} is ω-irresolute, so, $(\iota_a^{-1})^{-1}(B)$ is ω-open in X. Since $(\iota_a^{-1})^{-1} = \iota_a$, so, $\iota_a(B)$ is ω-open in X. Thus, $a * B$ is ω-open in X.

Conversely, Let $a * B$ be $\omega - open$ in X. From Corollary 3.1, we have ι_a is ω-irresolute, then $\iota_a^{-1}(a * B)$ is ω-open in X. Since $(\iota_a)^{-1} = \iota_a^{-1}$, so, $\iota_a^{-1}(a * B)$ is ω-open in X. Since $\iota_a^{-1}(a * B) = a^{-1} * (a * B) = B$, so, B is $\omega - open$ in X.

4. Let B be $\omega - closed$ in X. Then, by (1), $a * B = a * \omega ClB = \omega Cl(a * B)$, so, $a * B$ is ω-closed.

Conversely, suppose that $a * B$ is an ω-closed subset of X, so, $a * B = \omega Cl(a*B)$. But, from (1), we have $\omega Cl(a*B) = a*\omega ClB$, so $a*B = a*\omega ClB$. This implies that $a^{-1} * (a * B) = a^{-1} * (a * \omega ClB)$. Hence, $B = \omega ClB$. Thus, B is $\omega - closed$ in X.

5. Let $y = a * b \in A * \omega ClB$, where $a \in A$ and $b \in \omega ClB$. To show $y \in \omega Cl(A * B)$. Let G be any ω-open subset of X such that $y = a * b \in G$. By Proposition 3.2, there exists an ω-open subset V of X such that $b \in V$ and $a * V \subseteq G$, since $b \in V$ and $b \in \omega ClB$, so, $V \cap B \neq \phi$, so, $(a * V) \cap (a * B) \neq \phi$. Since $a*V \subset G$, so, $G \cap (a*B) \neq \phi$ and since $a*B \subseteq A*B$, so, $G \cap (A*B) \neq \phi$. Hence, $y \in \omega Cl(A * B)$. Thus, $A * \omega ClB \subseteq \omega Cl(A * B)$.

6. By (2), we have $A * \omega IntB = \bigcup_{a\in A}(a * \omega IntB) = \bigcup_{a\in A}(\omega Int(a * B) \subseteq \omega Int(\bigcup_{a\in A}(a * B)) = \omega Int(A * B)$.

7. Since $\omega IntA \subseteq A$, so, $\omega IntA*\omega IntB \subseteq A*\omega IntB$ and since $A*\omega IntB \subseteq \omega Int(A * B)$. So, by (6) $\omega IntA * \omega IntB \subseteq \omega Int(A * B)$.

8. Let $y \in \omega ClA * \omega ClB$. Then, $y = a * b$, for some $a \in \omega ClA$, $b \in \omega ClB$. Let G be any $\omega - open$ subset of X such that $y = a * b \in G$. By Proposition 3.2, there is an $\omega - open\, subset\, V$ of X such that $b \in V$ and $a * V \subseteq G$. Since $b \in \omega ClB$, so, $V \cap B = \phi$. Since $a*(V \cap B) = (a*V) \cap (a*B)$, so, $G \cap (a*B) = \phi$. Since $a * B \subseteq A * B$, then $G \cap (A * B) = \phi$. Therefore, y$\in \omega Cl(A * B)$. Hence, $\omega ClA * \omega ClB \subseteq \omega Cl(A * B)$.

9. Let B be $\omega - open$ in X. Then by (3) $a * B$ is $\omega - open$, for each $a \in A$. Since, the union of any family of $\omega - open$ sets is ω-open, so, $\bigcup_{a\in A}(a * B)$ is $\omega - open$. But, since $A * B = \bigcup_{a\in A}(a * B)$, so, $A * B$ is $\omega - open$.

10. Let B be ω-closed and A be a finite subset of X. Then, by (4) $a * B$ is ω-closed, for each $a \in A$. Since $A * B = \bigcup_{a\in A}(a * B)$ and the finite union of $\omega - closed$ is ω-closed, so, $A * B$ is ω-closed. □

Theorem 3.7. *Let $(H, *)$ be a subgroup of a group $(X, *)$ and τ be any topology on X.*

1. *If $\mu : X * X \to X$ is ω-continuous at the second variable, then $\mu_H : H * H \to H$ is ω-continuous at the second variable.*

2. *If $\mu : X{*}X \to X$ is ω-continuous at the first variable, then $\mu_H : H{*}H \to H$ is ω-continuous at the first variable.*

Proof. We prove part (1) and the proof of the second part is almost similar.

Let a be a fixed point of H such that $\mu_H(a,b) = a * b \in G$. Then, there is an open set O in X such that $O = G \cap H$ and $\mu(a,b) = \mu_H(a,b) = a * b \in O$. Since μ is ω-continuous at the second variable, so, by Definition 3.1, there is an ω-open subset V of X such that $b \in V$ and $a * V \subseteq O$. Then, by Theorem 2.2, $V \cap H$ is ω-open in H and $a*(V\cap H) = a*V \cap a*H = a*V \cap H \subseteq O \cap H = G$. Hence, $\mu_H : H * H \to H$ is ω-continuous at the second variable. □

Theorem 3.8. *Let τ be a topology on a group $(X, *)$ such that the multiplication map μ is ω-continuous at the second (first) variable. If S is a semigroup subset of X for which $\omega IntS \neq \phi$, then $\omega IntS$ is also a semigroup.*

Proof. Without loss of generality, we assume that μ is ω-continuous at the second variable. It is given that, $\omega IntS \neq \phi$. Let $a, b \in \omega IntS$, then, there is an $\omega - open$ subset V of X such that $b \in V \subseteq S$. Since S is a semigroup, so, $a * b \in a * V \subseteq S$. But, from (3) of Theorem 3.6 we have $a * V$ is $\omega - open$ in X, so, $a * b \in \omega IntS$. Also, since $\omega IntS \subseteq S$ and μ is associative on S, so, μ is associative on $\omega IntS$. Hence, $\omega IntS$ is a semigroup. □

Theorem 3.9. *Let H be subgroup of a group X. Let τ be any topology on X such that the multiplication map μ is ω-continuous at the second (first) variable and $\omega Int(H) \neq \phi$. If the function $f : X \to X$ give by $f(x) = x^{-1}$ for each $x \in X$ is ω-continuous then $\omega Int(H)$ is a subgroup of X.*

Proof. Without loss of generality, we assuming that μ is ω-continuous at the second variable. By what we have done in the proof of Theorem 3.8 for any $a, b \in \omega Int(H)$ we obtain that $a * b \in \omega IntH$. Also, for any $a \in \omega IntH$, we have an ω-open subset G of X such that $a \in G \subseteq H$, $f : X \to X$ is a bijective function and it is ω-continuous function. Since V is ω-open in X, so, $f^{-1}(V) = \{x : f(x) \in V\} = \{x : x^{-1} \in V\} = V^{-1}$ so, V^{-1} is ω-open in X. Since $a \in V \subseteq H$, so, $a^{-1} \in V^{-1} \subseteq H^{-1} = H$. Hence, $a^{-1} \in \omega IntH$. Therefore, $a * b^{-1} \in \omega IntH$. Hence, $\omega IntH$ is subgroup of G. □

Theorem 3.10. *Let τ be a topology on a group $(X, *)$ such that the multiplication map μ is ω-continuous at the second (first) variable. If S is a semigroup subset of X, then ωClS is a semigroup subset of X.*

Proof. We prove this result for the case that μ is ω-continuous at the second variable, we left the other because it has a similar proof. Since $\phi \neq S \subseteq \omega ClS$. So, $\omega ClS \neq \phi$. Let a, b be any two points of ωClS and V is any ω-open subset of X which contains $a * b$. Since μ is ω-continuous at the second variable, so, by Corollary 3.1, the left translation function l_a is an ω-irresolute, for each $a \in X$. Now, for $a, b, c \in \omega ClS$, we have $a, (b * c), (a * b), c \in \omega ClS$. Therefore,

$a*(b*c),(a*b)*c \in \omega ClS$. Since $(X,*)$ is a group and $a,b,c \in X$, so, $a*(b*c)=(a*b)*c$. This means that $a*(b*c)=(a*b)*c$ in ωClS. Therefore, ωClS is a semigroup subset of X. □

Theorem 3.11. *Let H be a subgroup of a group $(X,*)$. Let τ be a topology on X such that the function $f : X \to X$ given by $f(x) = x^{-1}$ is ω-continuous. If the multiplication map μ is ω-continuous at either first or second variable, then ωClH is a subgroup of X.*

Proof. Since every subgroup of a group is a semigroup, so, by Theorem 3.10, ωClH is a semigroup subset of X. For all $a, b \in \omega ClH$, we get $a*b \in \omega ClH$. Since $f : X \to X$ given by $f(x) = x^{-1}$ is a bijective $\omega - continuous$ function, so, f is ω-irresolute, for each $a \in \omega ClH$, and any $\omega - open$ subset V of X such that $a^{-1} \in V$, we have $f(a) \in V$. So, by $\omega - irresolute$ of f, there exists an ω-open subset U of X such that $a \in U$ and $f(U) \subseteq V$. Since $a \in U$ and $a \in \omega ClH$, so, $U \cap H \neq \phi$. Thus, $(U \cap H)^{-1} = \phi$. Since $(U \cap H)^{-1} = U^{-1} \cap H^{-1} = U^{-1} \cap H$, so, $U^{-1} \cap H \neq \phi$, but $U^{-1} \subseteq V$, so, $V \cap H \neq \phi$. Hence, $a^{-1} \in \omega ClH$. Therefore, for each $a, b \in \omega ClH$, we have $a, b^{-1} \in \omega ClH$, and so, $a*b^{-1} \in \omega ClH$. This means that ωClH is a subgroup of X. □

Remark 3.2. It is easy to prove the same result for a topology on the group $(X,*)$ which makes the multiplication map μ as $\omega - continuous$ at the first variable, but, we need only to replace a with b, $a*V$ with $V*a$.

Theorem 3.12. *Let $(X,*)$ be a group and τ be any topology on X such that the multiplication map μ is ω-continuous at each variable. If S is a normal set of X such that $\omega IntS \neq \phi$, then both $\omega Int(S)$ and $\omega Cl(S)$ are normal.*

Proof. Let $x \in X$. Then, $x^{-1} \in X$. Since $\omega Int(S)$ is ω-open and μ is ω-continuous at each variable, then by (3) of Theorem 3.6 and as μ is ω-continuous at the first variable, we obtain that $x*\omega Int(S)x^{-1}$ is ω-open in X and $\omega Int(x*\omega Int(S)*x^{-1}) = x*\omega Int(S)*x^{-1}$. Since S is a normal set, so $x*\omega Int(S)x^{-1} \subseteq x*S*X^{-1} \subseteq S$, so $\omega Int(x*\omega Int(S)x^{-1}) \subseteq \omega Int(S)$. Therefore, $x*\omega Int(S)x^{-1} \subseteq \omega Int(S)$. Hence, $\omega Int(S)$ is a normal subset of X.

Now, we have to show $\omega Cl(S)$ is also a normal subset of X. To make this end, let $y \in x*\omega Cl(S)*x^{-1}$ and G be any ω-open subset of X such that $y \in G$. Then, there is $s \in \omega Cl(S)$ such that $y = x*s*x^{-1}$ by Proposition 3.2 there exists an ω-open subset V of X such that $s*x^{-1} \in V$ and $x*V \subseteq G$. Again by Proposition 3.2 there is an ω-open subset U in X such that $s \in U$ and $U*x^{-1} \subseteq V$. That is, $x*U*x^{-1} \subseteq x*V \subseteq G$. Now, since $s \in U$ and $s \in \omega Cl(S)$, then $U \cap S \neq \phi$, so, $(x*U*x^{-1}) \cap (x*S*x^{-1}) \neq \phi$. Since $(x*U*x^{-1}) \subseteq G$ and $x*S*x^{-1} \subseteq S$, so, $G \cap S \neq \phi$. This implies that $y \in \omega Cl(S)$. Thus, $x*\omega Cl(S)*x^{-1} \subseteq \omega Cl(S)$. Hence, $\omega Cl(S)$ is a normal subset of X. □

Corollary 3.2. *Let τ be a topology on a group $(X,*)$ such that the multiplication map is ω-continuous at the first (second) variable. If H is a normal subgroup* *of X and the function $f : X \to X$ given by $f(x) = x^{-1}$ for all $x \in X$, is ω-continuous, then $\omega IntH \neq \phi$ and ωClH both are normal subgroup of X.*

Proof. The proof follows from Theorem 3.9, Theorem 3.11 and Theorem 3.12. □

Theorem 3.13. *Let $(X, *)$ be a group and τ be any topology on X. If the multiplication map is ω-continuous in the second variable, then for any $A \subseteq X$, we have $A * B$ is $\omega -$ open for any open set $B \subseteq X$.*

Proof. If B is open, then $Int(B) = B$. Let $a \in A$. Then, $a * B = a * Int(B) \subseteq \omega Int(a * B)$ by (1) of Theorem 3.4. Hence, $A * B = A * Int(B) = \bigcup_{a\in A} a * Int(B) \subseteq \bigcup_{a\in A} \omega Int(a * B)$, so, $\bigcup_{a\in A} \omega Int(a * B) \subseteq \omega Int(\bigcup_{a\in A} a * B) = \omega Int(A * B)$, since $\omega Int(A) \cup \omega Int(B) \subseteq \omega Int(A * B)$, $A * B \subseteq \omega Int(A * B)$. Hence, $A * B = \omega Int(A * B)$. Thus, $A * B$ is $\omega -$ *open*. □

Theorem 3.14. *Let the multiplication map μ of a group $X$with a topology τ on X is ω-continuous at the second (first) variable and $H \subseteq X$. Then:*

1. *If H is an ω-compact subset of X, then $a * H$ $(H * a)$ is a compact subset of X, for each $a \in X$.*
2. *If μ is ω-continuous at the second variable, then for each $a \in X$, where, H is ω-compact in X if and only if $a * H$ is ω-compact.*
3. *If μ is ω-continuous at the first variable, then for each $a \in X$, where, H is ω-compact in X if and only if $H * a$ is ω-compact.*

Proof. 1. Let H be an ω-compact subset of X and without loss of generality, we suppose that μ is ω-continuous at the second variable, so, by Theorem 3.5, ι_a is ω-continuous for each $a \in X$. Now, to show $a * H$ is compact. Let $\{\{V_\alpha\}_n; \alpha \in n\}$ be an open cover of $a * H$. Then, $(\iota_a)^{-1}(V_\alpha) = a^{-1} * V_\alpha$ is ω-open for each $\alpha \in \Lambda$. Since $H = (\iota_a)^{-1})(a * H) \subseteq (\iota_a)^{-1})(\bigcup_{\alpha\in\Lambda} V_\alpha) = \bigcup_{\alpha\in\Lambda}(\iota_a)^{-1})(V_\alpha) = \bigcup_{\alpha\in\Lambda} a^{-1} * V_\alpha$, so, $\{a^{-1} * (V_\alpha); \alpha \in \Lambda\}$ is an ω-open cover of H. So, by definition ω-compact, there exists a finite subset Λ_0 of Λ, such that $H \subseteq \bigcup_{\alpha\in\Lambda_0}(a^{-1} * V_\alpha)$. Hence, $a * H = \iota_a(H) \subseteq \iota_a(\bigcup_{\alpha\in\Lambda_0}(a^{-1} * V_\alpha) = a * (\bigcup_{\alpha\in\Lambda_0} a^{-1} * V_\alpha) = (a * a^{-1}) * (\bigcup_{\alpha\in\Lambda_0} V_\alpha = \bigcup_{\alpha\in\Lambda_0} V_\alpha$. Thus, $a * H$ is a compact subset of X.

2. Let H be an ω-compact subset of X and $\{G_N : N \in \Lambda\}$ is an ω-open cover of $a*H$ where a is an arbitrary point of X. Since μ is ω-continuous at the second variable, so, by Corollary 3.1 ι_a is an ω-irresolute function. Therefor $(\iota_a)^{-1}(G)$ is ω-open in X, for each $N \in \Lambda$. Since $(\iota_a)^{-1}(G) = a^{-1} * G_N$. So, $a^{-1} * G_N$ is ω-open in X for each $N \in \Lambda$. Since $H = (\iota_a)^{-1}(a * H) \subseteq (\iota_a)^{-1}(\bigcup_{N\in\Lambda} G_N) = \bigcup_{N\in\Lambda}(\iota_a)^{-1}(G_N) = \bigcup_{N\in\Lambda}(a^{-1} * G_N)$. So, $\{a^{-1} * G_N : N \in \Lambda\}$ is an ω-open cover of H. Since H is ω-compact, so, there exists a finite subset Λ_0 of Λ such that $H = U_{N\in\Lambda_0}(a^{-1} * G_N)$. So, $aH = \iota_a(H) \subseteq \iota_a(\bigcup_{N\in\Lambda_0} a^{-1} * G_N) = \bigcup_{N\in\Lambda_0} \iota a(a^{-1} * G_N) = \bigcup_{N\in\Lambda} G_N$. Hence, $a * H$ is ω-compact.

Conversely, let $a * H$ be an ω-compact subset of X where $a \in X$. To show H is ω-compact.

Let $\{O_N, N \in \Lambda\}$ be any ω-open set in H. Then $a * H = \iota_a(H) = (\iota_a)^{-1})^{-1}(H) = (\iota_a^{-1})^{-1}(\bigcup_{N\in\Lambda} O_N) = (\bigcup_{N\in\Lambda}((\iota_a^{-1})^{-1}(O_N)) = (\bigcup_{N\in\Lambda} \iota_a(O_N) = \bigcup_{N\in\Lambda}(a * O_N)$.

Since O_N is $\omega - open$ for each $N \in \Lambda$ and $a \in X$, so, by part (3) of Theorem 3.6, we have $a * O_N$ is ω-open in X, for each $N \in \Lambda$. Since $a * H$ is ω-compact, so, there exists a finite subset Λ_0 of Λ such that $a * H \subseteq \bigcup_{N\in\Lambda}(a * O_N)$. So, $H = a^{-1} * (a * H) = \iota_a^{-1}(a * H) = (\iota_a)^{-1}(a * H) \subseteq (\iota_a)^{-1}(\bigcup_{N\in\Lambda_0}(a * O_N))$= $(\bigcup_{N\in\Lambda_0}(\iota_a)^{-1})(a * O_N)$=$\bigcup_{N\in\Lambda_0} a^{-1} * (a * O_N) = \bigcup_{N\in\Lambda_0} O_N$. Hence, H is ω-compact.

3. The proof is similar to part 2 with only replacing ι_a with r_a. □

Theorem 3.15. *Let τ be a topology on a group $(X, *)$ and $a \in X$, $H \subseteq X$.*

1. *If μ is $\omega - continuous$ at the second variable, then H is lindelof if and only if $a * H$ is lindelof.*
2. *If μ is $\omega - continuous$ at the first variable, then H is lindelof if and only if $H * a$ is lindelof.*

Proof. The Proof is almost similar to the proof of parts (2) and (3) of the Theorem 3.14 by using Lemma 2.1. □

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