On k-perfect polynomials over $\mathbb{F}_2$

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Abstract. A polynomial A is called $k-$perfect over the finite field $\mathbb{F}_2$ if the sum of the k^{th} powers of all distinct divisors of A equals A^k, where k is a positive integer. We show that a $k-$perfect polynomial A over $\mathbb{F}_2$ must be even when $k = 2^n$, n is a non-negative integer, and we characterize all 2^n-perfect polynomials over $\mathbb{F}_2$ that are of the form $x^a(x+1)^b \prod_{i=1}^{r} P_i^{h_i}$, where each P_i is a Mersenne prime and a, b and h_i are positive integers.

Keywords: sum of divisors, multiplicative function, polynomials, finite fields, characteristic 2.

1. Introduction

Let n be a positive integer and let $\sigma(n)$ denote the sum of positive divisors of the integer n. We call the number n a $k-$super perfect number if $\sigma^k(n) = \underbrace{\sigma(\sigma(...(\sigma}_{k-\text{times}}(n)))) = 2n$. When $k = 1$, n is called a perfect number. An integer $M = 2^p - 1$, where p is a prime number, is called a Mersenne number. It is also well known that an even integer n is perfect if and only if $n = M(M+1)/2$ for some Mersenne prime number M. Suryanarayana [11] considered $k-$super perfect numbers in the case $k = 2$. Numbers of the form 2^{p-1} (p is prime) are

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2-super perfect if $2^{p-1}-1$ is a Mersenne prime. It is not known if there are odd $k-$super perfect numbers.

Researchers also studied the arithmetic function $\sigma_k(n)$ that finds the sum of the kth powers of the positive divisors of n. Recently, Luca and Ferdinands [10] showed that $\sigma_k(n)$ is divisible by n for infinitely many n when $k \geq 2$. Cai et al. [1] proved that if $n = 2^{a-1}p$ divides $\sigma_3(n)$, where $a > 1$ is an integer and p is an odd prime, then n is an even perfect number. Also, they proved that the converse is true when $n \neq 28$. Jiang [9] made an improvement to the result of Cai et al. They showed that $n = 2^{a-1}p^{b-1}$ divides $\sigma_3(n)$, where $a, b > 1$ are integers and p is an odd prime, if and only if n is an even perfect number other than 28. Chu [3] found a relation between an even perfect number n and $\sigma_k(n)$. He generalized the work of Cai et al. as given in the following theorem.

Theorem 1.1. *Let $k > 2$ be a prime such that $2^k - 1$ is a Mersenne prime. If $n = 2^{a-1}p$, where $a > 1$ and $p < 3{\cdot}2^{a-1} - 1$ is an odd prime. Then n divides $\sigma_k(n)$ if and only if n is an even perfect number other than $2^{k-1}(2^k - 1)$.*

Chu also generalized the work of Jiang as follows.

Theorem 1.2. *If $n = 2^{a-1}p^{b-1}$, where $a, b > 1$ and $p < 3{\cdot}2^{a-1} - 1$ is an odd prime. Then n divides $\sigma_5(n)$ if and only if n is an even perfect number other than 496.*

Chu conjectured if $k > 2$ is a prime such that $2^k - 1$ is a Mersenne prime and if $n = 2^{a-1}p^{b-1}$, where $a, b > 1$ and $p < 3.2^{a-1} - 1$ is an odd prime, then n divides $\sigma_k(n)$ if and only if n is an even perfect number other than $2^{k-1}(2^k - 1)$.

The present paper gives a polynomial analogue of the arithmetic function $\sigma_k(n)$. Let k be a positive integer and let A be a nonzero polynomial defined over the prime field $\mathbb{F}_2$. We denote by $\sigma_k(A)$ the sum of the k^{th} powers of the distinct divisors B of A. That is,

$$\sigma_k(A) = \sum_{B|A} B^k.$$

If $A \in \mathbb{F}_2[x]$ has the canonical decomposition $\prod_{i=1}^{r} P_i^{\alpha_i}$ where the primes $P_i \in \mathbb{F}_2[x]$ are distinct and $\alpha_i > 0$, then

$$\sigma_k(A) = \prod_{i=1}^{r} \frac{P_i^{k(\alpha_i+1)} - 1}{P_i^k - 1}.$$

In the case where $k = 1$, σ_k becomes the well-known σ function. For example, if $A = x(x+1)^2(x^2+x+1) \in \mathbb{F}_2[x]$ then

$$\begin{aligned}\sigma(A) &= \sum_{B|A} B \\ &= 1 + x + (x+1) + (x+1)^2 + (x^2+x+1) + x(x+1) + x(x+1)^2\end{aligned}$$

$$
\begin{aligned}
&+ x(x^2+x+1) + (x+1)(x^2+x+1) + (x+1)^2(x^2+x+1) \\
&+ x(x+1)(x^2+x+1) + x(x+1)^2(x^2+x+1) \\
&= x(x+1)^2(x^2+x+1)
\end{aligned}
$$

and

$$\sigma_4(A) = \sum_{B|A} B^4 = x^4(x+1)^8(x^2+x+1)^4.$$

Note that the function σ_k is multiplicative over $\mathbb{F}_2$.

Notation 1.1. *We use the following notations throughout the paper.*

- $\deg(A)$ *denotes the degree of the polynomial* A*.*
- $\overline{A}$ *is the polynomial obtained from* A *with* x *replaced by* $x+1$*, that is* $\overline{A}(x) = A(x+1)$*.*
- A^* *is the inverse of the polynomial* A *with* $\deg(A) = m$*, in this sense* $A^*(x) = x^m A(\frac{1}{x})$*.*
- P *and* Q *are distinct irreducible odd polynomials.*

A nonzero polynomial A defined over $\mathbb{F}_2$ is an even polynomial if it has a linear factor in $\mathbb{F}_2[x]$ else it is an odd polynomial. A polynomial T of the form $1 + x^a(x+1)^b$ with $\gcd(a,b) = 1$ is called a Mersenne polynomial, see [6]. The first five Mersenne polynomials over $\mathbb{F}_2$ are: $T_1 = 1+x+x^2$, $T_2 = 1+x+x^3$, $T_3 = 1+x^2+x^3, T_4 = 1+x+x^2+x^3+x^4$, $T_5 = 1+x^3+x^4$. Note that all these polynomials are irreducible, so we call them Mersenne primes.

The next definition is the main object of this study in which we introduce a new concept of $k-$perfect polynomials over $\mathbb{F}_2$.

Definition 1.1. *Let k be a positive integer. A polynomial A is called a $k-$perfect polynomial over $\mathbb{F}_2$ if $\sigma_k(A) = A^k$.*

A $1-$perfect polynomial A over $\mathbb{F}_2$ is a perfect polynomial, so we are interested in studying the case when $k > 1$. The polynomial $B = x(x+1)^2(x^2+x+1)$ is a $4-$perfect polynomial in $\mathbb{F}_2[x]$. Note that B is a perfect polynomial over $\mathbb{F}_2$. A natural question arise: Is there a relation between perfect polynomials and $k-$perfect polynomials in $\mathbb{F}_2[x]$? In Section 3, we answer this question and we find a relation between the sum of the divisors function $\sigma(A)$ and the sum of the powers of the divisors function $\sigma_k(A)$, $k > 1$, of the polynomial A over the finite field $\mathbb{F}_2$. We show that there are no odd 2^n-perfect polynomials over $\mathbb{F}_2$ and we characterize all even 2^n-perfect polynomials over $\mathbb{F}_2$ that have the form $x^a(x+1)^b \prod_{i=1}^{r} P_i^{h_i}$, where each P_i is a Mersenne prime and a, b and h_i are positive integers.

Our main result is given in the following theorem:

Theorem 1.3. *Let $a, b, t, h_i \in \mathbb{N}$ and let P_i be a Mersenne prime in $\mathbb{F}_2[x]$. Then, $A = x^a(x+1)^b \prod_{i=1}^{r} P_i^{h_i}$ is a 2^n-perfect polynomial over $\mathbb{F}_2$ for some $n \in \mathbb{N}$ if and only if $A \in \{x^{2^t-1}(x+1)^{2^t-1}, x^2(x+1)T_1, x(x+1)^2T_1, x^3(x+1)^4T_5, x^4(x+1)^3T_4, x^4(x+1)^4T_4T_5, x^6(x+1)^3T_2T_3, x^3(x+1)^6T_2T_3, x^6(x+1)^4T_2T_3T_5, x^4(x+1)^6T_2T_3T_5\}$.*

2. Preliminaries

The notion of perfect polynomials over $\mathbb{F}_2$ was introduced first by Canaday [2]. A polynomial A is perfect if $\sigma(A) = A$. Let $\omega(A)$ be the number of distinct irreducible polynomials that divide A. Canaday studied the case of even perfect polynomials with $\omega(A) \leq 3$. In the recent years, Gallardo and Rahavandrainy [4, 6, 7] showed the non-existence of odd perfect polynomials over $\mathbb{F}_2$ with either $\omega(A) = 3$ or with $\omega(A) \leq 9$ in the case where all the exponents of the irreducible factors of A are equal to 2. If the nonconstant polynomial A in $\mathbb{F}_2[x]$ is perfect, then $\omega(A) \geq 2$ (see [4], Lemma 2.3). Moreover, Canaday [2] showed that the only even perfect polynomials over $\mathbb{F}_2$ with exactly two prime divisors are $x^{2^n-1}(x+1)^{2^n-1}$ for some positive integers n.

It is well known that an even perfect number is exactly divisible by two distinct prime numbers but a non-trivial even perfect polynomial $A \in \mathbb{F}_2[x]$ may be divisible by more than 2 distinct primes as Gallardo and Rahavandrainy [6] gave some results with $\omega(A) \leq 5$. Although they did not give a general form of such polynomials in terms of Mersenne primes but all the non-trivial even perfect polynomials they found, with only two exceptions, have Mersenne primes as odd divisors.

The following two lemmas are useful.

Lemma 2.1 (Lemma 2.3 in [6]). *If $A{=}A_1A_2$ is perfect over $\mathbb{F}_2$ and if* $\gcd(A_1, A_2) = 1$*, then A_1 is perfect if and only if A_2 is perfect.*

Lemma 2.2 (Lemma 2.4 in [6]). *If A is perfect over $\mathbb{F}_2$, then the polynomial $\overline{A}$ is also perfect over $\mathbb{F}_2$*

In [5], Gallardo and Rahavandrainy gave a complete list for all even perfect polynomials with at most 5 irreducible factors as given in the following lemma.

Lemma 2.3. *The complete list of all even perfect polynomials over $\mathbb{F}_2$ with $\omega(A) \leq 5$ is:*

$\omega(A)$	A
0	0
1	1
2	$(x^2+x)^{2^n-1}$
3	$A_1 = x^2(x+1)T_1, A_2 = \overline{A_1}(x), A_3 = x^3(x+1)^4T_5, A_4(x) = \overline{A_3}$
4	$A_5 = x^2(x+1)(x^4+x+1)T_1^2, A_6 = \overline{A_5},$
	$A_7 = x^4(x+1)^4T_4T_5, A_8 = x^6(x+1)^3T_2T_3, C_9(x) = \overline{A_8}$
5	$A_{10} = x^6(x+1)^4T_2T_3T_5,\ A_{11} = \overline{A_{10}}$.

Lemma 2.4 (Proposition 5.1 in [6]). *If P is an odd irreducible polynomial in $\mathbb{F}_2[x]$, then $x(x+1)$ divides $\sigma(P^{2m-1})$ for $m \in \mathbb{N}$.*

The following lemma shows a nice relation between $\sigma_k(A)$ and $(\sigma(A))^k$ when A has exactly one prime factor.

Lemma 2.5. *Let $A = P^\alpha \in \mathbb{F}_2[x]$ with $\alpha \geq 1$. Then $\sigma_k(A) = \sigma(A)^k$ if and only if $k = 2^n$.*

Proof.

$$\sigma_{2^n}(A) = 1 + P^{2^n} + ... + P^{2^n\alpha} = (1 + P + ... + P^\alpha)^{2^n} = (\sigma(A))^{2^n}.$$

For the sufficient condition, the proof is done by contrapositive. Let $k = 2^n u$, $u > 1$ is odd, then $(\sigma(A))^k = (\sigma(A))^{2^n u} = (1 + P + ... + P^\alpha)^{2^n u} = (1 + P^{2^n} + ... + P^{2^n\alpha})^u \neq (1 + P^{2^n u} + ... + P^{2^n u\alpha}) = \sigma_k(A)$. □

Corollary 2.1. *Let $A = \prod_{i=1}^r P_i^{\alpha_i} \in \mathbb{F}_2[x]$, then $\sigma_{2^n}(A) = (\sigma(A))^{2^n}$.*

Lemma 2.6. *Let $A = P^\alpha \in \mathbb{F}_2[x]$ be an irreducible polynomial and $\alpha \geq 1$. Then A is not a factor of $\sigma_k(A)$.*

Proof. Assume that A divides $\sigma_k(A)$, then there exists a nonconstant $B \in \mathbb{F}_2[x]$ such that $\sigma_k(A) = AB$ with $\deg(B) < \deg(A^k)$. So, $1+P^k+...+P^{k(\alpha-1)}+P^{k\alpha} = P^\alpha B$ and $P\left(P^{k-1} + ... + P^{k(\alpha-1)-1} + P^{\alpha-1}(P^k + B)\right) = 1$. Hence, $P = 1$ and this contradicts the fact that P is prime in $\mathbb{F}_2[x]$. □

Lemma 2.7 (Lemma 2.6 in [8]). *Let m be a positive integer and let T be a Mersenne prime in $\mathbb{F}_2[x]$, then $\sigma(x^{2m})$ and $\sigma(T^{2m})$ are both odd and squarefree.*

Lemma 2.8. *If m and k are positive integers, then $\sigma_k(P^{2m-1})$ is divisible by $x(x+1)$.*

Proof. Let $2m = 2^h s$, where s is odd and $h \geq 1$. Then,

$$\begin{aligned}\sigma_k(P^{2m-1}) &= 1 + P^k + ... + P^{k\left(2^h s-1\right)} \\ &= (1 + P^k)^{2^h-1}\left(1 + P^k + ... + P^{k(s-1)}\right)^{2^h}\end{aligned}$$

But $x(x+1)$ divides $1 + P^k$, P is odd. This completes the proof. □

Lemma 2.9. *If m and k are positive integers, then $\sigma_k(P^{2m})$ is not divisible by $x(x+1)$.*

Proof. $\sigma_k(P^{2m}) = 1+P^k+...+P^{2km}$. So, $\sigma_k(P^{2m})(0) = 1 + \underbrace{P^k(0)+...+P^{2km}(0)}_{2m-\text{times}} = 1$ and x is not factor of $\sigma_k(P^{2m})$. Also, $\sigma_k(P^{2m})(1) = 1$ and hence $\sigma_k(P^{2m})$ is not divisible by $x+1$. The proof is now complete. □

Next we give some properties when $k = 2$.

Lemma 2.10. *Let t be a positive integer, then* $\sigma_2(x^{3.2^{t-1}-1}) = (1+x)^{2^t-2}T_1^{2^t}$.

Proof. We use induction. For $t = 1$, we have $\sigma_2(x^2) = (1+x+x^2)^2 = T_1^2$. Hence, the statement is true for $t = 1$. Now assume it is true for t, so

$$\begin{aligned}
\sigma_2\left(x^{3.2^t-1}\right) &= \left(1+x+...+x^{3.2^{t-1}-1}+x^{3.2^{t-1}}\left(1+x+...+x^{3.2^{t-1}-1}\right)\right)^2 \\
&= \left(1+x+...+x^{3.2^{t-1}-1}\right)^2\left(1+x^{3.2^{t-1}}\right)^2 \\
&= \sigma_2\left(x^{3.2^{t-1}-1}\right)\left(1+x^3\right)^{2^t} \\
&= (1+x)^{2^t-2}T_1^{2^t}\left((1+x)\,T_1\right)^{2^t} \\
&= (1+x)^{2^{t+1}-2}T_1^{2^{t+1}}.
\end{aligned}$$

We are done. □

Lemma 2.11. *Let t be a positive integer, then* $\sigma_2((1+x)^{3.2^{t-1}-1}) = x^{2^t-2}T_1^{2^t}$.

Lemma 2.12. *Let t be a positive integer, then* $\sigma_2(T_1^{2^t-1}) = (x^2+x)^{2(2^t-1)}$.

Proof. For $t = 1$, we have $\sigma_2(T_1) = (1+T_1)^2 = (x^2+x)^2$. Hence, the statement is true for $t = 1$. Now assume $\sigma_2(T_1^{2^t-1}) = (x^2+x)^{2(2^t-1)}$. And,

$$\begin{aligned}
\sigma_2\left(T_1^{2^{t+1}-1}\right) &= \left(1+T_1+...+T_1^{2^t-1}+T_1^{2^t}\left(1+T_1+...+T_1^{2^t-1}\right)\right)^2 \\
&= \left(1+T_1+...+T_1^{2^t-1}\right)^2\left(1+T_1^{2^t}\right)^2 \\
&= \sigma_2\left(T_1^{2^t-1}\right)(1+T_1)^{2^{t+1}} \\
&= (x^2+x)^{2(2^t-1)}\left(x^2+x\right)^{2^{t+1}} \\
&= (x^2+x)^{2(2^{t+1}-1)}.
\end{aligned}$$

The proof is complete. □

The following lemma follows directly from Lemmas 2.10, 2.11, and 2.12.

Lemma 2.13. *Let $t \in \mathbb{N}$ and let $A = x^aT_1^h$ or $A = (1+x)^aT_1^h$ be polynomials in $\mathbb{F}_2[x]$, where $a = 3.2^{t-1}-1$ and $h = 2^t-1$. Then* $\sigma_2(A) = x^{2h}(1+x)^{2(a-1)}T_1^{h+1}$.

Lemma 2.14. *If $a = 2^tu-1$ with u odd. Then,*

i- $\sigma_2(x^a) = (1+x)^{2^{t+1}-2}\left(\sigma\left(x^{u-1}\right)\right)^{2^{t+1}}$

ii- $\sigma_2(P^a) = (1+P)^{2^{t+1}-2}\left(\sigma\left(P^{u-1}\right)\right)^{2^{t+1}}$.

Lemma 2.15. *Let* $t \in \mathbb{N}$ *and let* $A = x^a T_1^h$ *or* $A = (1+x)^a T_1^h \in \mathbb{F}_2[x]$. *If* A *divides* $\sigma_2(A)$, *then* $a = 3.2^{t-1} - 1$ *and* $h = 2^t - 1$.

Definition 2.1. *Let* $A \in \mathbb{F}_2[x]$ *be a polynomial of degree* m. *Then,*

i. *A inverts into itself if* $A^* = A$.

ii. *A is said to be* $k-$*complete if there exists* $h \in \mathbb{N}^*$ *such that* $A = \sigma_k(x^h) = 1 + x^k + ... + x^{kh}$.

Lemma 2.16. *i.* *Any* $k-$*complete polynomial inverts to itself.*

ii. *If* $1 + x^k + ... + x^{km} = PQ$, *then* $P = P^*$ *and* $Q = Q^*$ *or* $P = Q^*$ *and* $Q = P^*$, *where* P *and* Q *are irreducible polynomials in* $\mathbb{F}_2[x]$.

Proof. i. Let A be a $k-$complete polynomial, then there exists $h \in \mathbb{N}$ such that

$$\begin{aligned} A &= \sigma_k(x^h) \\ &= 1 + x^k + ... + x^{kh} \\ A^* &= x^{kh} A\left(\frac{1}{x}\right) \\ &= x^{kh}\left(1 + \frac{1}{x^k} + ... + \frac{1}{x^{kh}}\right), A \text{ is } k-\text{complete} \\ &= A. \end{aligned}$$

Hence, A inverts to itself.

ii. If $1 + x^k + ... + x^{km} = PQ$, then PQ is $k-$complete. Using the above results, then PQ inverts to itself. Hence, $(PQ)^* = PQ = P^*Q^*$. Therefore, $P = P^*$ and $Q = Q^*$ or $P = Q^*$ and $Q = P^*$. □

3. Proof of Theorem 1.3

The following lemma is a direct consequence of Lemma 2.6.

Lemma 3.1. *The polynomial* $A = P^\alpha$, $\alpha \geq 1$, *is not a* $k-$*perfect polynomial over* $\mathbb{F}_2$, *for every* $k \geq 1$.

The preceding lemma shows that a $k-$perfect polynomial A over $\mathbb{F}_2$ has at least 2 prime factors.

Lemma 3.2. *Let* $m \leq n$ *be positive integers and let* $A \in \mathbb{F}_2[x]$, *then* $\sigma_{2^m}(A)$ *divides* $\sigma_{2^n}(A)$.

Proof.

$$\begin{aligned} \sigma_{2^n}(A) &= (\sigma(A))^{2^n} \\ &= (\sigma(A))^{2^m} (\sigma(A))^{2^{n-m}} \\ &= \sigma_{2^m}(A) (\sigma(A))^{2^{n-m}}. \end{aligned}$$ □

Notice that $\sigma_2(A)$ divides $\sigma_{2^n}(A)$ for any any $n \geq 1$. Hence, if A is a multi-perfect polynomial over $\mathbb{F}_2$, i.e. A divides $\sigma(A)$, then A is a $k-$multi-perfect polynomial over $\mathbb{F}_2$ when $k = 2^n$ for a positive integer n.

Lemma 3.3. *If $t \in \mathbb{N}$ and $A = x^a T_1^h$ or $A = (1+x)^a T_1^h$ be polynomials in $\mathbb{F}_2[x]$, where $a = 3.2^{t-1} - 1$ and $h = 2^t - 1$, then A divides $\sigma_{2^n}(A)$ for any $n \geq 1$.*

Proof. Since σ_2 divides σ_{2^n} and $\sigma_2(A) = x^{2h}(1+x)^{2(a-1)}T_1^{h+1}$ with $2h = a + 2^{t-1} - 1$. □

Lemma 3.4. *If $a = 2^t u - 1$ with u odd and $n \in \mathbb{Z}_{\geq 0}$). Then,*

i- $1 + x$ *divides* $\sigma_{2^n}(x^a)$

ii- $x(1 + x)$ *divides* $\sigma_{2^n}(P^a)$

Proof. We have $\sigma_2(A)$ divides $\sigma_{2^n}(A)$ and $1 + x$ divides $\sigma_2(A)$ (Lemma 2.14). □

Lemma 3.5. *If A is $k-$perfect over $\mathbb{F}_2$, then $\overline{A}$ is also $k-$perfect over $\mathbb{F}_2$.*

Proof. Let $A(x) = \prod_{i=1}^{r} P_i^{\alpha_i}(x)$, where the primes $P_i(x) \in \mathbb{F}_2[x]$. Since A is $k-$perfect, then

$$\sigma_k(A) = \prod_{i=1}^{r} \frac{P_i^{k(\alpha_i+1)} - 1}{P_i^k - 1} = A^k. \tag{1}$$

Let F_{2^t} be a splitting field for $A(x)$ over $\mathbb{F}_2$, then there exists $a_1, a_2, ..., a_k \in F_{2^t}$ such that for each i, $1 \leq i \leq k$, we have $P_i^{\alpha_i}(x) = \prod_{j=0}^{\beta_i - 1}(x - a_i^{2^j})^{\alpha_i}$, where $\deg(P_i(x)) = \beta_i$. Since $\gcd(P_i(x), P_j(x)) = 1$ over $\mathbb{F}_2$, for every $i \neq j$, then $\gcd(P_i(x), P_j(x)) = 1$ over F_{2^t}, for every $i \neq j$. Moreover,

$$P_i(x+1) = \prod_{j=0}^{\beta_i - 1}(x + 1 - a_i^{2^j}) = \prod_{j=0}^{\beta_i - 1}(x - (a_i - 1)^{2^j}).$$

Since $a_i - 1$ has degree β_i, it follows that each $Q_i(x) = P_i(x+1)$ is prime of degree β_i in $\mathbb{F}_2[x]$. We have $\gcd(Q_i(x), Q_j(x)) = 1$ in $\mathbb{F}_2[x]$, for every $i \neq j$, and hence the primes $Q_i(x)$ are distinct. Let $B(x) = \overline{A}(x) = \prod_{i=1}^{r} P_i^{\alpha_i}(x+1) = \prod_{i=1}^{r} Q_i^{\alpha_i}(x)$.

By substituting $B(x)$ in (1), we get

$$\begin{aligned}
\sigma_k(\overline{A}(x)) &= \sigma_k(B(x)) \\
&= \prod_{i=1}^{r} \frac{P_i^{k(\alpha_i+1)}(x+1) - 1}{P_i^k(x+1) - 1} \\
&= \prod_{i=1}^{r} \frac{Q_i^{k(\alpha_i+1)}(x) - 1}{Q_i^k(x) - 1} \\
&= B^k(x) \\
&= (\overline{A}(x))^k.
\end{aligned}$$

So, $B(x) = \overline{A}(x)$ is $k-$perfect over $\mathbb{F}_2$ □

Lemma 2.1 shows the relation between $\sigma_k(A)$ and $\sigma(A)$ when $k = 2^n$, and its important consequence, Theorem 3.1, completely characterizes all $k-$perfect polynomials over $\mathbb{F}_2$ when $k = 2^n$.

Theorem 3.1. *A is perfect over $\mathbb{F}_2$ if and only if A is 2^n-perfect over $\mathbb{F}_2$.*

Proof. Let $A = \prod_{i=1}^{r} P_i^{\alpha_i} \in \mathbb{F}_2[x]$ be a perfect polynomial over $\mathbb{F}_2$, where P_i is an irreducible polynomial, then

$$\sigma_{2^n}(A) = (\sigma(A))^{2^n} = A^{2^n}.$$

The converse is done by contrapositive. Assume that A is not perfect. Then,

$$\sigma_{2^n}(A) = (\sigma(A))^{2^n} \neq A^{2^n},$$

and we are done. □

Lemma 3.6. *Let $\omega(A) \geq 2$ and let A be a 2^n-perfect polynomial over $\mathbb{F}_2$, then $x(x+1)$ divides A.*

The proof of the following lemma can be done by a direct computation.

Lemma 3.7. *Let t be a positive integer, then the polynomial $x^{2^t-1}(x+1)^{2^t-1}$ is 2^n-perfect over $\mathbb{F}_2$.*

Lemma 3.8. *If $A = A_1A_2$ is 2^n-perfect over $\mathbb{F}_2$ and if gcd$(A_1, A_2) = 1$, then A_1 is 2^n-perfect if and only if A_2 is 2^n-perfect.*

The following lemma contains some interesting results from Canaday's paper (see [2], Lemma 6 and Theorem 8).

Lemma 3.9. *Let $A, B \in \mathbb{F}_2[x]$ and let $n, m \in \mathbb{N}$.*

(i) *If $\sigma(P^{2n}) = B^m A$, with $m > 1$ and $A \in \mathbb{F}_2[x]$ is nonconstant, then* $\deg(A)(P) > \deg(A)(B)$.

(ii) *If $\sigma\left(x^{2n}\right)$ has a Mersenne factor, then $n \in \{1, 2, 3\}$.*

Gallardo and Rahavandrainy [6] conjectured that $\sigma(T^{2m})$ is always divisible by a non-Mersenne prime, for any $m \in \mathbb{N}$, when $T = x^a(x+1)^b+1$ is a Mersenne prime with $a + b \neq 3$.

Lemma 3.10. *Let $A = x^a(x+1)^b \prod_i P_i^{h_i}$ be a 2^n-perfect polynomial over $\mathbb{F}_2$ with each P_i is a Mersenne prime. Then $h_i = 2^{c_i} - 1$, for every i.*

Proof. Assume that h_i is even for every i. $A = x^a(x+1)^b \prod_i P_i^{h_i}$ be a 2^n-perfect then there exists a non Mersenne prime S such that S divides $\sigma(P_i^{h_i})$. So, S divides $\sigma_{2^n}(A) = A^{2^n}$. Therefore, $S = x$ or $S = x+1$ and this contradicts Lemmas 2.8 and 2.9 as h_i must be odd. Now, suppose that $h_i + 1 = 2^{c_i}u$, u is odd and $c_i \in \mathbb{N}$. But $\sigma(P_i^{h_i}) = (1+P_i)^{2^{c_i}-1}\left(\sigma(P_i^{u-1})\right)^{2^{c_i}}$. If $u-1 \geq 2$, again there exists a non Mersenne prime W such that W divides $\sigma(P_i^{u-1})$. So, W divides $\sigma_{2^n}(A) = A^{2^n}$. By Lemma 2.9, $W \neq x$ and $W \neq x+1$. But any prime divisor of A which is not a Mersenne prime is either x or $x+1$, a contradiction. Hence, $u = 1$ and the result follows. □

Lemma 3.11. *Let $c_i \in \mathbb{N}$, and let $A = x^a(x+1)^b \prod_i P_i^{2^{c_i}-1}$ be a 2^n-perfect polynomial over $\mathbb{F}_2$ with each P_i is a Mersenne prime. Then, $P_i \in \{T_1, T_2, ..., T_5\}$, with $c_i = 1$ or 2.*

Proof. Since A is 2^n-perfect, then any irreducible factor Q of $\sigma(x^a)$ or $\sigma((1+x)^b)$ must divide A. So, $Q \in \{x, x+1, P_1, P_2, ...\}$. From Lemma 3.9(*ii.*), we have $Pi \in \{T_1, T_2, ..., T_5\}$. Now, we want to prove that $c_j \in \{1, 2\}$. Note that $\sigma(P_i^{2^{c_i}-1}) = (1+P_i)^{2^{c_i}-1}$ is not divisible by P_j, for any i, j. Moreover, if α_j are the exponents of P_j that are found in $\sigma(x^a)$ and in $\sigma((1+x)^b)$, then $\alpha_j \in \{0, 1, 2^r : r \in \mathbb{N}\}$ (Lemma 3.9(*ii.*)). Comparing exponents of P_j, we get $\alpha_j = 2^{c_j} - 1 \in \{0, 1, 2, 2^r, 2^r+1, 2^r+2^s : r, s \in \mathbb{N}\}$. Hence, $c_j = 1$ or 2. □

Lemma 3.12. *Let $c_i \in \mathbb{N}$, $P_i \in \{T_1, T_2, ..., T_5\}$, and $A = x^a(x+1)^b \prod_i P_i^{c_i}$ be a 2^n-perfect polynomial over $\mathbb{F}_2$ with $c_i \in \{1, 3\}$. Then a or b must be even.*

Proof. For contradictional purpose, assume that a and b are both odd. By Lemma 3.13, we have $a = 2^r u - 1$ and $b = 2^s v - 1$ for some $t, s \in \mathbb{N}$, and u and v are odd positive integers less than or equal to 7. But,

$$\sigma(x^a) = (x+1)^{2^t-1}(1+x+...+x^{u-1})^{2^t}$$

and

$$\sigma((1+x)^b) = x^{2^s-1}\left(1+(1+x)+...+(1+x)^{v-1}\right)^{2^s}.$$

Also, P_i is not a factor of $\sigma(P_j^{c_j}) = (1+P_j)^{c_j}$ for any i, j. Suppose that P_i is a factor of $1+x+...+x^{u-1}$ but is not a factor of $1+(1+x)+...+(1+x)^{v-1}$ for some i, with $u \geq 3$. Hence, $2^t = c_i = 2^{h_i} - 1$, a contradiction.

Now, assume that P_i is a factor of both $1+x+...+x^{u-1}$ and $1+(1+x)+...+(1+x)^{v-1}$, then $2^t + 2^s = c_i = 2^{h_i} - 1$, also a contradiction. Therefore, $u = 1$ and in a similar manner we get $v = 1$. So, $\sigma(x^a) = \sigma(x^{2^t-1}) = (x+1)^a$ and $\sigma\left((x+1)^b\right) = \sigma\left((x+1)^{2^s-1}\right) = x^b$. Hence, $a = b$ and $x^a(x+1)^b$ is a 2^n-perfect (Lemma 3.7). By Lemma 3.8, the polynomial $\prod_{i=1}^r P_i^{h_i}$ is also 2^n-perfect. This contradicts Lemma 3.1. □

Lemma 3.13. *Let $c_i \in \mathbb{N}$, $u \geq 1$ and a be odd integers and let $A = x^a(x+1)^b \prod_i P_i^{2^{c_i}-1}$ be a 2^n-perfect polynomial over $\mathbb{F}_2$, where each P_i is a Mersenne prime. Then, a is of the form $2^t u - 1$ with $u \leq 7$.*

Proof. Suppose that $a = 2^t u - 1$ with u is odd and $t \geq 1$. Since A is 2^n-perfect over $\mathbb{F}_2$, then

$$x^{2^n a}(x+1)^{2^n b} \prod_{i=1} P_i^{2^n(2^{c_i}-1)} = \left(\sigma(x^a)\sigma\left((x+1)^b\right) \prod_{i=1} \sigma\left(P_i^{2^{c_i}-1}\right) \right)^{2^n}.$$

But $\sigma(x^a) = 1 + x + ... + x^{2^t u - 1} = (1+x)^{2^t - 1}\sigma(x^{u-1})^{2^t}$. If $u > 2$, then as done in the proof of the preceding lemma we get $u - 1 \leq 6$ and hence the result. $\square$

Lemma 3.14. *Let $a, b, c_i \in \mathbb{N}$ such that a is even and let $A = x^a(x+1)^b \prod_{i=1}^m P_i^{2^{c_i}-1}$ be a 2^n-perfect polynomial over $\mathbb{F}_2$, where each P_i is a Mersenne prime. Then, $a \leq 6$.*

Proof. Let $a = 2m$. Since A is 2^n-perfect over $\mathbb{F}_2$, then

$$\begin{aligned} x^{2^{n+1}m}(x+1)^{2^n b} \prod_{i=1} P_i^{2^n(2^{c_i}-1)} &= A^{2^n} \\ &= \sigma_{2^n}(A) \\ &= \left(\sigma(x^{2m})\sigma\left((x+1)^b\right) \prod_{i=1} \sigma\left(P_i^{2^{c_i}-1}\right) \right)^{2^n}. \end{aligned}$$

But x and $x+1$ do not divide $\sigma(x^{2m})$ and P_i does not divide $\sigma\left(P_i^{2^{c_i}-1}\right)$ so P_i divides $\sigma(x^{2m})$. We are done by Lemma 3.9 (*ii.*). $\square$

3.1 Cases of the Proof

Let $A = x^a(x+1)^b \prod_{i=1}^r P_i^{h_i}$, where P_i, is a Mersenne prime be a 2^n-perfect over F_2. From Lemma 3.11, we have $h_i = 1$ or 3. By Lemma 3.12, we have a or b is even. To complete the proof of Theorem 1.3, we study the below cases:
Case 1. Both a and b are even:

In this case, we have

(2) $$1 + x + ... + x^a = P_{i_1} ... P_{i_s}.$$

Since the P_{i_j}'s are Mersenne primes, then $a, b \in \{2, 4, 6\}$. Since if A is a 2^n-perfect polynomial over F_2, then $\overline{A}$ is a 2^n-perfect polynomial over $\mathbb{F}_2$ so a and b can be chosen in the way $a \leq b$ and $a, b \in \{2, 4, 6\}$.

- If $a = b = 2$, then $1 + x + x^2 = 1 + (x+1) + (x+1)^2 = T_1$. Hence, $A = x^2(x+1)^2 T_1$ and $\sigma(A) = \sigma\left(x^2\right)\sigma\left((x+1)^2\right)\sigma(T_1) = (T_1)(T_1)(x(1+x)) = x(1+x)T_1^2 \neq A$. Therefore A is not perfect over $\mathbb{F}_2$ and hence A is not 2^n-perfect over $\mathbb{F}_2$ (Theorem 3.1).

- If $a = 2$ and $b = 4$, then $1 + x + x^2 = T_1$ and $1 + (x+1) + ... + (x+1)^4 = 1 + x^3(x+1) = T_5$. Hence, $A = x^2(x+1)^4 T_1 T_5$ and $\sigma(A) = \sigma\left(x^2\right)\sigma\left((x+1)^4\right)\sigma(T_1)\sigma(T_5) = (T_1)(T_5)(x(1+x))\left(x^3(1+x)\right) = x^4(1+x)^2 T_1 T_5 \neq A$. So, A is not 2^n-perfect over F_2 (Theorem 3.1).

- If $a = b = 4$, then $1 + x + ... + x^4 = T_4$ and $1 + (x + 1) + ... + (x + 1)^4 = 1 + x^3 + x^4 = T_5$. Hence, $A = x^4(x + 1)^4 T_4 T_5$ and $\sigma(A) = \sigma\left(x^4\right)\sigma\left((x+1)^4\right)\sigma(T_4)\sigma(T_5) = (T_4)(T_5)\left(x(1+x)^3\right)\left(x^3(1+x)\right) = x^4(1+x)^4 T_4 T_5 = A$. So, A is 2^n-perfect over $\mathbb{F}_2$ (Theorem 3.1).

- If $a = 2$ and $b = 6$, then $1 + x + x^2 = T_1$ and $1 + (x+1) + ... + (x+1)^6 = (1 + x + x^3)(1 + x^2 + x^3) = T_2 T_3$. Hence, $A = x^2(x+1)^6 T_1 T_2 T_3$ and

$$\begin{aligned} \sigma(A) &= \sigma\left(x^2\right)\sigma\left((x+1)^6\right)\sigma(T_1)\sigma(T_2)\sigma(T_3) \\ &= (T_1)(T_2T_3)(x(1+x))\left(x(1+x)^2\right)\left(x^2(1+x)\right) \\ &= x^4(1+x)^4 T_1 T_2 T_3 \\ &\neq A. \end{aligned}$$

  Therefore, A is not 2^n-perfect over $\mathbb{F}_2$.

- If $a = 4$ and $b = 6$, then $1+x+...+x^4 = T_4$ and $1+(x+1)+...+(x+1)^6 = T_2T_3$. Hence, $A = x^4(x+1)^6 T_2 T_3 T_4$ and $\sigma(A) = A$. So, A is 2^n-perfect over $\mathbb{F}_2$.

- If $a = b = 6$, then $1 + x + ... + x^6 = (1 + x + x^3)(1 + x^2 + x^3) = T_2T_3 = 1 + (x + 1) + ... + (x + 1)^6$. Hence, $A = x^6(x + 1)^6 T_2^2 T_3^2$ and $\sigma(A) = \sigma\left(x^6\right)\sigma\left((x+1)^6\right)\sigma\left(T_2^2\right)\sigma\left(T_3^2\right) = T_1^2 T_2^2 T_3^2 T_4 T_5 \neq A$. Therefore, A is not 2^n-perfect over $\mathbb{F}_2$.

Case 2. a is even and b is odd:

By Lemmas 3.13 and 3.14, we have $a \in \{2, 4, 6\}$ and $b = 2^t u - 1$ for some $t \in \mathbb{Z}_{\geq 1}$ and $u \in \{1, 3, 5, 7\}$.

- If $u = 1$ and $a = 2$, then $\sigma(x^2) = T_1, \sigma((x + 1)^{2^t-1}) = x^{2^t-1}$, and $\sigma(T_1) = x(x+1)$. Hence, $2^t-1+1 = b+1 \leq a = 2$. Thus, $t = 1$ and $A = x^2(x+1)T_1$.

- If $u = 1$ and $a = 4$, then $\sigma(x^4) = T_4, \sigma((x + 1)^{2^t-1}) = x^{2^t-1}$, and $\sigma(T_4) = x(x+1)^3$. Hence, $2^t-1+1 = b+1 \leq a = 4$. Thus, $t \leq 2$ and $3 \leq b = 2^t-1$, so $t = 2$ and $A = x^4(x + 1)^3 T_4$.

- If $u = 1$ and $a = 6$, then $\sigma(x^6) = T_2T_3, \sigma((x + 1)^{2^t-1}) = x^{2^t-1}$, $\sigma(T_2) = x(x + 1)^2$ and $\sigma(T_3) = x^2(x + 1)$. Hence, $2^t - 1 + 2 + 1 = b + 3 \leq a = 6$. Thus, $t \leq 2$ and $3 \leq b = 2^t - 1$, so $t = 2$ and $A = x^6(x + 1)^3 T_2 T_3$.

- If $u = 3$ and $a = 2$, then $\sigma(x^2) = T_1, \sigma((x + 1)^{3.2^t-1}) = x^{2^t-1}T_1^{2^t}$. Hence, $T_1^{2^t+1}$ divides $\sigma(A) = A$ but $T_1^{2^t+2}$ does not divide $\sigma(A) = A$. By Lemma 3.11, we have $2^t + 1 \in \{1, 3\}$ and thus $t = 1$ and $A = x^2(1 + x)^5 T_1$. But $\sigma(x^2(1 + x)^5 T_1) \neq x^2(1 + x)^5 T_1$ and hence A is not 2^n-perfect over F_2.

- If $u = 3$ and $a = 4$, then $\sigma(x^4) = T_4$. Since T_1 does not divide $\sigma(x^4)$, then $T_1^{2^t}$ divides $\sigma(A) = A$ but $T_1^{2^t+1}$ does not divide $\sigma(A) = A$. By Lemma 3.11, we have $2^t \in \{1, 3\}$, a contradiction.

- The case $u = 3$ and $a = 6$ is similar to the preceding one.
- If $u = 5$ and $a \in \{2, 6\}$, then $\sigma((x+1)^{5.2^t-1}) = x^{2^t-1}T_4^{2^t}$. Since T_4 does not divide $\sigma(x^a)$, then $T_4^{2^t}$ divides $\sigma(A) = A$ where $T_1^{2^t+1}$ does not divide $\sigma(A) = A$. By Lemma 3.11, we have $2^t \in \{1, 3\}$, a contradiction.
- If $u = 5$ and $a = 4$, then $\sigma(x^4) = T_4$. Since $T_4^{2^t+1}$ divides A and $T_1^{2^t+2}$ does not divide A. By Lemma 3.11, we have $2^t + 1 \in \{1, 3\}$. Thus $t = 1$ and $A = x^4(1+x)^9T_1^3$. But $\sigma(x^4(1+x)^9T_1^3) \neq x^4(1+x)^9T_1^3$. Hence, A is not 2^n-perfect over F_2.
- If $u = 7$ and $a \in \{2, 4\}$, then $\sigma((x+1)^{7.2^t-1}) = x^{2^t-1}T_2^{2^t}T_3^{2^t}$. Since T_2 and T_3 do not divide $\sigma(x^a)$, then $T_2^{2^t}$ divides A and $T_2^{2^t+1}$ does not divide $\sigma(A) = A$. By Lemma 3.11, we have $2^t \in \{1, 3\}$, a contradiction.
- If $u = 7$ and $a = 6$, then $\sigma\left(x^6\right) = T_2T_3$. So, $T_2^{2^t+1}$ (resp. $T_3^{2^t+1}$) divides A and $T_2^{2^t+1}$(resp. $T_3^{2^t+1}$) does not divide A. By Lemma 3.11, we have $2^t+1 \in \{1, 3\}$. Thus $t = 1$ and $A = x^6(1+x)^{13}T_2^3T_3^3$. But $\sigma\left(x^6(1+x)^{13}T_2^3T_3^3\right) \neq x^6(1+x)^{13}T_2^3T_3^3$. Hence, A is not 2^n-perfect over F_2.

The proof of Theorem 1.3 is now complete

4. Conclusion

We show the non existence of odd 2^n-perfect, $n \in \mathbb{N}$, polynomials over $\mathbb{F}_2$. A characterization of 2^n-perfect polynomials A over the prime field with two elements that are divisible by x, $x+1$, and Mersenne primes is given.

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