Prime-valent one-regular graphs of order $18p$

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Abstract. A graph is *one-regular* and *arc-transitive* if its full automorphism group acts on its arcs regularly and transitively, respectively. In this paper, we classify connected one-regular graphs of prime valency and order $18p$ for each prime p. As a result there are two infinite families of such graphs, one is the cycle C_{18p} with valency two and the other is the normal Cayley graph on the generalized dihedral group $(\mathbb{Z}_{3p}\times\mathbb{Z}_3)\rtimes\mathbb{Z}_2$ with valency three and $p \equiv 1 \pmod 6$.
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1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [21, 22] or [2, 3], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be G-*vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An s-*arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be (G,s)-*arc-transitive* or (G,s)-*regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G,s)-arc-

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transitive graph is said to be (G, s)-*transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is called G-*symmetric*. A graph X is simply called s-*arc-transitive*, s-*regular* or s-*transitive* if it is $(\mathrm{Aut}(X), s)$-arc-transitive, $(\mathrm{Aut}(X), s)$-regular or $(\mathrm{Aut}(X), s)$-transitive, respectively.

We denote by C_n and K_n the cycle and the complete graph of order n, respectively. Denote by D_{2n} the dihedral group of order $2n$. As we all known that there is only one connected 2-valent graph of order n, that is, the cycle C_n, which is 1-regular with full automorphism group D_{2n}. Let p be a prime. Classifying s-transitive and s-regular graphs has received considerable attention. The classification of s-transitive graphs of order p and $2p$ was given in [6] and [7], respectively. Pan [20] characterized the prime-valent s-transitive graphs of square free order. Kutnar [17] classified cubic symmetric graphs of girth 6 and Oh [19] determined arc-transitive elementary abelian covers of the Pappus graph. The classification of pentavalent and heptavalent s-transitive graphs of order $18p$ was given in [1] and [13], respectively.

For 2-valent case, s-transitivity always means 1-regularity, and for cubic case, s-transitivity always means s-regularity by Miller [11]. However, for the other prime-valent case, this is not true, see for example [14] for pentavalent case and [15] for heptavalent case. Thus, characterization and classification of prime-valent s-regular graphs is very interesting and also reveals the s-regular global and local actions of the permutation groups on s-arcs of such graphs. In particular, 1-regular action is the most simple and typical situation. In this paper, we classify prime-valent one-regular graphs of order $18p$ for each prime p.

2. Preliminary results

Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [18, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$ and prime valency $q \geq 3$, and let N be a normal subgroup of G. Then, one of the following holds:*

(1) *N is transitive on $V(X)$;*
(2) *X is bipartite and N is transitive on each part of the bipartition;*
(3) *N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected q-valent G/N-symmetric graph.*

To extract a classification of connected prime-valent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [7], we introduce the graphs

$G(2p,q)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V=\{0,1,\cdots,p-1\}$ and $V'=\{0',1',\cdots,(p-1)'\}$. Let q be a positive integer dividing $p-1$ and $H(p,q)$ the unique subgroup of Z_p^* of order q. Define the graph $G(2p,q)$ to have vertex set $V\cup V'$ and edge set $\{xy' \mid x-y\in H(p,q)\}$.

Proposition 2.2. *Let X be a connected q-valent symmetric graph of order $2p$ with p,q primes. Then, X is isomorphic to K_{2p} with $q=2p-1$, $K_{p,p}$ or $G(2p,q)$ with $q \,\big|\, (p-1)$. Furthermore, if $(p,q)\neq(11,5)$ then $\mathrm{Aut}(G(2p,q)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_q)\rtimes \mathbb{Z}_2$; if $(p,q)=(11,5)$ then $\mathrm{Aut}(G(2p,q))=\mathrm{PGL}(2,11)$.*

The following proposition is about the prime-valent symmetric graphs of order $6p$ with p a prime, which is deduced from [20, Theorem 1.2].

Proposition 2.3. *Let p and q be two primes. If $q>7$, then there is no q-valent symmetric graph of order $6p$ admitting a solvable arc-transitive automorphism group.*

The following proposition is the famous "N/C-Theorem", see for example [16, Chapter I, Theorem 4.5]).

Proposition 2.4. *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H)$ of H.*

From [10, p.12-14], we can deduce the non-abelian simple groups whose orders have at most three different prime divisors.

Proposition 2.5. *Let G be a non-abelian simple group. If the order $|G|$ has at most three different prime divisors, then G is called K_3-simple group and isomorphic to one of the following groups.*

Table 1: **Non-abelian simple $\{2,3,p\}$-groups**

Group	Order	Group	Order
A_5	$2^2\cdot3\cdot5$	$\mathrm{PSL}(2,17)$	$2^4\cdot3^2\cdot17$
A_6	$2^3\cdot3^2\cdot5$	$\mathrm{PSL}(3,3)$	$2^4\cdot3^3\cdot13$
$\mathrm{PSL}(2,7)$	$2^3\cdot3\cdot7$	$\mathrm{PSU}(3,3)$	$2^5\cdot3^3\cdot7$
$\mathrm{PSL}(2,8)$	$2^3\cdot3^2\cdot7$	$\mathrm{PSU}(4,2)$	$2^6\cdot3^4\cdot5$

3. Classification

This section is devoted to classifying prime-valent one-regular graphs of order $18p$ for each prime p. Let q be a prime. In what follows, we always denote by X a connected q-valent one-regular graph of order $18p$. Set $A=\mathrm{Aut}(X)$, $v\in V(X)$. Then, the vertex stabilizer $A_v\cong\mathbb{Z}_q$ and hence $|A|=18pq$.

Now, we first deal with the case $q \leq 7$. Clearly, any connected graph of order $18p$ and valency two is isomorphic to the cycle C_{18p}. Thus, for $q = 2$, $X \cong C_{18p}$ and $A \cong D_{36p}$. Let $q = 3$. Then, by [17, Theorem 1.2] and [19, Theorem 3.4], $X \cong CF_{18p}$ is a $\mathbb{Z}_p$-cover of the Pappus graph and also a normal Cayley graph of a generalized dihedral group $(\mathbb{Z}_{3p}\times\mathbb{Z}_3)\rtimes\mathbb{Z}_2$ with $p \equiv 1 (\text{mod } 6)$. This implies that $A \cong ((\mathbb{Z}_{3p}\times\mathbb{Z}_3)\rtimes\mathbb{Z}_2) \rtimes \mathbb{Z}_3$. If $q = 5$ or 7, then by [1, Theorem 4.1] for $q = 5$ and [13, Theorem 3.1] for $q = 7$, there is no q-valent one-regular graph of order $18p$. Thus, in what follows we deal with the case $q > 7$. The next lemma is about the case $p = 2$.

Lemma 3.1. *Let X be a connected q-valent one-regular graph of order 36. Then, $X \cong C_{36}$.*

Proof. Since $|V(X)| = 36$, we have that $p = 2$. If $q \leq 7$, then by the above argument, the only possibility is $q = 2$ and X is isomorphic to the cycle C_{36}.

Let $q > 7$. Then, $|A| = 2^2{\cdot}3^2{\cdot}q$. If A is non-solvable, then A has a composition factor isomorphic to a non-abelian simple group and hence this composition factor has order dividing $|A| = 2^2{\cdot}3^2{\cdot}q$. This forces that this composition factor is a K_3-simple group. By Proposition 2.5, A has a composition factor isomorphic to A_5 and $q = 5$, contrary to our assumption. Thus, A is solvable. Let N be a minimal normal subgroup of A. Then, $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_3$, $\mathbb{Z}_3^2$ or $\mathbb{Z}_q$. Clearly, N is not transitive on $V(X)$. By Proposition 2.1, X_N is a q-valent symmetric graph of order $36/|N|$. Note that, $q > 7$ and there is no connected regular graph of odd order and odd valency. Thus, N is not isomorphic to $\mathbb{Z}_2^2$ or $\mathbb{Z}_q$.

Suppose that $N \cong \mathbb{Z}_2$. Then, X_N has order 18 and valency q. Since $q > 7$ is a prime, by [8], X_N is isomorphic to Pappus graph with $q = 3$ or the complete graph K_{18} with $q = 17$. For the former, X is a cubic symmetric graph of order 36. However, by [9], there is no cubic symmetric graph of order 36, a contradiction. For the latter, $A/N \lesssim \mathrm{Aut}(K_{18}) \cong \mathrm{S}_{18}$. Recall that $|A| = 2^2{\cdot}3^2{\cdot}q$. We have $|A/N| = 18{\cdot}17$. However, by Magma [4], S_{18} has no subgroup of order $18{\cdot}17$, a contradiction.

Suppose that $N \cong \mathbb{Z}_3$. Then, X_N is a q-valent symmetric graph of order 12. By [8], $X_N \cong K_{12}$ with $q = 11$ because $q > 7$. It follows that $A/N \lesssim \mathrm{Aut}(K_{12}) \cong \mathrm{S}_{12}$. However, $|A/N| = 12{\cdot}11$ and by Magma [4], S_{12} has no subgroup of order $12{\cdot}11$, a contradiction.

Suppose that $N \cong \mathbb{Z}_3^2$. Then, X_N is a q-valent symmetric graph of order 4. Clearly, the only symmetric graphs of order 4 are C_4 with valency 2 and K_4 with valency 3. This is impossible because the valency $q > 7$. □

Finally, we treat with the case $p \geq 3$ and $q > 7$.

Lemma 3.2. *Let $p \geq 3$ and $q > 7$. Then, there is no new graph.*

Proof. Since $p \geq 3$ and $q > 7$, we have that $|A| = 18pq = 2{\cdot}3^2{\cdot}p{\cdot}q$ is twice an odd integer. It follows that A has a normal subgroup of odd order and index 2. By Feit-Thompson's Theorem [12, Theorem], any group of odd order is solvable and so A is also solvable. Let N be a minimal normal subgroup of A. Then, N is also solvable and hence N is isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_3^2$, $\mathbb{Z}_p$, $\mathbb{Z}_q$ or $\mathbb{Z}_p^2$ with $p = q$. By Proposition 2.1, X_N is a q-valent symmetric graph of order $9p$, $6p$, $2p$ or 18. Since there is no connected regular graph of odd order and odd valency, we have that $N \ncong \mathbb{Z}_2$. If $p \neq q$ and $N \cong \mathbb{Z}_q$, then X_N has order $18p/q$. This is impossible because q cannot divide $18p$. If $p = q$ and $N \cong \mathbb{Z}_p^2$, then $N_v \cong \mathbb{Z}_q = \mathbb{Z}_p$. However, by Proposition 2.1, X_N has order 18 and N is semiregular on $V(X)$. This forces that $N_v = 1$, a contradiction. Thus, $N \cong \mathbb{Z}_3$, $\mathbb{Z}_3^2$, $\mathbb{Z}_p$.

Let $N \cong \mathbb{Z}_3$. Then, X_N is a q-valent symmetric graph of order $6p$ and $A/N \lesssim \mathrm{Aut}(X_N)$. Recall that A is solvable. Thus, A/N is also solvable and acts arc-transitively on X_N. However, by Proposition 2.3, there is no q-valent symmetric graph admitting a solvable arc-transitive automorphism group with $q > 7$, a contradiction.

Let $N \cong \mathbb{Z}_p$. Then, X_N is a q-valent symmetric graph of order 18. By [8], there is only one q-valent symmetric graph of order 18 with $q > 7$, that is, the complete graph K_{18} and hence $q = 17$. It follows that $A/N \lesssim \mathrm{Aut}(K_{18}) \cong \mathrm{S}_{18}$ and $|A/N| = 2{\cdot}3^2{\cdot}17$. However, S_{18} has no subgroup of order $2{\cdot}3^2{\cdot}17$ by Magma [4], a contradiction.

Let $N \cong \mathbb{Z}_3^2$. Then, X_N is a q-valent symmetric graph of order $2p$. By Proposition 2.2, X_N is isomorphic to K_{2p} with $q = 2p-1$ a prime, $K_{p,p}$ with $q = p$ or $G(2p, q)$ with $q \mid (p-1)$.

Suppose that $X_N \cong K_{2p}$. Then, A/N has order $2{\cdot}p{\cdot}q$ and acts 2-transitively on $V(X_N)$. By Burnside's Theorem [5, p.192, Theorem IX], any 2-transitive permutation group is either almost simple or affine. Since A is solvable, A/N is also solvable. It forces that A/N is affine and hence A/N has a normal subgroup $M/N \cong \mathbb{Z}_p$. Note that, $N \cong \mathbb{Z}_3^2$. By Proposition 2.4, $M/C_M(N) \lesssim \mathrm{Aut}(N) \cong \mathrm{Aut}(\mathbb{Z}_3^2) \cong \mathrm{GL}(2,3)$. Since $|\mathrm{GL}(2,3)| = 48$ and $q = 2p-1 > 7$, we have that $C_M(N) = M$ and hence $M \cong \mathbb{Z}_3^2 \times \mathbb{Z}_p$. It follows that M has a characteristic subgroup $K \cong \mathbb{Z}_p$. The normality of M in A implies that K is also normal in A. By Proposition 2.1, X_K is a q-valent symmetric graph of order 18 with $q > 7$, and by [8], $X_K \cong K_{18}$ with $q = 17$. Recall that $q = 2p-1$. This forces that $p = 9$ is not a prime, a contradiction.

Suppose that $X_N \cong K_{p,p}$. Then, $p = q$ and $|A/N| = 2{\cdot}p^2$. Since $p > 7$, we have that A/N has a normal subgroup M/N of order p^2. Note that, $A/N \lesssim \mathrm{Aut}(K_{p,p}) \cong \mathrm{S}_p \,\mathrm{wr}\, \mathrm{S}_2$. Thus, a Sylow p-subgroup of A/N is isomorphic to $\mathbb{Z}_p^2$ and so $M/N \cong \mathbb{Z}_p^2$. By Proposition 2.4, $M/C_M(N) \lesssim \mathrm{Aut}(N) \cong \mathrm{GL}(2,3)$. Since $|\mathrm{GL}(2,3)| = 48$ and $p > 7$, we have that $C_M(N) = M$. This forces that $M \cong \mathbb{Z}_p^2 \times \mathbb{Z}_3^2$ has a characteristic subgroup $P \cong \mathbb{Z}_p^2$. By Proposition 2.1, X_P has order 18 and hence P is semiregular on $V(X)$. Clearly, this is impossible because $q = p$ and $P_v \cong \mathbb{Z}_p$.

Suppose that $X_N \cong G(2p, q)$. Then, $q \mid (p-1)$ and $A/N \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$. Similarly, by Proposition 2.4, we can easily deduce that A has a normal subgroup $P \cong \mathbb{Z}_p$. It follows that the quotient graph X_P has order 18 and is isomorphic to K_{18}. With a similar argument as the case "$N \cong \mathbb{Z}_p$", we have A/P has order $2{\cdot}3^2{\cdot}17$ and cannot be embedded in $\mathrm{Aut}(K_{18}) \cong \mathrm{S}_{18}$, a contradiction. □

Combining the above arguments with the cases $q = 2, 3, 5, 7$, and Lemmas 3.1-3.2, we have the following result.

Theorem 3.1. *Let p, q be two primes and X a connected q-valent one-regular graph of order $18p$. Then, the only possibilities are $q = 2, 3$ and furthermore,*

(1) *for $q = 2$, $X \cong C_{18p}$ and $A \cong D_{36p}$;*
(2) *for $q = 3$, $X \cong CF_{18p}$ and $A \cong ((\mathbb{Z}_{3p} \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3$ with $p \equiv 1 (\mathrm{mod}\ 6)$.*

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